Coding for Random Access in Wireless Networks

by

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Abstract

Wireless networks in the near future face a formidable challenge of accommodating a dense set of infrequently communicating devices characterized by small data payloads and strict latency and energy constraints. In such a scenario, providing energy efficient random-access access becomes a challenge. Information theoretic analysis of such systems becomes imperative to understand the gap from optimality of the methods of random-access currently employed.

In this thesis we discuss the trade-off between the required energy-per-bit to achieve a target probability of error (per-user) and the number of active users. Previous works in this regard focused on the AWGN channel model. In this thesis we consider the issue of Rayleigh fading. Specifically, we use random coding with a subspace projection based decoder to get finite blocklength bounds from which we arrive at the trade-off. Further we justify the use of our decoder by proving its asymptotic optimality for the channel under consideration. We also show that the required energy-per-bit increases from around 0-2 dB (for AWGN) to around 8-12 dB under fading.

Thesis Supervisor: Yury Polyanskiy
Title: Professor of Electrical Engineering and Computer Science

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[1] Sangam is the Indian graduate students association at MIT

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Chapter 1

Introduction

One of the significant challenges faced by wireless networks in the future is to handle a massive number of occasionally communicating terminals where the energy requirements are very strict. This is also known as massive machine-type communication (mMTC) and there has been significant discussions on this problem in the next generation (5G) communities. MTC applications look towards having hundreds of thousands of devices connected to a single base station communicating sporadically with small data payloads [8]. For instance, the internet of things (IoT) is characterized, among other things, by the network's support for communication of massive number of devices and long battery lifetimes [25]. In general the characteristics of such MTC networks include very high node density, short packets, sporadic transmission, strict energy constraints and limited computational capability [22]. In such a scenario, access management plays a critical role. In this thesis, we focus on the energy-efficiency aspect of such systems.

Currently, there is an active discussion of possible transmission schemes for mMTC in 3GPP standardization committee. The main candidates are multi-user shared access (MUSA, [38]), sparse coded multiple access (SCMA, [23]) and resource shared multiple access (RSMA, [15] [1]).In MUSA, fixed complex spreading sequences of short length and successive interference cancellation (SIC) decoder are employed. SCMA is a modification of the low-density signature (LDS [16]) scheme in which sparse spreading sequences are used. Due to sparsity, iterative decoding using message passing algorithm [16] [18] [35] can be used. SCMA differs from LDS in the code used for spreading: the latter uses repetition coding (e.g. from QAM) whereas the former uses a specifically designed multidimensional codebook to map the incoming bits which results in enhanced gain over the latter. SCMA codebooks are designed by casting it as an optmization problem [23] [32], and different codebooks are allocated for each user [23]. Coming to the third scheme, RSMA is characterized by long pseudo-random spreading sequences along with a low-rate code. RSMA has the advantage of low complexity over SCMA. In ideal scenarios, the performance of SCMA is better than RSMA but with significantly complex receiver; but the advantages of SCMA are not so significant in a practical situation (with imperfect power control, for example) [2].

In current systems (such as LTE) the initial uncoordinated multiple-access problem is addressed by using a low-rate physical random-access channel (PRACH), using which the nodes register themselves with the base station (BS) [3]. Subsequent communication is managed centrally by the BS. Simple slotted ALOHA [30] [3] scheme is used for random-access. But is this does not scale to high-node density. This is the problem we are addressing in this work.

Since ALOHA is known to have low utilization ($\approx 37\%$) [30], a recent improvement, *coded slotted ALOHA* (CSA), was introduced in [9] where users repeat their packet in different slots. Interfering packets can be successively cancelled by using the additive nature of the channel. Furthering this idea, in [20], interference cancellation is linked to iterative decoding of graph based codes. By optimizing the probability distribtuion of the repetition rate, it is shown that the utilization improves to around 80%. In [11] a *grant free* random access scheme called asynchronous ALOHA was developed using similar ideas of packet repetition. In [4], more progress in asynchronous ALOHA have been made towards low-complexity IoT devices.

According to what we are aware of, at the heart of most of the ALOHA based methods lies the ability to decode only uncollided information packets, and hence any (non-orthogonal) collision is declared as an error. The idea and analysis of using multi-user detectors for resolving small order collsions has appeared many times (see [14], for example or more recently in [20, Appendix A]). More recently, a concrete scheme to resolve higher order collisions called *T–fold ALOHA* was proposed in [24] in which up to T collisions are decoded using a specially constructed code, and its performance in terms of energy-per-bit on an additive white Gaussian noise (AWGN) channel was compared against the slotted ALOHA scheme and the finite blocklength (FBL) bounds developed in [27]. In this work, we follow similar lines to develop FBL bounds using random coding and T–fold ALOHA on a quasi-static Rayleigh fading channel, and show that this gives better trade-off than slotted ALOHA (which is just 1-fold ALOHA) (see figure 6-1).

The model we consider in this thesis follows from [27] and [24]. Consider a single base-station (receiver) and a large, potentially unbounded, number of transmitters wanting to communicate with the base-station. Let $K_a \geq 1$ be a fixed integer. This represents the number of active users – at any given time exactly K_a of the users are transmitting. Further, the message of each user is of size k bits and it is transmitted over n channel uses, which is the blocklength. Typical values of these parameters that we consider are $k = 100$ bits and $n = 30000$. Hence we are looking at theoretically evaluating the performance for small payloads in the FBL regime. As mentioned in [27], the goal of this model is to be able to take the total number of users to be infinite, and hence schemes like ALOHA would become an achievability. Moreover, any FBL bound would give us a way to compare all such achievability schemes against one another, and a gap from from the information theoretic bound would clearly depict how far the current schemes are from the theoretical achievability.

Coming back to the channel model, we consider a quasi-static Rayleigh fading channel with additive white Gaussian noise (AWGN) (2.6). Since we are considering a situation with potentially unbounded number of users, it is necessary that all users use the same codebook. Further, as in a vanilla AWGN MAC, each user is subject to a maximum power constraint P (2.8). We consider the case of the so called no channel state information (no-CSI) where neither the transmitters nor the receiver have knowledge of the realization of the fading coefficients. This makes sense in our model since the payload is small and the set of active users can change, it may not be feasible to estimate the channel reliably. Since we are dealing with the quasi-static case, the fading coefficients remain fixed for the entire block of transmission. The decoder at the receiver is supposed to output an estimate of the list of the messages that were sent. Due to the common codebook, the decoding is done upto permutation of sent messages, and hence user identification is ruled out (which also makes sense when the number of users is infinite). The rationale for this is that the identity of users can be embedded in the messages the users send. For instance in LTE PRACH where the users contend for resources by sending preambles, it is enough if the base station is able to decode the list of preambles that were sent. Finally the error metric is the expected fraction of incorrectly decoded messages.

We review some of the related works. As mentioned before, this thesis continues the line of work initiated in [27] where FBL bounds were developed for the random-access situation in an AWGN channel. Further, this bound was compared against existing schemes like ALOHA, TDMA etc. A low complexity scheme using cocatinated codes and T–fold aloha was introduced in [24]. Although this is much better than ALOHA, there is a significant gap from the FBL bound. Serial interference cancellation along with interleaved LDPC codes was considered in [33] towards reducing this gap, and further improvements in the LDPC part of [33] was done in [21]. The error metric considered in all of these works is the per-user probability of error. The idea of per-user error error can be traced back to [5]. In [5], the average fraction of users that cannot be decoded was analyzed for a quasi-static K–user MAC (with CSIR) in the regime of $n \to \infty$ and both $K = 2$ and $K \gg 1$. The analysis for large K was carried out using the fact that the ordered statistics of fading coefficients *crys-*

tallize to some constants. We use this result as a benchmark to compare our bounds.

The main contributions of this thesis are that we develop FBL bounds for the quasi-static Rayleigh fading random access MAC under the no-CSI assumption [1]. We use this bound to find the minimum *energy-per-bit* $E_b/N_0 = nP/k$ to achieve a target probability of error. This random coding achievability bound uses a subspace projection based decoder inspired from [37] which doesn't need the knowledge of the realization of the fading coefficients. Further we show that this decoder achieves the ϵ-capacity region of the quasi-static MAC under the classical joint probability of error. We also show that this decoder achieves the same asymptotics as that of the joint decoder in [5] under per-user error. We also develop a simple converse bound based on the converse from [37] and a modification to the meta-converse theorem from [28]. We would also like to mention here that in a recent unplished work [17] with our collaborators a low-complexity iterative decoding scheme is developed based on LDPC codes [13,29,31] and a belief propagation based decoder that shows significant performance compared to the theoretical predictions. So, although not part of the work in the thesis we include these results in our plots so as to compare our theoretical predictions to an actual coding scheme [2].

[1] Part of this work appears in an unpublished work [17] with the author of the thesis as one of the co-authors. The FBL bounds, T-fold ALOHA and the converse in [17] were developed by the present author.

[2] The practical coding scheme based on LDPC codes in [17] were developed by Alexey A. Frolov (Skoltech, Moscow, al.frolov@skoltech.ru) and Yury Polyanskiy (MIT, yp@mit.edu). We have used the this data as an additional graph in our plots, and it is labeled with a name containing "LDPC".

Chapter 2

Definitions and System Model

In this chapter we introduce the definitions of a code and also describe the system model.

For a positive integer m, let $[m] = \{1, 2, ..., m\}$. We denote by $\mathcal{CN}(\mu, \Sigma)$ the complex normal distribution with mean μ and covariance matrix Σ (and pseudo-covariance 0).

2.1 Definitions

Definition 1 (MAC). *Fix an integer $K \geq 1$. A multiple access channel (MAC) with K users is a sequence triples $\left(\times_{i=1}^{K} \mathcal{X}_i^n, \mathcal{Y}^n, P_{Y|X_1^n,...,X_2^n}\right)_n$ where $\mathcal{X}_i^n$ is the alphabet of user i, $\mathcal{Y}^n$ is the output alphabet, and $P_{Y^n|X_1^n,...,X_2^n} : \times_{i=1}^{K} \mathcal{X}_i^n \to \mathcal{Y}^n$ is a probability transition kernel.*

Next we define the random access MAC with K_a active users.

Definition 2 (RAC). *Fix an integer $K_a \geq 1$. A random access MAC (RAC) with K_a active users is a sequence of triples $\left(\mathcal{X}^n, \mathcal{Y}^n, P_{Y^n|X_1^n,...,X_{K_a}^n}\right)_n$ where $\mathcal{X}^n$ and $\mathcal{Y}^n$ denote the common input and output alphabets respectively, and $P_{Y^n|X_1^n,...,X_{K_a}^n} : (\mathcal{X}^n)^{K_a} \to \mathcal{Y}^n$ is a probability transition kernel. Further, it is assumed that kernel is permutation invariant: for any permutation π on $[K_a]$ the distribution $P_{Y^n|X_1^n,...,X_{K_a}^n}(\cdot|x_1^n, ..., x_{K_a}^n)$ coincides with $P_{Y^n|X_1^n,...,X_{K_a}^n}(\cdot|x_{\pi(1)}^n, ..., x_{\pi(K_a)}^n)$.*

We note that the above definition of RAC allows for unbounded number of users but only K_a of them are active at any time.

For the sake of brevity, we denote the kernel in the above definitions by $P_{Y^n|X^n}$ or just $P_{Y|X}$ when what we are referring to is clear from the context.

Now we give various definitions of a code for both MAC and RAC. There are four possible variants for each: same vs non-same codebook and joint vs per-user (average) probability of error. But we do not define for all the eight combinations; we just deal with non-same codebook for the MAC and same-codebook for the RAC.

Definition 3. *An $((M_1, M_2, ..., M_K), n)$ code for a K user MAC $P_{Y^n|X^n}$ is a set of (possibly randomized) maps $\{f_i : [M_i] \to \mathcal{X}_i^n\}_{i=1}^K$ (the encoding functions) and $g : \mathcal{Y}^n \to \prod_{i=1}^K [M_i]$ (the decoder).*

Definition 4. *An (M, n) code for a K_a active user RAC $P_{Y^n|X^n}$ is a set of (possibly randomized) maps $f : [M] \to \mathcal{X}^n$ (the encoding functions) and $g : \mathcal{Y}^n \to \binom{[M]}{K_a}$ (the decoder). Here $\binom{[M]}{K_a}$ denotes the subsets of $[M]$ of size K_a.*

Notice that in def. 4, the decoder's output is just a list of K_a messages. This is due to the fact that users employ the same codebook and the channel is permutation invariant. We emphasize here that in the random access setting, we avoid the user identification problem since we allow a setting where the total number of users could be taken as infinite. Further, in a practical setting, the messages can contain the headers for user identification. Hence our main focus is on data transmission rather than user identification.

Definition 5 (Non-same codebook, joint error)**.** *An $((M_1, M_2, ..., M_K), n, \epsilon)_J$ code for the MAC $P_{Y^n|X^n}$ is an $((M_1, M_2, ..., M_K), n)$ code such that if for $j \in [K]$, $X_j = f(W_j)$ constitute the input to the channel and W_j is chosen uniformly (and independently of other W_i, $i \neq j$) from $[M_j]$ then the average (joint) probability of error satisfies*

$$\mathbb{P}\left[\bigcup_{j\in[K]} \left\{W_j \neq (g(Y))_j\right\}\right] \leq \epsilon \tag{2.1}$$

where Y is the channel output. If there are input constraints where each $X_i \in F_i \subset \mathcal{X}_i^n$, then we define an $((M_1, M_2, ..., M_K), n, \epsilon, (F_1, ..., F_K))_J$ code as $((M_1, M_2, ..., M_K), n, \epsilon)_J$ code where each codeword satisfies the input constraint.

From now on, we do not explicitly state the cost constraint in the definition.

Definition 6 (Non-same codebook, per-user error). *An $((M_1, M_2, ..., M_K), n, \epsilon)_{PU}$ code for the MAC $P_{Y^n|X^n}$ is an $((M_1, M_2, ..., M_K), n)$ code such that if for $j \in [K]$, $X_j = f(W_j)$ constitute the input to the channel and W_j is chosen uniformly (and independently of other W_i, $i \neq j$) from $[M_j]$ then the average (per-user) probability of error satisfies*

$$\frac{1}{K}\sum_{j=1}^{K} \mathbb{P}\left[W_j \neq (g(Y))_j\right] \leq \epsilon \tag{2.2}$$

where Y is the channel output.

Definition 7 (Same codebook, per-user error [27]). *An (M, n, ϵ) random-access code for the K_a user RAC $P_{Y^n|X^n}$ is an (M, n) code such that if $W_1, ..., W_{K_a}$ are chosen independently and uniformly from $[M]$ and $X_j = f(W_j)$ then the average (per-user) probability of error satisfies*

$$\frac{1}{K_a}\sum_{j=1}^{K_a} \mathbb{P}\left[E_j\right] \leq \epsilon \tag{2.3}$$

where $E_j \triangleq \{W_j \notin g(Y^n)\} \cup \{W_j = W_i \text{ for some } i \neq j\}$ and Y is the channel output.

Observe that according to this definition, collision results in an error. The rationale is that the probability of a collision is at most $\frac{\binom{K_a}{2}}{M}$ which is small in a practical situation. For example, we consider the scenario where each user has a payload of 100 bits and the number of active users K_a of order 100.

Next we have the definition of ϵ-achievability of codes from [15].

Definition 8 (ϵ–achievability [15]). *Fix $\epsilon > 0$. Let $R_1, R_2, ..., R_K$ be non-negative real numbers. We say that a rate tuple $(R_1, ..., R_K)$ is joint (or per-user) ϵ–achievable for a MAC $P_{Y^n|X^n}$ of there exists a sequence of $\left((M_1^{(n)}, M_2^{(n)}, ..., M_K^{(n)}), n, \epsilon_n\right)_{J \text{ \{or } PU \text{ resp.\}}}$*

codes such that

$$\limsup_{n\to\infty} \epsilon_n \le \epsilon \tag{2.4a}$$

$$\forall i \in [K], \liminf_{n\to\infty} \frac{1}{n} \log M_i^{(n)} \ge R_i \tag{2.4b}$$

Note that the above definition holds for both joint and per-user probabilities of error for the case of non-same codebook.

Using joint ϵ–achievability, we can talk of the ϵ–capacity region for the MAC.

Definition 9 (ϵ-capacity region C_ϵ [15]). *The joint ϵ–capacity region for the MAC $P_{Y^n|X^n}$ is defined as the set of all rate tuples that are ϵ-achievable. That is*

$$C_\epsilon = \{(R_1, ..., R_K : \forall i, R_i \ge 0, \text{ and } (R_1, ..., R_K) \text{ is } \epsilon\text{–achievable})\}. \tag{2.5}$$

2.2 System model

In this thesis, the focus is exclusively on the quasi-static fading MAC (or RAC) which is described below.

1. **K–user fading AWGN MAC (K–MAC):** The channel law $P_{Y^n|X^n}$ is described by

$$Y^n = \sum_{i=1}^{K} H_i X_i^n + Z^n \tag{2.6}$$

   where $X_i^n \in \mathbb{C}^n$, $Z^n \sim \mathcal{CN}(0, I_n)$, and $H_i \overset{iid}{\sim} \mathcal{CN}(0,1)$ are the fading coefficients which are independent of $\{X_i^n\}$ and Z^n.

2. **K_a–user random-access fading AWGN MAC (K_a–MAC):** This is a K_a active user RAC (def. 2) with the channel law given by

$$Y^n = \sum_{i=1}^{K_a} H_i X_i^n + Z^n \tag{2.7}$$

where $X_i^n \in \mathbb{C}^n$, $Z^n \sim \mathcal{CN}(0, I_n)$, and $H_i \overset{iid}{\sim} \mathcal{CN}(0,1)$ are the fading coefficients which are independent of $\{X_i^n\}$ and Z^n.

We emphasize that the fading coefficients remain fixed for the entire duration of the transmission, and hence quasi-static.

Further, for both the above models, we assume that there is a maximum power constraint:

$$\|X_i^n\|^2 \leq nP. \tag{2.8}$$

In the rest of the thesis we drop the superscript n where it may not cause confusion.

Chapter 3

K_a–MAC: Achievability and converse

In this chapter, we provide some achievability and converse bounds for the K_a–MAC. For the achievability part, we use random coding with subspace projection based decoding. The converse is a simple list-decoding version of the converse in [37]. The results provided here are for the case of no channel state information (no-CSI) either at the transmitters or the receiver.

3.1 Achievability

Before stating the achievability result, we describe the encoder and the decoder for random coding.

3.1.1 Encoder

As with a vanilla Gaussian MAC, we use random coding with either

1. **Spherical codebook:** For each message, a vector uniformly distributed on the $\sqrt{nP}$–complex sphere is independently generated. That is $X_i \overset{iid}{\sim} Unif\left(\sqrt{nP}(\mathcal{CS})^{n-1}\right)$ where $((\mathcal{CS}))^{n-1}$ denotes the unit sphere in $\mathbb{C}^n$.

   or

2. **Gaussian codebook:** For each message a $\mathcal{CN}(0, P'I_n)$ vector is independently generated. That is $X_i \overset{iid}{\sim} \mathcal{CN}(0, P'I_n)$ where $P' < P$. For a message W_j of user

j, if $\|X(W_j)\|^2 > nP$ then that user sends 0.

Also observe that a set at most $n-1$ codewords are linearly independent almost surely.

3.1.2 Decoder: Projection decoding

Inspired from [37], we use a projection based decoder. The idea is the following. Suppose there were no additive noise. Then the received vector will lie in the subspace spanned by the sent codewords no matter what the fading coefficients are. So a decoder that outputs a list of K_a codewords which form the subspace, such that projection of Y onto to this subspace is maximum is a natural choice. Formally, let C denote a set of vectors in $\mathbb{C}^n$. Denote P_C as the orthogonal projection operator onto the subspace spanned by C.

Let $\mathcal{C}$ denote the common codebook. Then, upon receiving Y from the channel, the decoder outputs $g(Y)$ given by

$$\begin{aligned} g(Y) &= \{f^{-1}(c) : c \in \hat{C}\} \\ \hat{C} &= \arg \max_{C \subset \mathcal{C}: |C| = K_a} \|P_C Y\|^2 \end{aligned} \tag{3.1}$$

where f is the encoding function.

Another rationale for using projection decoding is that, when the receiver does not have the channel state information, projection onto subspaces is a natural thing to do. Since the receiver is trying to find the closest subspace (from a collection of subspaces) to the received vector, the channel coefficients are implicitly estimated as a function of the codewords spanning the subspace: they are precisely the coordinates of projection of Y in the basis of codewords constituting that subspace. Further, we show later that projection decoding achieves the ϵ–capacity of the K–MAC.

3.1.3 Achievability bounds

In this sub-section we state our main achievability results.

Theorem 3.1.1. *Fix $P' < P$. Then there exists an (M, n, ϵ) random access code for the K_a-MAC satisfying power constraint P (see (2.8)) and*

$$\epsilon \leq \frac{1}{K_a} \sum_{t=1}^{K_a} t\mathbb{P}\left[F_t\right] + p_0 \tag{3.2}$$

where, if $S, \hat{S}$ denote the indices corresponding to sent and decoded codewords respectively, $F_t = \{|S \setminus \hat{S}| = t\}$, $p_0 = \frac{\binom{K_a}{2}}{M}$ for the Spherical codebook and $p_0 = \frac{\binom{K_a}{2}}{M} + K_a\mathbb{P}\left[\frac{P'}{2}\sum_{i\in[2n]} W_i^2 > nP\right]$, $W_i \overset{iid}{\sim} \mathcal{N}(0,1)$ for the Gaussian codebook [27].

Further, assume w.l.o.g that messages $\{1, 2, ..., K_a\}$ were sent and c_i is the codeword corresponding to $i \in [K_a]$. For $1 \leq t \leq K_a$, let $R_1 = \log\left(\binom{M-K_a}{t}\right)$ and $R_2 = \log\left(\binom{K_a}{t}\right)$. For $\delta > 0$, let $\epsilon(\delta) = 1 - \exp\left(-\left(\delta + \frac{R_1}{n-Ka} + (t-1)\frac{\log(n-K_a)}{n-K_a}\right)\right)$. Then

$$\mathbb{P}\left[F_t\right] \leq \inf_{\delta>0}\left[e^{R_2-(n-K_a)\delta} + \mathbb{P}\left[\bigcup_{\substack{S_0\subset[K_a]\\|S_0|=t}}\left\{\frac{\|Y\|^2 - \left\|P_{c_{[[K_a]]}}Y\right\|^2}{\|Y\|^2 - \left\|P_{c_{[S_0^c]}}Y\right\|^2} \geq 1-\epsilon(\delta)\right\}\right]\right] \tag{3.3}$$

where $c_{[S]} = \{c_i : i \in S\}$.

Proof. The proof of (3.2) follows from the proof of theorem 1 in [27]. Next, we bound $\mathbb{P}\left[F_t\right]$ for the projection decoder.

The common codebook is generated by choosing $c_i \overset{iid}{\sim} Unif\left(\sqrt{nP}(\mathcal{CS})^{n-1}\right)$ or $c_i \overset{iid}{\sim} \mathcal{CN}(0, P'I_n), i \in [M]$. Therefore codebook size is M. Note that, following [27], the users now select K_a messages without replacement from $[M]$ (this is accounted by p_0). W.l.o.g assume that $S = \{1, 2, ..., K_a\}$ is list of messages that were sent. Therefore the send codewords are $c_{[S]} = c_{[[K_a]]}$. Let $A_1 = \{S_0 \subset [K_a] : |S_0| = t\}$ and $A_2 = \{S_0 \subset [M] \setminus [K_a] : |S_0| = t\}$. Then we have

$$F_t = \mathbb{P}\left[|S \setminus \hat{S}| = t\right] \subset \bigcup_{S_0\in A_1 S_0'\in A_2} F(S_0, S_0') \tag{3.4}$$

where $F(S_0, S_0') = \left\{ \left\| P_{c_{[S_0^c]}, c_{[S_0']}} Y \right\|^2 > \left\| P_{c_{[S_0^c]}, c_{[S_0]}} Y \right\|^2 \right\}$ and $S_0^c = [Ka] \setminus S_0$. Further note that $P_{c_{[S_0^c]}, c_{[S_0]}} Y = P_{c_{[[K_a]]}} Y$.

Claim 1. *For any $S_0 \in A_1$ and $S_0' \in A_2$, conditioned on $c_{[K_a]}$, $H_{[K_a]} = \{H_i : i \in [K_a]\}$ and Z, the law of $\left\| P_{c_{[S_0^c]}, c_{[S_0']}} Y \right\|^2$ is same as the law of $\left\| P_{c_{[S_0^c]}} Y \right\|^2 + \left\| (I - P_{c_{[S_0^c]}}) Y \right\|^2 \mathrm{Beta}(t, n - K_a)$ where $\mathrm{Beta}(a, b)$ is a beta distributed random variable with parameters a and b.*

Proof. Note that $P_{c_{[S_0^c]}, c_{[S_0']}} Y = P_{c_{[S_0^c]}} Y + P_{c_{[S_0^c]}, c_{[S_0']}} P^{\perp}_{c_{[S_0^c]}} Y$. Further $P_{c_{[S_0^c]}, c_{[S_0']}} P^{\perp}_{c_{[S_0^c]}} Y = P_{\left(P^{\perp}_{c_{[S_0^c]}} c_{[S_0']}\right)} P^{\perp}_{c_{[S_0^c]}} Y$. Hence $\left\| P_{c_{[S_0^c]}, c_{[S_0']}} Y \right\|^2 = \left\| P_{c_{[S_0^c]}} Y \right\|^2 + \left\| P_{\left(P^{\perp}_{c_{[S_0^c]}} c_{[S_0']}\right)} P^{\perp}_{c_{[S_0^c]}} Y \right\|^2$.

Now conditioned on $c_{[K_a]}$ and $H_{[K_a]}$, $P^{\perp}_{c_{[S_0^c]}} Y$ is a fixed $n - K_a + t$ dimensional vector. So, $\left\| P_{\left(P^{\perp}_{c_{[S_0^c]}} c_{[S_0']}\right)} P^{\perp}_{c_{[S_0^c]}} Y \right\|^2$ is the squared length of the projection of a fixed vector in $\mathbb{C}^{n-K_a+t}$ (defined by the orthogonal projection operator $P^{\perp}_{c_{[S_0^c]}}$) of length $\left\| P^{\perp}_{c_{[S_0^c]}} Y \right\|^2$ onto a random t–dimensional subspace defined by the orthogonal projection operator $P_{\left(P^{\perp}_{c_{[S_0^c]}} c_{[S_0']}\right)}$. Further, the law of the squared length of the orthogonal projection of a fixed unit vector in $\mathbb{C}^d$ onto a random t–dimensional subspace is same as the law of the squared length of the orthogonal projection of a random unit vector in $\mathbb{C}^d$ onto a fixed t–dimensional subspace, which is $\mathrm{Beta}(t, d - t)$ (see for e.g. [37, Eq. 79]). Hence the conditional law of $\left\| P_{\left(P^{\perp}_{c_{[S_0^c]}} c_{[S_0']}\right)} P^{\perp}_{c_{[S_0^c]}} Y \right\|^2$ is $\left\| (I - P_{c_{[S_0^c]}}) Y \right\|^2 \mathrm{Beta}(t, n - K_a)$. ☐

Hence we have

$$\mathbb{P}\left[F(S_0, S_0') | c_{[K_a]}, H_{[K_a]}, Z\right] = F_\beta \left(\frac{\|Y\|^2 - \left\| P_{c_{[[K_a]]}} Y \right\|^2}{\|Y\|^2 - \left\| P_{c_{[S^c]}} Y \right\|^2}; n - K_a, t \right) \tag{3.5}$$

where $F_\beta(x; a, b)$ is the cdf of beta distribution $\mathrm{Beta}(a, b)$. Further, from [36], we have

$$F_\beta(x; n - K_a, t) \leq (n - K_a)^{t-1} x^{n-K_a}. \tag{3.6}$$

Let $g(Y, c_{[K_a]}, S_0) = \frac{\|Y\|^2 - \left\|P_{c_{[[K_a]]}} Y\right\|^2}{\|Y\|^2 - \left\|P_{c_{[S_0^c]}} Y\right\|^2}$. Then, using ideas similar to random coding union (RCU) bound [26], we have,

$$\begin{aligned}
\mathbb{P}[F_t] &\le \mathbb{P}\left[\bigcup_{S_0 \in A_1, S_0' \in A_2} F(S_0, S_0')\right] \le \mathbb{E}\left[\min\left\{1, \sum_{S_0, S_0'} \mathbb{P}\left[F(S_0, S_0') | c_{[K_a]}, H_{[K_a]}, Z\right]\right\}\right] \\
&= \mathbb{E}\left[\min\left\{1, \sum_{S_0} \binom{M - K_a}{t} F_\beta\left(g(Y, c_{[K_a]}, S_0); n - K_a, t\right)\right\}\right] \\
&\le \mathbb{E}\left[\min\left\{1, \sum_{S_0} e^{R_1}(n - K_a)^{(t-1)} g(Y, c_{[K_a]}, S_0)^{(n-K_a)}\right\}\right]. \qquad (3.7)
\end{aligned}$$

where the summations are over $S_0 \in A_1$ and $S_0' \in A_2$ (we do not write A_1 and A_2 for the sake of brevity).

For $\delta > 0$ define the event E_t as

$$\begin{aligned}
E_t &= \bigcap_{S_0} \left\{ -\log(g(Y, c_{[K_a]}, S_0)) - \frac{R_1}{n - K_a} - (t-1)\frac{\log(n - K_a)}{n - K_a} > \delta \right\} \\
&= \bigcap_{S_0} \left\{ g(Y, c_{[K_a]}, S_0) < e^{-\left(\frac{R_1}{n-K_a} + (t-1)\frac{\log(n-K_a)}{n-K_a} + \delta\right)} \right\} \\
&= \bigcap_{S_0} \left\{ g(Y, c_{[K_a]}, S_0) < 1 - \epsilon(\delta) \right\} \qquad (3.8)
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathbb{P}[F_t] &\le \sum_{S_0} e^{-(n-K_a)\delta} + \mathbb{P}[E_t^c] \\
&\le e^{R_2 - (n-K_a)\delta} + \mathbb{P}\left[\bigcup_{S_0} \left\{ g(Y, c_{[K_a]}, S_0) \ge 1 - \epsilon \right\}\right]. \qquad (3.9)
\end{aligned}$$

Since this true for any $\delta > 0$, we are done. □

We make few more observations. Since $(c_1, ..., c_{K_a})$ were the sent codewords, we have $Y = \sum_{i=1}^{K_a} H_i c_i + Z$. But, for any $S_0 \subset [K_a], |S_0| = t$, Y can be written as $Y = v_1 + v_2 + Z_1 + Z_2 + Z_3$ where

$$Z_1 = P_{c_{[S_0^c]}} P_{c_{[[K_a]]}} Z \qquad (3.10a)$$

$$Z_2 = P^{\perp}_{c_{[S_0^c]}} P_{c_{[[K_a]]}} Z \tag{3.10b}$$

$$Z_3 = P^{\perp}_{c_{[[K_a]]}} Z \tag{3.10c}$$

$$v_1 = P_{c_{[S_0^c]}} \sum_{i \in [K_a]} H_i c_i \tag{3.10d}$$

$$v_2 = P^{\perp}_{c_{[S_0^c]}} \sum_{i \in S_0} H_i c_i. \tag{3.10e}$$

Hence

$$\mathbb{P}[F_t] \leq \inf_{\delta>0} e^{R_2-(n-K_a)\delta} + \mathbb{P}\left[\frac{\|Z_3\|^2}{\|Z_3\|^2 + \min_{S_0}\|Z_2+v_2\|^2} \geq 1-\epsilon\right]. \tag{3.11}$$

One way to compute (3.11) is through a union bound which we state next. But as we shall see, the bound is loose. Apart from the union bound, it is not straightforward how to compute (3.11) numerically since computing the minimum over all subsets of $[K_a]$ of size t is unfeasible. However, for small K_a (say $K_a \leq 4$) we can perform Monte-Carlo simulation of the bound.

Theorem 3.1.2 (Union bound). *For $1 \leq t \leq K_a$, let $n' = n - K_a$ and $\tilde{n} = n - K_a + t$. Then following the same notation as in theorem 3.1.1, we have*

$$\mathbb{P}[F_t] \leq p_t = \inf_{\delta>0, 0<\gamma\leq 1, r>0, \delta_1>0} e^{R_2}\left[e^{-(n-K_a)\delta} + e^{-n'\left(1+r-\sqrt{2r+1}\right)} + e^{-\frac{\tilde{n}}{2}\gamma^2} + e^{-\frac{1}{4}\frac{\delta_1^2}{2\alpha+2\delta_1-2t}} + \mathbb{P}\left[\chi_2(2t) \leq \frac{\delta_1 - 2t + \alpha}{(1-\gamma)\tilde{n}P'}\right]\right]. \tag{3.12}$$

where $\chi_2(2t)$ is a chi-square distributed random variable with $2t$ dimensions, and $\alpha \equiv \alpha(r,\delta) = \frac{2n'\epsilon(1+r)}{1-\epsilon}$

Proof. From (3.11), we have

$$\mathbb{P}[F_t] \leq e^{R_2}\left(e^{-(n-K_a)\delta} + \mathbb{P}\left[\frac{\|Z_3\|^2}{\|Z_3\|^2 + \|Z_2+v_2\|^2} \geq 1-\epsilon\right]\right) \tag{3.13}$$

where S_0 now represents a generic size t subset of $[K_a]$.

Here, the common codebook is generated by choosing $c_i \overset{iid}{\sim} \mathcal{CN}(0, P'I_n), i \in [M]$. We have, $\|Z_3\|^2 \sim \frac{1}{2}W_3$ where $W_3 \sim \chi_2(2n')$. Let $Z_2' = \sqrt{2}Z_2$ and $v_2' = \sqrt{2}v_2$. Then we have

$$g(Y, c_{[K_a]}, S_0) = \frac{\|Z_3\|^2}{\|Z_2 + v_2\|^2 + \|Z_3\|^2} = \frac{W_3}{W_3 + \|Z_2' + V_2'\|^2}. \tag{3.14}$$

Therefore, for $r > 0$,

$$\begin{aligned}\mathbb{P}\left[g(Y, c_{[K_a]}, S_0) \geq 1 - \epsilon\right] &\leq \mathbb{P}\left[W_3 > (1+r)\mathbb{E}\left[W_3\right]\right] \\ &\quad + \mathbb{P}\left[\|Z_2' + v_2'\|^2 \leq \frac{\epsilon}{1-\epsilon}(1+r)\mathbb{E}\left[W_3\right]\right] \\ &\leq e^{-n'\left(1+r-\sqrt{2r+1}\right)} + \mathbb{P}\left[\|Z_2' + v_2'\|^2 \leq \alpha\right] \end{aligned} \tag{3.15}$$

where $\alpha = \frac{\epsilon}{1-\epsilon}(1+r)2n'$, and the last bound follows from the upper tail bound (A.2) with $\lambda = 0$.

Conditioned on $c_{[K_a]}$ and $H_{[K_a]}$, v_2' is fixed and hence $v_2' + Z_2' \sim \mathcal{CN}(v_2', 2P^{\perp}_{c_{[S_0^c]}} P_{c_{[[K_a]]}})$. Therefore, upon conditioning, $\|v_2' + Z_2'\|^2 \sim \chi_2'(\|v_2'\|^2, 2t)$, where $\chi_2'(\lambda, d)$ represents the non-central chi-squared distribution with non-centrality parameter λ and dimension d. Now we use lower tail in (A.3) to bound $\mathbb{P}\left[\|Z_2' + v_2'\|^2 \leq \alpha\right]$ as follows.

Let $\delta_1 > 0$. Let $B = \{\|v_2'\|^2 > \alpha - 2t + \delta_1\}$. We have

$$\begin{aligned}\mathbb{P}\left[\|Z_2' + v_2'\|^2 \leq \alpha\right] &\leq \mathbb{E}\left[\mathbb{P}\left[\|Z_2' + v_2'\|^2 \leq \alpha, B \mid c_{[K_a]}, H_{[K_a]}\right]\right] + \mathbb{P}\left[B^c\right] \\ &\leq \mathbb{E}\left[e^{-\frac{1}{4}\frac{\left(\|v_2'\|^2 + 2t - \alpha\right)^2}{2t + 2\|v_2'\|^2}} 1[B]\right] + \mathbb{P}\left[B^c\right]. \end{aligned} \tag{3.16}$$

The function $f_1(\lambda) = e^{-\frac{1}{4}\frac{(\lambda+2t-\alpha)^2}{2t+2\lambda}}$ is monotonically decreasing for $\lambda > 0$ if $\alpha < \lambda + 2t$. Hence on B, we have $e^{-\frac{1}{4}\frac{\left(\|v_2'\|^2 + 2t - \alpha\right)^2}{2t + 2\|v_2'\|^2}} \leq e^{-\frac{1}{4}\frac{\delta_1^2}{2\alpha + 2\delta_1 - 2t}}$.

So we have

$$\mathbb{P}\left[\|Z_2' + v_2'\|^2 \leq \alpha\right] \leq e^{-\frac{1}{4}\frac{\delta_1^2}{2\alpha + 2\delta_1 - 2t}} + \mathbb{P}\left[B^c\right] \tag{3.17}$$

To bound $\mathbb{P}[B^c]$, observe that $\|v_2'\|^2 \sim \frac{1}{2}P'HU$ where $H \sim \chi_2(2t)$ and $U \sim \chi_2(2\tilde{n})$ (H and U are independent). So, for $0 < \gamma < 1$,

$$\begin{aligned}
\mathbb{P}[B^c] &= \mathbb{P}\left[\frac{1}{2}P'HU \le \alpha - 2t + \delta_1\right] \\
&\le \mathbb{P}[U < (1-\gamma)\mathbb{E}[U]] + \mathbb{P}\left[H \le \frac{\alpha+\delta_1-2t}{(1-\gamma)\mathbb{E}[U]}\frac{1}{2}P'\right] \\
&\le e^{-\frac{\mathbb{E}[U]}{4}\gamma^2} + \mathbb{P}\left[H \le \frac{\alpha+\delta_1-2t}{(1-\gamma)\mathbb{E}[U]}\frac{2}{P'}\right] \qquad (3.18)
\end{aligned}$$

where the last inequality follows from the tail bound (A.3) with $\lambda = 0$.

Therefore we have

$$\mathbb{P}[B^c] \le e^{-\frac{\tilde{n}}{2}\gamma^2} + \mathbb{P}\left[H \le \frac{\alpha+\delta_1-2t}{(1-\gamma)\tilde{n}}\frac{2}{P'}\right] \qquad (3.19)$$

Finally combining (3.13), (3.15), (3.17) and (3.19), and optimizing over δ, r, γ and δ_1 we get (3.12). □

Next we discuss yet another achievability bound using the so called T–fold ALOHA method introduced in [24]

T–fold ALOHA

Let $T, n_1 \in \mathbb{N}$ such that $T < K_a$ and $n_1 < n$. Here T represents the maximum number of collisions that we decode before an error is declared. The time frame of length n is split into $L = n/n_1$ slots of length n_1. The common codebook is of blocklength n_1. Each user independently and uniformly picks a slot to transmit his message. We assume that the decoder has the knowledge of the number of users transmitting in each slot. This is not that much of an issue since for e.g. the decoder can try to decode all possible T or use energy detection. Suppose there is a code that can resolve at most T collisions. Then the decoder tries to decode in the slots where there are at most T–collisions, but declares an error if there are more. So we can use the random coding to resolve upto T-collisions, and we can achieve the following probability of error per user:

$$\epsilon_T \leq 1 - \sum_{t=1}^{T} \binom{K_a - 1}{t-1} \left(\frac{1}{L}\right)^{t-1} \left(1 - \frac{1}{L}\right)^{K_a - t} + \sum_{t=1}^{T} P_e(M, n_1, t, LP) \binom{K_a - 1}{t-1} \left(\frac{1}{L}\right)^{t-1} \left(1 - \frac{1}{L}\right)^{K_a - t} \tag{3.20}$$

where $P_e(M, n_1, t, P) = \epsilon$ of an (M, n_1, ϵ) code used over the $K_a = t$ fading MAC with power constraint P. This is easy to see since in the slot that a particular user is transmitting, the probability that there are exactly $t-1$ of the remaining users also transmitting is given by $\binom{K_a-1}{t-1} \left(\frac{1}{L}\right)^{t-1} \left(1 - \frac{1}{L}\right)^{K_a - t}$.

We will later see that attempting to evaluate (3.11) directly for large K_a does not result in a good performance (although better than the union bound) since the user with the smallest fading gain creates a bottleneck. But evaluating (3.11) for $K_a \leq T$ where T is small and then appealing to T–fold ALOHA results in a very good performance.

3.2 Converse bound

In this section we describe a simple converse bound based on results from [37] and the meta-converse from [26]. But first, we will discuss a list-decoding version of the meta-converse.

3.2.1 Meta-converse for list decoding

Following the notation of [26], let $(A, B, P_{Y|X})$ be a random transformation. That is, the input and output alphabets are A and B respectively, and the channel is given by the transition kernel $P_{Y|X}$.

Definition 10. *An (M, K_a) code for the random transformation $(A, B, P_{Y|X})$ is defined by a pair (f, g) with function (encoder) $f : [M] \to A$ and transition kernel (decoder) $g : B \to \cup_{t=1}^{K_a} \binom{[M]}{t} \equiv [M_{K_a}]$ where $\binom{[M]}{t} = \{S \subset [M] : |S| = t\}$. For this* *code, the probability of error is defined as*

$$\epsilon(f, g) = \frac{1}{M} \sum_{j=1}^{M} 1 - \mathbb{P}[j \in g(Y)|X = f(j)] \tag{3.21}$$

where, with abuse of notation, g denotes a $[M_{K_a}]$ valued random variable distributed according to kernel $g \equiv g(\cdot|y)$.

Next, we define some notations on binary hypothesis testing from [26]. Let W be a random variable that can take one of the two distributions P and Q on the same alphabet W. A randomized test between the two distributions is a transition kernel $P_{Z|W} : \mathsf{W} \to \{0, 1\}$ where 1 indicates P. The optimal performance is given by

$$\beta_\alpha(P, Q) = \inf_{\substack{P_{Z|W}: \\ P(Z=1) \geq \alpha}} Q(Z = 1) \tag{3.22}$$

where $P(Z = 1) = \sum_{w \in \mathsf{W}} P_{Z|W}(1|w)P(w)$ and $Q(Z = 1) = \sum_{w \in \mathsf{W}} P_{Z|W}(1|w)Q(w)$. Similar to the meta-converse theorem in [26], we have the following.

Theorem 3.2.1. *Let $(A, B, P_{Y|X})$ and $(A, B, Q_{Y|X})$ be two random transformations and fix an (f, g) code (here f can also be randomized). Let ϵ and ϵ' be the error probabilities (as in defn. 10) under the transformations P and Q, respectively. Let $P_X = Q_X$ be the distribution induced by f on A. Then*

$$\beta_{1-\epsilon}(P_{XY}, Q_{XY}) \leq 1 - \epsilon' \tag{3.23}$$

Proof. Let W and $\hat{W}$ be the random variables denoting the input to the encoder and output of the (list) decoder. Then we have the following two joint distributions

$$P_{WXY\hat{W}}(w, x, y, \hat{w}) = \frac{1}{M} f(x|w) P_{Y|X}(y|x) g(\hat{w}|y) \tag{3.24a}$$

$$Q_{WXY\hat{W}}(w, x, y, \hat{w}) = \frac{1}{M} f(x|w) Q_{Y|X}(y|x) g(\hat{w}|y) \tag{3.24b}$$

where $W \sim Unif[M]$. Define the random variable Z as

$$Z = 1\{W \in \hat{W}\}. \tag{3.25}$$

Claim 2. $P_{Z|XY} = Q_{Z|XY}$

Proof.

$$\begin{aligned}
\mathbb{P}[Z = 1|X, Y] &= \sum_{j=1}^{M} \mathbb{P}\left[W = j, j \in \hat{W}|X, Y\right] \\
&= \sum_{j=1}^{M} \mathbb{P}[W = j|X]\, \mathbb{P}\left[j \in \hat{W}|Y\right] \\
&= \sum_{j=1}^{M} \mathbb{P}[W = j|X]\, g(\{S \in [M_{K_a}] : j \in S\}|Y).
\end{aligned} \tag{3.26}$$

We note that the last expression is for both P and Q. □

So $P_{Z|XY}$ defines a transition kernel from $A \times B$ to $\{0, 1\}$ and hence is a binary hypothesis test between P_{XY} and Q_{XY} with

$$\sum_{x \in A} \sum_{y \in B} P_{Z|XY}(1|x, y) P_{XY}(x, y) = 1 - \epsilon \tag{3.27}$$

$$\sum_{x \in A} \sum_{y \in B} P_{Z|XY}(1|x, y) Q_{XY}(x, y) = 1 - \epsilon'. \tag{3.28}$$

Consequently, by (3.23),

$$\beta_{1-\epsilon}(P_{XY}, Q_{XY}) \leq 1 - \epsilon' \tag{3.29}$$

□

Next, we have the converse bound.

Theorem 3.2.2 (Converse). *Any (M, K_a) code for a random transformation $(A, B, P_{Y|X})$*

with probability of error ϵ satisfies

$$M \leq K_a \sup_{P_X} \inf_{Q_Y} \frac{1}{\beta_{1-\epsilon}(P_{XY}, P_X \times Q_Y)} \tag{3.30}$$

where P_X ranges over all distributions on A (or, if there is a cost constraint, then on the constraint set F), and Q_Y over all distributions on B.

Proof. Let P_X be the distribution induced by the encoder. Choose $Q_{Y|X} = Q_Y$ in Theorem 3.2.1 for an arbitrary distribution Q_Y. Let ϵ' be the probability of error under Q. Since the input is independent of the decoder output under Q, we have

$$\begin{aligned}
\epsilon' = \mathbb{P}\left[W \notin \hat{W}\right] &= \sum_{i \in [M]} \mathbb{P}\left[W = i\right] \mathbb{P}\left[i \notin \hat{W}\right] \\
&= \frac{1}{M} \sum_{i \in [M]} \left(1 - \mathbb{P}\left[i \in \hat{W}\right]\right) \\
&\geq 1 - \frac{K_a}{M}
\end{aligned} \tag{3.31}$$

since $\sum_{i \in [M]} 1[i \in \hat{W}] \leq K_a$, *a.s.*

Hence from Theorem 3.2.1, we have

$$\begin{aligned}
\frac{K_a}{M} &\geq \sup_{Q_Y} \beta_{1-\epsilon}(P_{XY}, P_X \times Q_Y) \\
&\geq \inf_{P_X} \sup_{Q_Y} \beta_{1-\epsilon}(P_{XY}, P_X \times Q_Y)
\end{aligned} \tag{3.32}$$

□

3.2.2 Converse for the K_a–MAC

Theorem 3.2.3. *Let*

$$L_n = n \log(1 + PG) + \sum_{i=1}^{n} \left(1 - |\sqrt{PG} Z_i - \sqrt{1 + PG}|^2\right) \tag{3.33}$$

$$S_n = n \log(1 + PG) + \sum_{i=1}^{n} \left(1 - \frac{|\sqrt{PG} Z_i - 1|^2}{1 + PG}\right) \tag{3.34}$$

where $G = \|H\|^2$ *and* $Z_i \overset{iid}{\sim} \mathcal{CN}(0,1)$*. Then for every* n *and* $0 < \epsilon < 1$*, any* $(M, n-1, \epsilon)$ *code for the quasi static* K_a *MAC satisfies*

$$\log(M) \leq \log(K_a) + \log \frac{1}{\mathbb{P}\left[L_n \geq n\gamma_n\right]} \tag{3.35}$$

where γ_n *is the solution of*

$$\mathbb{P}\left[S_n \leq n\gamma_n\right] = \epsilon. \tag{3.36}$$

Proof. We note that a converse bound for the case where full CSI is available at receiver (and/or transmitter) is a converse for the no-CSI case as well. Further, by symmetry on the users, it is sufficient to get a lower bound on the probability that a particular user's message is not in the decoded list. Finally we can assume that the decoder has the knowledge of the codewords of all other users. Formally, let Y be the received vector and let $L(Y)$ be the list of codewords output by the decoder (we use list of codewords or messages interchangeably). The size of the list is $|L(Y)| \leq K_a$. Then we have the following implications:

$$\begin{aligned}
&\frac{1}{K_a}\sum_{t=1}^{K_a} \mathbb{P}\left[X_t \notin L(Y)\right] \geq 1-\epsilon \\
\iff\ & \mathbb{P}\left[X_1 \notin L(Y)\right] \geq 1-\epsilon && (3.37)\\
\Longleftarrow\ & \mathbb{P}\left[X_1 \notin L(Y, H_1)\right] \geq 1-\epsilon && (3.38)\\
\Longleftarrow\ & \mathbb{P}\left[X_1 \notin L(Y, H_{[K_a]}, X_{[K_a]\setminus\{1\}})\right] \geq 1-\epsilon && (3.39)
\end{aligned}$$

where (3.38) and (3.39) represents the case when decoder has access to the fading realization of user 1 and interference from all other users respectively.

Now, given $H_{[K_a]}$ and $X_{[K_a]\setminus\{1\}}$ at the receiver, the channel is equivalent to

$$Y_1 = H_1 X_1 + Z$$

where H_1 and Z are same as before, the decoder outputs a list of messages $\hat{W} =$

$L(Y_1, H_1)$ of size at most K_a and the probability of error is $\mathbb{P}\left[W_1 \notin \hat{W}\right]$ where $W_1 \sim unif[M]$ is the users message. Observe that this is similar to the case dealt in [37], but the decoder is performing list-decoding. So the remainder of the proof similar to the one in [37] (also note that [28, Lemma 39] holds here) with the usual meta-converse replaced by theorem 3.2.2.

$\square$

Chapter 4

Asymptotics of the projection decoder: Joint probability of error

In this chapter we discuss the asymptotics of the projection decoder under the classical joint probability of error in the case of non-same codebook. We show that the projection decoder achieves ϵ–capacity of the K–MAC.

4.1 ϵ–capacity region for K–MAC

Note that for a fixed finite number of users, the ϵ–capacity of the quasi-static channel does not depend on whether or not the channel state information (CSI) is available at the receiver since the fading coefficients can be reliably estimated with negligible rate penalty as $n \to \infty$ [6] [5]. Hence from this fact and using [15, Theorem 5] it is easy to see that, for $0 \leq \epsilon < 1$, the ϵ–capacity region of the K–MAC defined in (2.6) is given by

$$C_\epsilon = \{R = (R_1, ..., R_K) : \forall i, R_i \geq 0 \text{ and } P_0(R) \leq \epsilon\} \tag{4.1}$$

where the outage probability $P_0(R)$ is given by

$$P_0(R) = \mathbb{P}\left[\bigcup_{S\subset[K],S\neq\emptyset}\left\{\log\left(1+P\sum_{i\in S}|H_i|^2\right) \leq \sum_{i\in S} R_i\right\}\right] \tag{4.2}$$

4.2 Achievability

Before we state the main result, we adapt the projection decoder in (3.1) to the case of non-same codebook.

Encoder

As before, we use random coding with either

1. **Spherical codebook:** Each user, for each message independently generates a vector uniformly distributed on the $\sqrt{nP}$–complex sphere. That is $X_i \overset{iid}{\sim} Unif\left(\sqrt{nP}(\mathcal{CS})^{n-1}\right)$.
   or

2. **Gaussian codebook:** Each user, for each message independently generates a $\mathcal{CN}(0, P'I_n)$ vector. That is $X_i \overset{iid}{\sim} \mathcal{CN}(0, P'I_n)$ where $P' < P$. If a codeword generated violates the power constraint (2.8) then that user sends 0.

Also observe that a set at most $n-1$ codewords are linearly independent almost surely.

Decoder

Let C denote a set of vectors in $\mathbb{C}^n$. Denote P_C as the orthogonal projection operator onto the subspace spanned by C. Let $\mathcal{C}_1, ..., \mathcal{C}_K$ denote the codebooks of the K users respectively. We have $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ *a.s.* by the random generation of codebooks. Upon receiving Y from the channel the decoder outputs $g(Y)$ which is given by

$$\begin{aligned} g(Y) &= \left(f_1^{-1}(\hat{c}_1), ..., f_K^{-1}(\hat{c}_K)\right) \\ (\hat{c}_1, ...\hat{c}_K) &= \arg \max_{(c_i \in \mathcal{C}_i)_{i=1}^K} \left\|P_{\{c_i : i \in [K]\}} Y\right\|^2 \end{aligned} \tag{4.3}$$

where f_i are the encoding functions.

Now we present our main theorem in this section

Theorem 4.2.1 (Projection decoding achieves C_ϵ). *For the K-MAC given by (2.6), projection decoding achieves C_ϵ. That is, let $R \in C_\epsilon$ given by (4.1) and (4.2). Then for the K-MAC there exists a sequence of $\left(\left(M_1^{(n)}, ..., M_K^{(n)}\right), n, \epsilon_n\right)_J$ codes satisfying the power constraint (2.8), with the decoder being the projection decoder (4.3) , such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log\left(M_i^{(n)}\right) \geq R_i, \forall i \in [K] \tag{4.4}$$

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon \tag{4.5}$$

Proof. Let $\eta_i > 0, i \in [K]$. Choose $M_i^{(n)} = \lceil e^{n(R_i - \eta_i)} \rceil, \forall i \in [K]$. Let user j generate a spherical codebook of size M_j and power P independently across codewords and independent of other users. Hence the channel inputs are given by $X_i^{(n)} \overset{iid}{\sim} \left(\sqrt{nP}(\mathcal{CS})^{n-1}\right)$. Let $\{\mathcal{C}_i\}_{i=1}^K$ denote the codebooks of the K users with $|\mathcal{C}_i| = M_i$. We will drop the superscript n for brevity.

Suppose the codewords $(c_1, c_2, ..., c_K) \in \mathcal{C}_1 \times \mathcal{C}_2 ... \times \mathcal{C}_K$ were actually sent. Then by (3.1), error occurs iff $\exists (c_1', c_2', ..., c_K') \in \mathcal{C}_1 \times \mathcal{C}_2 ... \times \mathcal{C}_K$ such that $(c_1', c_2', ..., c_K') \neq (c_1, c_2, ..., c_K)$ and

$$\left\|P_{c_1', ..., c_K'} Y\right\|^2 > \|P_{c_1, ..., c_K} Y\|^2 . \tag{4.6}$$

This can be equivalently written as follows. Let $S \subset [K]$ be such that

$$i \in [S] \iff \hat{c}_i \neq c_i \tag{4.7}$$

where $(\hat{c}_i)_{i=1}^K$ denote the decoded codewords.

Let $c_{[S]} \equiv \{c_i : i \in [S]\}$. Then, error occurs iff $\exists S \subset [K]$ and $S \neq \emptyset$, and $\exists \{c_i' : i \in [S], c_i' \neq c_i\}$ such that

$$\left\|P_{c'_{[S]}, c_{[S^c]}} Y\right\|^2 > \left\|P_{c_{[[K]]}} Y\right\|^2 . \tag{4.8}$$

So, the average probability of error is given by

$$\epsilon_n = \left[\bigcup_{\substack{S \subset [K] \\ S \neq \emptyset}} \bigcup_{\substack{\{c'_i \in \mathcal{C}_i : \\ i \in S, c'_i \neq c_i\}}} \left\{ \left\| P_{c'_{[S]}, c_{[S^c]}} Y \right\|^2 > \left\| P_{c_{[[K]]}} Y \right\|^2 \right\} \right]$$

$$= \left[\bigcup_{t \in [K]} \bigcup_{\substack{S \subset [K] \\ |S| = t}} \bigcup_{\substack{\{c'_i \in \mathcal{C}_i : \\ i \in S, c'_i \neq c_i\}}} \left\{ \left\| P_{c'_{[S]}, c_{[S^c]}} Y \right\|^2 > \left\| P_{c_{[[K]]}} Y \right\|^2 \right\} \right] \tag{4.9}$$

Using ideas similar to the Random Coding Union (RCU) bound [26], we have

$$\epsilon_n \leq \mathbb{E} \left[\min \left\{ 1, \sum_{t \in [K]} \sum_{S \in [K] : |S| = t} \left(\prod_{j \in S} (M_j - 1) \right) \right. \right.$$
$$\left. \left. \mathbb{P} \left[\left\| P_{c'_{[S]}, c_{[S^c]}} Y \right\|^2 > \left\| P_{c_{[[K]]}} Y \right\|^2 \mid c_{[K]}, H_{[K]}, Z \right] \right\} \right] \tag{4.10}$$

where primes denote unsent codewords and $H_{[K]} = \{H_i : i \in [K]\}$. Unsent codeword c'_i here means that it is independent of the actual channel inputs/output and distributed with the same law as X_i.

Claim 3. *For $t \in [K]$ and $S \subset [K]$ with $|S| = t$,*

$$P \left[\left\| P_{c'_{[S]}, c_{[S^c]}} Y \right\|^2 > \left\| P_{c_{[[K]]}} Y \right\|^2 \mid c_{[K]}, H_{[K]}, Z \right]$$
$$= F \left(\frac{\|Y\|^2 - \left\| P_{c_{[[K]]}} Y \right\|^2}{\|Y\|^2 - \left\| P_{c_{[S^c]}} Y \right\|^2} ; n - K, t \right) \tag{4.11}$$

where $F(x; a, b)$ is the cdf of beta distribution $Beta(a, b)$. Further, from [37], we have

$$F(x; n - K, t) \leq (n - K)^{t-1} x^{n-K} \tag{4.12}$$

Proof. Same as the proof of claim 1 $\square$

Letting $g(Y, c_{[K]}, S) = \frac{\|Y\|^2 - \left\| P_{c_{[[K]]}} Y \right\|^2}{\|Y\|^2 - \left\| P_{c_{[S^c]}} Y \right\|^2}$ and $M_S = \prod_{j \in S} (M_j - 1)$, we have the following from (4.10), (4.11) and (4.12)

$$\epsilon_n \leq \mathbb{E}\left[\min\left\{1, \sum_{t\in[K]}\sum_{\substack{S\subset[K]\\|S|=t}} \exp\left(-(n-K)\left[-(t-1)\frac{\log(n-K)}{n-K}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\frac{\log(M_S)}{n-K} - \log(g(Y, c_{[K]}, S))\right]\right)\right\}\right] \tag{4.13}$$

Let $\delta > 0$ and let E_1 be the following event

$$E_1 = \bigcap_{t\in[K]}\bigcap_{\substack{S\in[K]\\|S|=t}} \left\{-\log(g(Y, c_{[K]}, S)) - (t-1)\frac{\log(n-K)}{n-K} - \frac{\log(M_S)}{n-K} > \delta\right\} \tag{4.14}$$
$$= \bigcap_{t\in[K]}\bigcap_{\substack{S\in[K]\\|S|=t}} \{-\log(g(Y, c_{[K]}, S)) > \tilde{\gamma}_n\}$$
$$= \bigcap_{t\in[K]}\bigcap_{\substack{S\in[K]\\|S|=t}} \{g(Y, c_{[K]}, S) < \gamma_n\} \tag{4.15}$$

where $\tilde{\gamma}_n = (t-1)\frac{\log(n-K)}{n-K} + \frac{\log(M_S)}{n-K} + \delta$ and $\gamma_n = e^{-\tilde{\gamma}_n}$. Note that γ_n depends on S and t.

Then, from (4.13) we have the following

$$\epsilon_n \leq \mathbb{E}\left[\min\left\{1,\right.\right.$$
$$\sum_{t\in[K]}\sum_{\substack{S\subset[K]\\|S|=t}} \exp\left(-(n-K)\left[-(t-1)\frac{\log(n-K)}{n-K}\right.\right.$$
$$\left.\left.\left.\left.-\frac{\log(M_S)}{n-K} - \log(g(Y, c_{[K]}, S))\right]\right)\right\}(1[E_1] + 1[E_1^c])\right]$$
$$\leq \sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}} e^{-(n-K)\delta} + \mathbb{P}[E_1^c]$$
$$= \sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}} e^{-(n-K)\delta} + \mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{S\subset[K]:|S|=t} g(Y, c_{[K]}, S) \geq \gamma_n\right]. \tag{4.16}$$

Hence, as $n \to \infty$, it is the second term in the above expression that potentially dominates.

Claim 4. *For $t \in [K]$, $S \subset [K]$ with $|S| = t$, we have*

$$\mathbb{P}\left[g(Y, c_{[K]}, S) \geq \gamma_n\right] = \mathbb{P}\left[\frac{\|Y\|^2 - \left\|P_{c_{[[K]]}} Y\right\|^2}{\|Y\|^2 - \left\|P_{c_{[S^c]}} Y\right\|^2} \geq \gamma_n\right]$$
$$\leq \mathbb{P}\left[\left\|(1-\gamma_n)P^{\perp}_{c_{[S^c]}} Z - \gamma_n P^{\perp}_{c_{[S^c]}} \sum_{i \in S} H_i c_i\right\|^2 \right.$$
$$\left. \geq \gamma_n \left\|P^{\perp}_{c_{[S^c]}} \sum_{i \in S} H_i c_i\right\|^2\right] \tag{4.17}$$

where $P^{\perp}_{c_{[S^c]}}$ represents the orthogonal projection onto the orthogonal complement of the space spanned by $c_{[S^c]}$.

Proof. See appendix. □

To evaluate the above probability, we condition on $c_{[K]}$ and $H_{[K]}$. For ease of notation, we will not explicitly write the conditioning.

Since $Z \sim \mathcal{CN}(0, I_n)$, we have $Z - \frac{\gamma_n}{1-\gamma_n}\sum_{i \in S} H_i c_i \sim \mathcal{CN}(-\frac{\gamma_n}{1-\gamma_n}\sum_{i \in S} H_i c_i, I_n)$. Hence $P^{\perp}_{c_{[S^c]}}\left(Z - \frac{\gamma_n}{1-\gamma_n}\sum_{i \in S} H_i c_i\right) \sim \mathcal{CN}(-\frac{\gamma_n}{1-\gamma_n} P^{\perp}_{c_{[S^c]}} \sum_{i \in S} H_i c_i, P^{\perp}_{c_{[S^c]}})$. Now using the fact that if $W = P + iQ \sim \mathcal{CN}(\mu, \Gamma, 0)$ then

$$\begin{bmatrix} P \\ Q \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} Re(\mu) \\ Im(\mu) \end{bmatrix}, \frac{1}{2}\begin{bmatrix} Re(\Gamma) & -Im(\Gamma) \\ Im(\Gamma) & Re(\Gamma) \end{bmatrix}\right), \tag{4.18}$$

we can show the following

Lemma 4.2.2. *Conditioned on $H_{[K]}$ and $c_{[K]}$, we have*

$$\left\|P^{\perp}_{c_{[S^c]}}\left(Z - \frac{\gamma_n}{1-\gamma_n}\sum_{i \in S} H_i c_i\right)\right\|^2$$
$$\sim \frac{1}{2}\chi'_2\left(2\left\|\frac{\gamma_n}{1-\gamma_n} P^{\perp}_{c_{[S^c]}} \sum_{i \in S} H_i c_i\right\|^2, 2(n-K+t)\right) \tag{4.19}$$

where $\chi_2'(\lambda, k)$ represents a non-central chi-squared random variable with non-centrality parameter λ and dimension k. Hence its conditional expectation is

$$\mu = n' + \lambda \tag{4.20}$$

where $\lambda = \frac{\gamma_n^2}{(1-\gamma_n)^2} \left\| P^{\perp}_{c_{[S^c]}} \sum_{i \in S} H_i c_i \right\|^2$ and $n' = n - K + t$.

Proof. See appendix. □

Hence we have

$$\begin{aligned}
&\mathbb{P}\left[\bigcup_{t\in[K]} \bigcup_{\substack{S\in[K] \\ |S|=t}} \left\| P^{\perp}_{c_{[S^c]}} \left(Z - \frac{\gamma_n}{1-\gamma_n} \sum_{i\in S} H_i c_i \right) \right\|^2 - \mu \geq \frac{\gamma_n}{(1-\gamma_n)^2} \left\| P^{\perp}_{c_{[S^c]}} \sum_{i\in S} H_i c_i \right\|^2 - \mu \right] \\
&= \mathbb{P}\left[\bigcup_{t\in[K]} \bigcup_{\substack{S\in[K] \\ |S|=t}} \left\| P^{\perp}_{c_{[S^c]}} \left(Z - \frac{\gamma_n}{1-\gamma_n} \sum_{i\in S} H_i c_i \right) \right\|^2 - \mu \geq \frac{\gamma_n}{(1-\gamma_n)} \left\| P^{\perp}_{c_{[S^c]}} \sum_{i\in S} H_i c_i \right\|^2 - n' \right] \\
&= \mathbb{E}\left[\mathbb{P}\left[\left\{ \bigcup_{t\in[K]} \bigcup_{\substack{S\in[K] \\ |S|=t}} \left\{ \frac{1}{2}\chi_2'(2\lambda, 2n') - (\lambda + n') \geq \gamma \right\} \right\} \middle| c_{[K]}, H_{[K]} \right] \right]
\end{aligned} \tag{4.21}$$

where $\gamma = \frac{\gamma_n}{(1-\gamma_n)} \left\| P^{\perp}_{c_{[S^c]}} \sum_{i\in S} H_i c_i \right\|^2 - n'$ and $\lambda = \frac{\gamma_n^2}{(1-\gamma_n)^2} \left\| P^{\perp}_{c_{[S^c]}} \sum_{i\in S} H_i c_i \right\|^2$. Hence $\lambda = \frac{\gamma_n}{1-\gamma_n}(\gamma + n')$. Note that γ_n, γ ,λ all depend on t and S.

Now, note that

$$\begin{aligned}
\gamma &= \frac{\gamma_n}{(1-\gamma_n)} \left\| P^{\perp}_{c_{[S^c]}} \sum_{i\in S} H_i c_i \right\|^2 - n' \\
&= \frac{n'}{1-\gamma_n} \left(\gamma_n \left(1 + \frac{\left\| P^{\perp}_{c_{[S^c]}} \sum_{i\in S} H_i c_i \right\|^2}{n'} \right) - 1 \right) \\
&= n'\gamma^1
\end{aligned} \tag{4.22}$$

where $\gamma^1 = \frac{1}{1-\gamma_n}\left(\gamma_n\left(1+\frac{\left\|P_{c_{[S^c]}}^{\perp}\sum_{i\in S} H_i c_i\right\|^2}{n'}\right)-1\right)$. Hence

$$\lambda = \frac{\gamma_n}{1-\gamma_n}(\gamma + n') = n'\frac{\gamma_n}{1-\gamma_n}(\gamma^1+1). \tag{4.23}$$

Let $\delta_1 > 0$. Let $E_{11} = \bigcap_{t\in[K]}\bigcap_{\substack{S\in[K]\\|S|=t}}\{\gamma^1 > \delta_1\}$. From (4.21) we have

$$\begin{aligned}
&\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\frac{1}{2}\chi_2'(2\lambda,2n')-(\lambda+n')\geq\gamma\right\}\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[1\left\{\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\frac{1}{2}\chi_2'(2\lambda,2n')-(\lambda+n')\geq\gamma\right\}\right\}\middle|c_{[K]},H_{[K]}\right]\right]\\
&=\mathbb{E}\left[\mathbb{P}\left[\left\{\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\frac{1}{2}\chi_2'(2\lambda,2n')-(\lambda+n')\geq\gamma\right\}\right\}\middle|c_{[K]},H_{[K]}\right](1[E_{11}]+1[E_{11}^c])\right]\\
&\leq\mathbb{E}\left[\mathbb{P}\left[\left\{\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\frac{1}{2}\chi_2'(2\lambda,2n')-(\lambda+n')\geq\gamma\right\}\right\}\middle|c_{[K]},H_{[K]}\right]1[E_{11}]\right]+\mathbb{P}\left[E_{11}^c\right]\\
&\leq\sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}}\mathbb{E}\left[\mathbb{P}\left[\left\{\frac{1}{2}\chi_2'(2\lambda,2n')-(\lambda+n')\geq\gamma\right\}\middle|c_{[K]},H_{[K]}\right]1[E_{11}]\right]+\mathbb{P}\left[E_{11}^c\right]\\
&\leq\sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}}\mathbb{E}\left[\mathbb{P}\left[\left\{\frac{1}{2}\chi_2'(2\lambda,2n')-(\lambda+n')\geq\gamma\right\}\middle|c_{[K]},H_{[K]}\right]1[\gamma^1>\delta_1]\right]+\mathbb{P}\left[E_{11}^c\right]\\
&\leq\sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}}\mathbb{E}\left[\exp\left(-n'f_n(\gamma^1)\right)1[\gamma^1>\delta_1]\right]+\mathbb{P}\left[E_{11}^c\right] \qquad (4.24)
\end{aligned}$$

where the last inequality follows from (A.2) with

$$\begin{aligned}
f_n(x) &= x+1+\frac{2\gamma_n}{1-\gamma_n}(1+x)\\
&\quad-\sqrt{1+\frac{2\gamma_n}{1-\gamma_n}(1+x)}\sqrt{2x+1+\frac{2\gamma_n}{1-\gamma_n}(1+x)}
\end{aligned} \tag{4.25}$$

Now, we claim the f_n is monotonic:

Claim 5. *For $0 < \gamma_n < 1$ and $x > 0$, $f_n(x)$ is a monotonically increasing function of x.*

Proof. See appendix $\square$

Hence we have

$$\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\frac{1}{2}\chi_2'(2\lambda, 2n') - (\lambda + n') \geq \gamma\right]$$
$$\leq \sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}}\exp(-n' f_n(\delta_1)) + \mathbb{P}\left[E_{11}^c\right]. \tag{4.26}$$

So, we have the following proposition

Proposition 1. *If $0 < \gamma_n < 1$ for all $t \in [K]$, $S \subset [K]$ with $|S| = t$ (γ_n depends on S and t) then we have*

$$\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\frac{\|Y\|^2 - \left\|P_{c_{[[K]]}}Y\right\|^2}{\|Y\|^2 - \left\|P_{c_{[S^c]}}Y\right\|^2} \geq \gamma_n\right]$$
$$\leq \sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}}\exp(-n' f_n(\delta_1)) + \mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\gamma_n\left(1 + \frac{\left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_i c_i\right\|^2}{n'}\right) - 1 \leq \delta_1\right]. \tag{4.27}$$

Proof.

$$\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\frac{\|Y\|^2 - \left\|P_{c_{[[K]]}}Y\right\|^2}{\|Y\|^2 - \left\|P_{c_{[S^c]}}Y\right\|^2} \geq \gamma_n\right]$$
$$\leq \sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}}\exp(-n' f_n(\delta_1)) + \mathbb{P}\left[E_{11}^c\right]$$

$$
\begin{aligned}
&= \sum_{t\in[K]} \sum_{\substack{S\in[K]\\|S|=t}} \exp(-n' f_n(\delta_1)) + \\
&\mathbb{P}\left[\bigcup_{t\in[K]} \bigcup_{\substack{S\in[K]\\|S|=t}} \frac{1}{1-\gamma_n} \left(\gamma_n \left(1 + \frac{\left\| P^{\perp}_{c[S^c]} \sum_{i\in S} H_i c_i \right\|^2}{n'} \right) - 1 \right) \le \delta_1 \right] \\
&\le \sum_{t\in[K]} \sum_{\substack{S\in[K]\\|S|=t}} \exp(-n' f_n(\delta_1)) + \mathbb{P}\left[\bigcup_{t\in[K]} \bigcup_{\substack{S\in[K]\\|S|=t}} \gamma_n \left(1 + \frac{\left\| P^{\perp}_{c[S^c]} \sum_{i\in S} H_i c_i \right\|^2}{n'} \right) - 1 \le \delta_1 \right].
\end{aligned}
\tag{4.28}
$$

□

Now, we need to upper bound

$$
\mathbb{P}\left[\bigcup_{t\in[K]} \bigcup_{\substack{S\in[K]\\|S|=t}} \gamma_n \left(1 + \frac{\left\| P^{\perp}_{c[S^c]} \sum_{i\in S} H_i c_i \right\|^2}{n'} \right) - 1 \le \delta_1 \right].
$$

We have

$$
\begin{aligned}
&\left\| P^{\perp}_{c[S^c]} \sum_{i\in S} H_i c_i \right\|^2 = \sum_{i\in S} |H_i|^2 \left\| P^{\perp}_{c[S^c]} c_i \right\|^2 + \\
&2 \sum_{i<j:i,j\in S} Re\left(\left\langle P^{\perp}_{c[S^c]} c_i, P^{\perp}_{c[S^c]} c_j \right\rangle H_i \bar{H}_j \right).
\end{aligned}
\tag{4.29}
$$

Further,

$$
\left\langle P^{\perp}_{c[S^c]} c_i, P^{\perp}_{c[S^c]} c_j \right\rangle = \langle c_i, c_j \rangle - \left\langle P_{c[S^c]} c_i, P_{c[S^c]} c_j \right\rangle \tag{4.30}
$$

Hence we get

$$
\begin{aligned}
\left| Re \left\langle P^{\perp}_{c[S^c]} c_i, P^{\perp}_{c[S^c]} c_j \right\rangle \right| &\le \left| \left\langle P^{\perp}_{c[S^c]} c_i, P^{\perp}_{c[S^c]} c_j \right\rangle \right| \\
&\le |\langle c_i, c_j \rangle| + \left| \left\langle P_{c[S^c]} c_i, P_{c[S^c]} c_j \right\rangle \right| \\
&\le |\langle c_i, c_j \rangle| + \left\| P_{c[S^c]} c_j \right\| \left\| P_{c[S^c]} c_i \right\|
\end{aligned}
$$

$$= nP\left(|\langle \hat{c}_i, \hat{c}_j\rangle| + \left\|P_{c_{[S^c]}}\hat{c}_i\right\| \left\|P_{c_{[S^c]}}\hat{c}_j\right\|\right) \tag{4.31}$$

where hats denote corresponding normalized vectors. Since these unit vectors are high dimensional, their dot products and projection onto a smaller, fixed dimension surface is very small. Indeed, we have the following two lemmas.

Lemma 4.2.3. *If $e_1, e_2 \overset{iid}{\sim} Unif((\mathcal{CS})^{n-1})$, then for any $\delta_2 > 0$, we have*

$$\mathbb{P}\left[| \langle e_1, e_2\rangle | > \delta_2\right] \leq 4e^{-\frac{n\delta_2^2}{2}} \tag{4.32}$$

Proof. First, lets take $e_1, e_2 \overset{iid}{\sim} S^{n-1}$. Let x be a fixed unit vector in $\mathbb{R}^n$. Due to symmetry, we have $\mathbb{P}\left[\langle e_1, x\rangle \geq 0\right] = 1/2$. Hence, by Levy's Isoperimetric inequality on the sphere [19], we have

$$\mathbb{P}\left[\langle e_1, x\rangle > \delta_2\right] \leq e^{-n\delta_2^2/2}. \tag{4.33}$$

Again by symmetry, and then taking x as e_2, we have

$$\mathbb{P}\left[|\langle e_1, e_2\rangle| > \delta_2\right] \leq 2e^{-n\delta_2^2/2}. \tag{4.34}$$

Now uniform distribution on $(\mathcal{CS})^{n-1}$ is same as the uniform distribution on S^{2n-1}, and for complex vectors $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ we have $Re\langle z_1, z_2\rangle = x_1^T x_2 + y_1^T y_2 = (x_1, y_1)^T(x_2, y_2)$. Hence if $e_1, e_2 \overset{iid}{\sim} (\mathcal{CS})^{n-1}$, and $u_1, u_2 \overset{iid}{\sim} S^{2n-1}$ then $Re\langle e_1, e_2\rangle$ has same law as $\langle u_1, u_2\rangle$. Hence we have

$$\mathbb{P}\left[|Re\langle e_1, e_2\rangle| > \delta_2\right] \leq 2e^{-\frac{2n\delta_2^2}{2}}. \tag{4.35}$$

Also, $Im\langle z_1, z_2\rangle = x_1^T y_2 - y_1^T x_2$. Hence $Im\langle e_1, e_2\rangle$ has the same law as $Re\langle e_1, e_2\rangle$. Hence we have

$$\begin{aligned}
&\mathbb{P}\left[| \langle e_1, e_2\rangle | > \delta_2\right] \\
&= \mathbb{P}\left[| \langle e_1, e_2\rangle |^2 > \delta_2^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left[|Re\langle e_1, e_2\rangle|^2 + |Im\langle e_1, e_2\rangle|^2 > \delta_2^2\right] \\
&\leq \mathbb{P}\left[|Re\langle e_1, e_2\rangle| > \frac{\delta_2}{\sqrt{2}}\right] + \mathbb{P}\left[|Im\langle e_1, e_2\rangle| > \frac{\delta_2}{\sqrt{2}}\right] \\
&\leq 4e^{-\frac{n\delta_2^2}{2}}.
\end{aligned} \tag{4.36}
$$

□

Next we have a similar lemma for low dimensional projections from [34, Lemma 5.3.2]

Lemma 4.2.4 ([34]). *Let $x \sim Unif(S^{n-1})$ and P be a projection to an m dimensional subspace of $\mathbb{R}^n$. Then for any $\delta_3 > 0$, we have*

$$
\mathbb{P}\left[\left|\|Px\| - \sqrt{\frac{m}{n}}\right| > \delta_3\right] \leq 2e^{-cn\delta_3^2} \tag{4.37}
$$

where c is some absolute constant. Hence, by symmetry, the result remains true if P is a uniform random projection, independent of x.

Now we need to prove that a similar result holds for the complex variable case as well. We have the following lemma

Lemma 4.2.5. *Let $z \sim Unif(\mathcal{CS})^{n-1}$ and P be a projection to an m dimensional subspace V of $\mathbb{C}^n$. Then for any $\delta_3 > 0$, we have*

$$
\mathbb{P}\left[\left|\|Pz\| - \sqrt{\frac{m}{n}}\right| > \delta_3\right] \leq 2e^{-2cn\delta_3^2} \tag{4.38}
$$

where c is some absolute constant. Hence, by symmetry, the result remains true if P is a uniform random projection, independent of z.

Proof. Consider $\|Pz\|$. Let U be the unitary change of basis matrix which converts V to first m coordinates. Hence $\|Pz\| = \|UPz\|$. Therefore we can just consider the orthogonal projection onto first m coordinates. Hence the projection matrix P is real. Let $e_1, ..., e_m$ be the standard basis corresponding to the first m coordinates. Let A be the $n \times m$ matrix whose columns are $e_1, ..., e_m$. Then $P = AA^*$ ($*$ denotes conjugate transpose). Since A is real, we have $Re(Pz) = AA^*Re(z)$ and $Im(Pz) = AA^*Im(z)$. Now, if $z \sim Unif((\mathcal{CS})^{n-1})$ then $Re(z)$ has same law as $Im(z)$. Hence $Re(Pz)$ has same law as $Im(Pz)$. Further $A^* = A^T$. Also note that, if $z = x+iy$ then $\|Pz\|^2 = z^*AA^*Z = x^TAA^Tx + y^TAA^Ty = \begin{bmatrix} x^T & y^T \end{bmatrix} \begin{bmatrix} AA^T & 0 \\ 0 & AA^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \left\| \hat{P} \begin{bmatrix} x \\ y \end{bmatrix} \right\|$ where $\hat{P}$ denotes the orthogonal projection from $\mathbb{R}^{2n}$ to a $2m$ dimensional subspace. Hence $\|Pz\|^2$ has the same law as that of the projection of a uniform random vector on S^{2n-1} to a $2m$ dimensional subspace. Hence using lemma 4.2.4, we have

$$\mathbb{P}\left[\left|\|Pz\| - \sqrt{\frac{m}{n}}\right| > \delta_3\right] \leq 2e^{-2cn\delta_3^2} \tag{4.39}$$

$\square$

Since $H_i \sim \mathcal{CN}(0,1)$, we have $|H_i|^2 \sim \frac{1}{2}\chi_2(2) = \exp(1)$ where $\chi_2(d)$ denotes the chi-squared distribution with d degrees of freedom and $\exp(1)$ represents an exponentially distributed random variable with rate 1. Therefore, for $\nu \geq 0$,

$$\mathbb{P}\left[|H_i|^2 \geq \nu\right] = e^{-\nu} \tag{4.40}$$

Now, we are in a position to bound

$$\mathbb{P}\left[\bigcup_{t\in[K]} \bigcup_{\substack{S\in[K] \\ |S|=t}} \gamma_n \left(1 + \frac{\left\|P^{\perp}_{c_{[S^c]}} \sum_{i\in S} H_i c_i\right\|^2}{n'}\right) - 1 \leq \delta_1\right].$$

For $S \subset [K]$ with $|S| = t$, define the events E_2, E_3 and E_4 as follows:

$$E_2(S,t) = \bigcap_{i\in S} \left\{\left|\left\|P_{c_{[S^c]}}\hat{c}_i\right\| - \sqrt{\frac{K-t}{n}}\right| \leq \delta_3\right\} \tag{4.41a}$$

$$E_3 = \bigcap_{i<j:i,j\in[K]} \{|\langle \hat{c}_i, \hat{c}_j\rangle| \leq \delta_2\} \tag{4.41b}$$

$$E_4 = \bigcap_{i\in[K]} \{|H_i|^2 \leq \nu\} \tag{4.41c}$$

where we choose $\delta_2 = n^{-\frac{1}{3}} = \delta_3$ and $\nu = n^{\frac{1}{4}}$. Hence we have

$$\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}} \gamma_n\left(1+\frac{\left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_i c_i\right\|^2}{n'}\right)-1\leq\delta_1\right]$$
$$\leq \mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\gamma_n\left(1+\frac{\left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_i c_i\right\|^2}{n'}\right)-1\leq\delta_1, E_2(S,t), E_3, E_4\right\}\right]+$$
$$\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left(E_2^c(S,t)\cup E_3^c\cup E_4^c\right)\right]$$
$$\leq \mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\gamma_n\left(1+\frac{\left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_i c_i\right\|^2}{n'}\right)-1\leq\delta_1, E_2(S,t), E_3, E_4\right\}\right]+$$
$$\mathbb{P}\left[E_3^c\right]+\mathbb{P}\left[E_4^c\right]+\sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}}\mathbb{P}\left[E_2^c(S,t)\right]. \tag{4.42}$$

Using lemmas 4.2.3 and 4.2.5 and eq. (4.40), we have

$$\mathbb{P}\left[E_3^c\right]+\mathbb{P}\left[E_4^c\right]+\sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}}\mathbb{P}\left[E_2^c(S,t)\right]\leq 2Ke^{-cn\delta_3^2}+Ke^{-\nu}+\sum_{\substack{S\in[K]\\|S|=t}}2t(t-1)e^{-\frac{n\delta_2^2}{2}}. \tag{4.43}$$

Note that the above quantity goes to 0 as $n\to\infty$ due to the choice of δ_2, δ_3 and ν. Also, the choice of parameters is not the optimum. Nevertheless, this is enough to prove the result.

Further, observe that on the sets E_2, E_3 and E_4, we have from (4.31)

$$\left|Re\left\langle P^{\perp}_{c_{[S^c]}}c_i, P^{\perp}_{c_{[S^c]}}c_j\right\rangle H_i\bar{H}_j\right|\leq\nu\left[\delta_2+\left(\delta_3+\sqrt{\frac{K-t}{n}}\right)^2\right]=O(n^{-\frac{1}{12}}) \tag{4.44a}$$
$$|H_i|^2\left\|P_{c_{[S^c]}}\hat{c}_i\right\|^2\leq\nu\left[\delta_3+\sqrt{\frac{K-t}{n}}\right]^2=O(n^{-\frac{5}{12}}) \tag{4.44b}$$

So we have

$$\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\gamma_n\left(1+\frac{\left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_i c_i\right\|^2}{n'}\right)-1\leq\delta_1, E_2(S,t), E_3, E_4\right\}\right]$$

$$=\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\gamma_n\left[1+\frac{nP}{n'}\left\{\sum_{i\in S}|H_i|^2\|\hat{c}_i\|^2-\sum_{i\in S}|H_i|^2\left\|P_{c_{[S^c]}}\hat{c}_i\right\|^2\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+2\sum_{i<j:i,j\in S}Re\left(\left\langle P^{\perp}_{c_{[S^c]}}\hat{c}_i, P^{\perp}_{c_{[S^c]}}\hat{c}_j\right\rangle H_i\bar{H}_j\right)\right\}\right]-1\leq\delta_1, E_2(S,t), E_3, E_4\right\}\right]$$

$$\leq\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\gamma_n\left[1+\frac{nP}{n'}\left\{\sum_{i\in S}|H_i|^2\|\hat{c}_i\|^2-t\nu\left(\delta_3+\sqrt{\frac{K-t}{n}}\right)^2\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-t(t-1)\left(\delta_2+\left(\delta_3+\sqrt{\frac{K-t}{n}}\right)^2\right)\right\}\right]-1\leq\delta_1\right\}\right]$$

$$=\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\gamma_n\left[1+\frac{nP}{n'}\sum_{i\in S}|H_i|^2\right]\leq 1+\delta_1+\gamma_n O(n^{-\frac{1}{12}})\right\}\right]$$

$$\leq\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\gamma_n\left[1+\frac{nP}{n'}\sum_{i\in S}|H_i|^2\right]\leq 1+\delta_1+O(n^{-\frac{1}{12}})\right\}\right]$$

$$\leq\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\gamma_n\left[1+P\sum_{i\in S}|H_i|^2\right]\leq 1+\delta_1+O(n^{-\frac{1}{12}})\right\}\right]$$

$$=\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\log\left[1+P\sum_{i\in S}|H_i|^2\right]\leq -\log(\gamma_n)+\log(1+\delta_1+O(n^{-1/12}))\right\}\right]$$

$$=\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\log\left[1+P\sum_{i\in S}|H_i|^2\right]\leq \tilde{\gamma}_n+\log(1+\delta_1+O(n^{-1/12}))\right\}\right]$$

$$= \mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\log\left[1+P\sum_{i\in S}|H_i|^2\right]\le\gamma_n'\right\}\right] \tag{4.45}$$

where $\gamma_n' = \tilde{\gamma}_n + \log(1+\delta_1+O(n^{-1/12}))$, and O depends on K and t.

Let $\delta_n = \log(1+\delta_1+O(n^{-1/12}))$. We have $\frac{\log(M_S)}{n-K} = \left(\sum_{i\in S}(R_i-\eta_i)\right)(1+o(1))$.

By the choice of $M_i^{(n)}$, for sufficiently large n, sufficiently small δ and δ_1, we have

$$\begin{aligned}
&\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\log\left[1+P\sum_{i\in S}|H_i|^2\right]\le\gamma_n'\right\}\right]\\
&=\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\log\left[1+P\sum_{i\in S}|H_i|^2\right]\le(t-1)\frac{\log(n-K)}{n-K}+\frac{\log(M_S)}{n-K}+\delta+\delta_n\right\}\right]\\
&=\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\log\left[1+P\sum_{i\in S}|H_i|^2\right]\le(t-1)\frac{\log(n-K)}{n-K}\right.\right.\\
&\qquad\left.\left.+\left(\sum_{i\in S}(R_i-\eta_i)\right)(1+o(1))+\delta+\delta_n\right\}\right]\\
&\le\mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\log\left[1+P\sum_{i\in S}|H_i|^2\right]\le\left(\sum_{i\in S}R_i\right)\right\}\right] && (4.46)
\end{aligned}$$

Finally combining everything, we have

$$\epsilon_n \le \mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\log\left[1+P\sum_{i\in S}|H_i|^2\right]\le(t-1)\frac{\log(n-K)}{n-K}\right.\right.$$

$$+\left(\sum_{i\in S}(R_i-\eta_i)\right)(1+o(1))+\delta+\delta_n)\Bigg\}\Bigg]+$$

$$2Ke^{-cn^{1/3}}+Ke^{-n^{1/4}}+\sum_{t\in[K]}\sum_{\substack{S\in[K]\\|S|=t}}\left[e^{-\delta(n-K)}+e^{-nf_n(\delta_1)}+2t(t-1)e^{-\frac{n^{1/3}}{2}}\right]. \tag{4.47}$$

Therefore for this choice of $\left(M_i^{(n)}\right)$, from (4.46) we have

$$\begin{aligned}
&\limsup_{n\to\infty}\epsilon_n \\
&\leq \mathbb{P}\left[\bigcup_{t\in[K]}\bigcup_{\substack{S\in[K]\\|S|=t}}\left\{\log\left[1+P\sum_{i\in S}|H_i|^2\right]\leq\left(\sum_{i\in S}R_i\right)\right\}\right] \\
&\leq \epsilon
\end{aligned} \tag{4.48}$$

Since $\eta_i>0$ were arbitrary, we are done. That is (4.5) is also satisfied.

$\square$

Chapter 5

Asymptotics of the projection decoder: per-user probability of error

In this chapter, we analyze the asymptotics of the projection decoder under the per-user probability of error in the case of non-same codebook. Our benchmark in this case is the Shamai-Bettesh asymptotic bound from [5]. The authors provide an asymptotic bound ($n \to \infty$) on the probability of error per user in the case of symmetric rate and large K. The idea is the following. The joint decoder that they use knows the realization of fading coefficients and users are ranked according to this information. The decoder first tries to decode all users. If it fails (i.e., the rate vector is not inside the instantaneous full capacity region), it drops the user with least fading coefficient and tries to decode the remaining $K - 1$ users. The dropped user forms part of the noise. This process continues iteratively, and the fraction of users that were not decoded is precisely the outage/probability of error per-user. Since the case under discussion is for large K, the order statistics of the absolute value of fading coefficients crystallize (i.e., become almost non-random) and hence analytical expressions can be derived for outage in terms of spectral efficiency (kK/n) and total power.

Assuming channel state information at the receiver (CSIR), we show that in the general non-symmetric case, the projection decoder (suitably modified to use CSIR) achieves the asymptotic bound as that of [5] generalized to the non-symmetric situation.

5.1 Achievability

Just to recall, we consider the quasi-static K–MAC with CSIR. We modify the projection decoder to use CSIR as follows. The decoder works in two stages. The first stage finds the following set

$$D \in \arg\max \left\{ |D| : D \subset [K], \forall S \subset D, S \neq \emptyset, \sum_{i \in S} R_i < \log\left(1 + \frac{P \sum_{i \in S} |H_i|^2}{1 + P \sum_{i \in D^c} |H_i|^2}\right) \right\} \tag{5.1}$$

where D is chosen to contain users with largest fading coefficients. The second stage is similar to (3.1) but decodes only those users in D. Formally, let ε denote an error symbol. The decoder output $g_D(Y) \in \prod_{i=1}^{k} \mathcal{C}_i$ is given by

$$\begin{aligned} (g_D(Y))_i &= \begin{cases} f_i^{-1}(\hat{c}_i) & i \in D \\ \varepsilon & i \notin D \end{cases} \\ (\hat{c}_i)_{i \in D} &= \arg \max_{(c_i \in \mathcal{C}_i)_{i \in D}} \left\| P_{\{c_i : i \in D\}} Y \right\|^2 \end{aligned} \tag{5.2}$$

where f_i are the encoding functions. Our error metric is the average per-user probability of error (2.2).

We recall the result on the per-user probability of error as $n \to \infty$ from [5], for the joint decoder, in the general non-symmetric rate case:

$$\begin{aligned} P_e^S(R) = 1 - \frac{1}{K} \mathbb{E} \sup \Bigg\{ &|D| : D \subset [K], \forall S \subset D, S \neq \emptyset, \\ &\sum_{i \in S} R_i < \log\left(1 + \frac{P \sum_{i \in S} |H_i|^2}{1 + P \sum_{i \in D^c} |H_i|^2}\right) \Bigg\} \end{aligned} \tag{5.3}$$

where $R = (R_1, ..., R_K)$ and the maximizing set, among all those that achieve the maximum, is chosen to contain the users with largest fading coefficients.

The following theorem is the main result of this section.

Theorem 5.1.1. *For the K-MAC, if $P_e^S(R) < \epsilon$, then there exists a sequence of $\left((M_1^{(n)}, M_2^{(n)}, ..., M_K^{(n)}), n, \epsilon_n\right)_{PU}$ codes with the decoder given by* (5.1) *and* (5.2) *such* *that*

$$\liminf_{n\to\infty} \frac{1}{n} \log \left(M_i^{(n)} \right) \geq R_i, \forall i \in [K] \tag{5.4}$$

$$\limsup_{n\to\infty} \epsilon_n \leq \epsilon \tag{5.5}$$

Proof. Let $P_e^S(R) < \epsilon$ and $\eta_i > 0, i \in [K]$. Choose $M_i^{(n)} = \lceil e^{n(R_i - \eta_i)} \rceil, \forall i \in [K]$. User i generates M_i codewords $\{c_j^i : j \in [M_i]\} \overset{iid}{\sim} \mathcal{CN}(0, P_n' I_n)$ independent of other users, where $P_n' = \frac{P}{1+n^{-\frac{1}{3}}}$. For the (random) message $W_i \in [M_i]$, user i transmits $X_i = c_{W_i}^i 1\{\|c_{W_i}^i\|^2 > nP\}$. The channel model is given in (2.6) and the decoder is given by (5.1) and (5.2). The per-user probability of error is given by (2.2)

$$P_e = \mathbb{E}\left[\frac{1}{K} \sum_{j=1}^{K} 1\left\{ W_j \neq (g_D(Y))_j \right\} \right]. \tag{5.6}$$

Similar to the proof of [27, Theorem 1], we change the measure over which $\mathbb{E}$ is taken in (5.6) to the one where $X_i = c_{W_i}^i$ at the cost of adding a total variation distance. Hence the probability of error under this change of measure becomes

$$P_e \leq p_1 + p_0$$

with

$$p_0 = K\mathbb{P}\left[\|w\|^2 > n\frac{P}{P_n'} \right] \tag{5.7}$$

$$p_1 = \mathbb{E}\left[\frac{1}{K} \sum_{j=1}^{K} 1\left\{ W_j \neq (g_D(Y))_j \right\} \right] \tag{5.8}$$

where $w \sim \mathcal{CN}(0, I_n)$ and, with abuse of notation, $\mathbb{E}$ in p_1 is taken over the new measure. It can be easily seen that by the choice of P_n' and lemma A.0.1, $p_0 \to 0$ as $n \to \infty$. From now on, we exclusively focus on bounding p_1.

p_1 can also be written as

$$\begin{aligned} p_1 &= \frac{1}{K}\mathbb{E}\left[\sum_{i\in D} 1\left\{W_j \neq (g_D(Y))_j\right\} + |D^c|\right] \\ &= 1 - \frac{\mathbb{E}\left[|D|\right]}{K} + \frac{1}{K}\mathbb{E}\left[\sum_{i\in D} 1\left\{W_j \neq (g_D(Y))_j\right\}\right] \end{aligned} \tag{5.9}$$

where D is given by (5.1), because, for $i \in D^c$, $1\left\{W_j \neq (g_D(Y))_j\right\} = 1,\ a.s.$ Define p_2 as

$$p_2 = \mathbb{P}\left[\sum_{i\in D} 1\left\{W_j \neq (g_D(Y))_j\right\} > 0\right]. \tag{5.10}$$

So, it's enough to show that $p_2 \to 0$ as $n \to \infty$. This is because, if $p_2 \to 0$, then the non-negative random variables $A_n = \sum_{i\in D} 1\left\{W_j \neq (g_D(Y))_j\right\}$ converge to 0 in probability. Since $A_n \leq K$, $a.s$, we have, by dominated convergence, $\mathbb{E}\left[A_n\right] = \mathbb{E}\left[\sum_{i\in D} 1\left\{W_j \neq (g_D(Y))_j\right\}\right] \to 0$. To this end, we upper bound p_2.

Let $c = (c_1, ..., c_K) \in \mathcal{C}_1 \times ... \times \mathcal{C}_K$ be the tuple of sent codewords. Let $K_1 = |D|$. Let $c_{(D)}$ denote the ordered tuple corresponding to indices in D. That is, if $i_1 < i_2 < ... < i_{K_1}$ are the elements of D, then $(c_{(D)})_j = c_{i_j}, \forall j \in [K_1]$. Then p_2 can also be written as

$$\begin{aligned} p_2 &= \mathbb{P}\left[\sum_{i\in D} 1\left\{W_j \neq (g_D(Y))_j\right\} > 0\right] && (5.11) \\ &= \mathbb{P}\left[\exists S \subset D, S \neq \emptyset : \forall i \in S, (g_D(Y))_i \neq W_i\right] && (5.12) \\ &= \mathbb{P}\left[\exists c'_{(D)} \neq c_{(D)} : \left\|P_{c'_{[D]}} Y\right\|^2 > \left\|P_{c_{[D]}} Y\right\|^2\right] && (5.13) \\ &= \mathbb{P}\left[\exists S \subset D, S \neq \emptyset \text{ s.t } \forall i \in S, \exists c'_i \neq c_i : \left\|P_{c'_{[S]}, c_{[S^c]}} Y\right\|^2 > \left\|P_{c_{[D]}} Y\right\|^2\right] && (5.14) \\ &= \mathbb{P}\left[\bigcup_{t\in[K_1]} \bigcup_{\substack{S\subset D \\ |S|=t}} \bigcup_{\substack{c'_i \in \mathcal{C}_i \setminus \{c_i\} \\ i\in S}} \left\{\left\|P_{c'_{[S]}, c_{[S^c]}} Y\right\|^2 > \left\|P_{c_{[D]}} Y\right\|^2\right\}\right]. && (5.15) \end{aligned}$$

Let $\delta > 0$, $g(Y, c_{[K]}, S, D) = \frac{\|Y\|^2 - \left\|P_{c_{[D]}} Y\right\|^2}{\|Y\|^2 - \left\|P_{c_{[S^c]}} Y\right\|^2}$, $M_S = \prod_{j\in S}(M_j - 1)$, $\tilde{\gamma}_n = (t -$

$1)\frac{\log(n-K_1)}{n-K_1} + \frac{\log(M_S)}{n-K_1} + \delta$ and $\gamma_n = e^{-\tilde{\gamma}_n}$. Note that, since D is random, both M_S and γ_n are random. But in the symmetric case only M_S is not random. Now, following steps similar to (4.10), (4.11), (4.13) and (4.16), we have

$$p_2 \leq \mathbb{E}\left[\sum_{t\in[K_1]}\sum_{\substack{S\subset D\\|S|=t}} e^{-(n-K_1)\delta}\right] + \mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{S\subset D:|S|=t}\{g(Y,c_{[K]},S,D)\geq\gamma_n\}\right] \tag{5.16}$$

$$\leq \sum_{t\in[K]}\sum_{\substack{S\subset[K]\\|S|=t}} e^{-(n-K)\delta} + \mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{S\subset D:|S|=t}\{g(Y,c_{[K]},S,D)\geq\gamma_n\}\right]. \tag{5.17}$$

So, the first term goes to 0 as $n\to\infty$.

Let $Z_D = Z + \sum_{i\in D^c} H_i c_i$. It can be easily seen that, similar to (4.17), we have

$$\begin{aligned}&\mathbb{P}\left[g(Y,c_{[K]},S,D)\geq\gamma_n\right]\\ &\leq \mathbb{P}\left[\left\|(1-\gamma_n)P^{\perp}_{c_{[S^c]}}Z_D - \gamma_n P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_i c_i\right\|^2 \geq \gamma_n\left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_i c_i\right\|^2\right].\end{aligned} \tag{5.18}$$

Now, conditional of $H_{[K]}$ and $c_{[D]}$, $Z_D \sim \mathcal{CN}(0,(1+P'_n\sum_{i\in D^c}|H_i|^2))$. Hence $P^{\perp}_{c_{[S^c]}}\left(Z_D - \frac{\gamma_n}{1-\gamma_n}\sum_{i\in S}H_i c_i\right) \sim \mathcal{CN}(-\frac{\gamma_n}{1-\gamma_n}P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_i c_i, (1+P'_n\sum_{i\in D^c}|H_i|^2)P^{\perp}_{c_{[S^c]}})$. Therefore

$$\left\|P^{\perp}_{c_{[S^c]}}\left(Z_D - \frac{\gamma_n}{1-\gamma_n}\sum_{i\in S}H_i c_i\right)\right\|^2 \sim \left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)\frac{1}{2}\chi'_2(2\lambda, 2n') \tag{5.19}$$

where

$$\lambda = \frac{\left\|\frac{\gamma_n}{1-\gamma_n}P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_i c_i\right\|^2}{(1+P'_n\sum_{i\in D^c}|H_i|^2)} \tag{5.20}$$

$$n' = n - K_1 + t. \tag{5.21}$$

Let

$$\gamma = \frac{\gamma_n}{(1-\gamma_n)} \frac{\left\| P^{\perp}_{c_{[S^c]}} \sum_{i\in S} H_i c_i \right\|^2}{\left(1 + P'_n \sum_{i\in D^c} |H_i|^2\right)} - n' \tag{5.22}$$

$$\gamma^1 = \frac{1}{1-\gamma_n} \left(\gamma_n \left(1 + \frac{\left\| P^{\perp}_{c_{[S^c]}} \sum_{i\in S} H_i c_i \right\|^2}{n'\left(1 + P'_n \sum_{i\in D^c} |H_i|^2\right)} \right) - 1 \right) \tag{5.23}$$

Hence $\gamma = n'\gamma^1$ and $\lambda = \frac{\gamma_n}{1-\gamma_n} n'(1+\gamma^1)$. So, similar to (4.21), we have

$$\begin{aligned} &\mathbb{P}\left[\bigcup_{t\in[K_1]} \bigcup_{S\subset D:|S|=t} \{ g(Y, c_{[K]}, S, D) \geq \gamma_n \} \right] \\ &\leq \mathbb{P}\left[\bigcup_{t\in[K_1]} \bigcup_{\substack{S\subset D\\ |S|=t}} \left\{ \frac{1}{2}\chi'_2(2\lambda, 2n') - (\lambda + n') \geq \gamma \right\} \right] \end{aligned} \tag{5.24}$$

Let $\delta_1 > 0$ and $E_{11} = \bigcap_{t\in[K_1]} \bigcap_{\substack{S\subset D\\ |S|=t}} \{\gamma^1 > \delta_1\} \in \sigma(H_{[K]}, c_{[D]})$.

Now, similar to (4.26), we have

$$\mathbb{P}\left[\bigcup_{t\in[K_1]} \bigcup_{\substack{S\subset D\\ |S|=t}} \left\{ \frac{1}{2}\chi'_2(2\lambda, 2n') - (\lambda + n') \geq \gamma \right\} \right] \leq \mathbb{E}\left[\sum_{t\in[K_1]} \sum_{\substack{S\subset D\\ |S|=t}} \exp(-n' f_n(\delta_1)) \right] + \mathbb{P}\left[E_{11}^c\right]. \tag{5.25}$$

where f_n (now a random function) was defined in (4.25). So, again by claim 5 and dominated convergence, the first term in (5.25) converges to 0 as $n \to \infty$. Next, we upper bound the second term $\mathbb{P}[E_{11}^c]$.

Similar to (4.28), we have

$$\mathbb{P}\left[E_{11}^c\right] = \mathbb{P}\left[\bigcup_{t\in[K_1]} \bigcup_{\substack{S\subset D\\ |S|=t}} \{\gamma^1 \leq \delta_1\} \right] \leq$$

$$\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\gamma_n\left(1+\frac{\left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_ic_i\right\|^2}{n'\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right)-1\leq\delta_1\right]. \tag{5.26}$$

Let $\hat{c}_i = c_i/\|c_i\|$. Let $\delta_2 > 0$, $\delta_3 > 0$, $\delta_4 > 0$ and $\nu > 1$. Define the events

$$E_2(S,t)=\bigcap_{i\in S}\left\{\left|\left\|P_{c_{[S^c]}}\hat{c}_i\right\|-\sqrt{\frac{K_1-t}{n}}\right|\leq\delta_3\right\} \tag{5.27a}$$

$$E_3=\bigcap_{i<j:i,j\in[K]}\{|\langle\hat{c}_i,\hat{c}_j\rangle|\leq\delta_2\} \tag{5.27b}$$

$$E_4=\bigcap_{i\in[K]}\{|H_i|^2\leq\nu\} \tag{5.27c}$$

$$E_5=\bigcap_{i\in[K]}\left\{|\,\|c_i\|-\sqrt{nP'_n}\leq\delta_4\sqrt{nP'}|\right\} \tag{5.27d}$$

and choose $\delta_2 = O(n^{-\frac{1}{3}}) = \delta_3 = \delta_4$ and $\nu = O(n^{1/4})$.

Using these events we can bound $\mathbb{P}\left[E_{11}^c\right]$ as

$$\begin{aligned}&\mathbb{P}\left[E_{11}^c\right]\leq\\&\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\gamma_n\left(1+\frac{\left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S}H_ic_i\right\|^2}{n'\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right)-1\leq\delta_1,E_2(S,t),E_3,E_4,E_5\right\}\right]+\\&\mathbb{P}\left[E_3^c\right]+\mathbb{P}\left[E_4^c\right]+\mathbb{P}\left[E_5^c\right]+\mathbb{E}\left[\sum_{t\in[K_1]}\sum_{\substack{S\subset D\\|S|=t}}\mathbb{P}\left[E_2^c(S,t)|H_{[K]}\right]\right].\end{aligned} \tag{5.28}$$

From [34, Theorem 3.1.1], we have

$$\mathbb{P}\left[E_5^c\right]\leq 2Ke^{-c_1n\delta_4^2} \tag{5.29}$$

for some constant $c_1 > 0$. So, from lemma 4.2.3, lemma 4.2.5, (4.40) and (5.29), we have

$$
\begin{aligned}
&\mathbb{P}\left[E_{11}^c\right] \leq \\
&\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\gamma_n\left(1+\frac{\left\|P^{\perp}_{c[S^c]}\sum_{i\in S}H_ic_i\right\|^2}{n'\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right)-1\leq\delta_1, E_2(S,t), E_3, E_4, E_5\right\}\right]+ \\
&2Ke^{-cn\delta_3^2}+Ke^{-\nu}+2Ke^{-c_1n\delta_4^2}+\sum_{t\in[K]}\sum_{\substack{S\subset[K]\\|S|=t}}4e^{-\frac{n\delta_2^2}{2}}
\end{aligned}
\quad . \tag{5.30}
$$

So, by the chose of $\delta_i, i \in \{2,3,4\}$ and ν, the exponential terms in the last expression go to 0 as $n \to \infty$.

Now, arguing similar to (4.45), we get

$$
\begin{aligned}
&\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\gamma_n\left(1+\frac{\left\|P^{\perp}_{c[S^c]}\sum_{i\in S}H_ic_i\right\|^2}{n'\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right)-1\leq\delta_1, E_2(S,t), E_3, E_4, E_5\right\}\right] \\
&=\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\gamma_n\left[1+\frac{1}{n'\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\left\{\sum_{i\in S}|H_i|^2\|c_i\|^2-\sum_{i\in S}|H_i|^2\left\|P_{c[S^c]}c_i\right\|^2\right.\right.\right.\right. \\
&\quad\left.\left.\left.\left.+2\sum_{i<j:i,j\in S}Re\left(\left\langle P^{\perp}_{c[S^c]}c_i, P^{\perp}_{c[S^c]}c_j\right\rangle H_i\bar{H}_j\right)\right\}\right]-1\leq\delta_1, E_2(S,t), E_3, E_4, E_5\right\}\right] \qquad (5.31) \\
&\leq\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\gamma_n\left[1+\left\{\frac{nP'_n(1-\delta_4)^2\sum_{i\in S}|H_i|^2}{n'\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right.\right.\right.\right. \\
&\quad-(1+\delta_4)^2\left(\frac{nP'_n}{n-K}t\nu\left(\delta_3+\sqrt{\frac{K_1-t}{n}}\right)^2\right. \\
&\quad\left.\left.\left.\left.\left.+\frac{nP'_nt(t-1)}{n-K}\left(\delta_2+\left(\delta_3+\sqrt{\frac{K_1-t}{n}}\right)^2\right)\right)\right\}\right]-1\leq\delta_1\right\}\right] \qquad (5.32)
\end{aligned}
$$

$$\leq \mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\gamma_n\left[1+\left\{\frac{nP'_n\sum_{i\in S}|H_i|^2}{n'\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}-\frac{nP'_nK\nu O(\delta_4)}{n-K}\right.\right.\right.\right.$$

$$-\frac{nP'_n}{n-K}K(1+\delta_4)^2\nu\left(\delta_3+\sqrt{\frac{K}{n}}\right)^2$$

$$\left.\left.\left.\left.-\frac{nP'_nK^2(1+\delta_4)^2}{n-K}\left(\delta_2+\left(\delta_3+\sqrt{\frac{K}{n}}\right)^2\right)\right\}\right]-1\leq\delta_1\right\}\right] \tag{5.33}$$

$$=\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\gamma_n\left[1+\frac{nP'_n\sum_{i\in S}|H_i|^2}{n'\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right]\leq 1+\delta_1+\gamma_nO(n^{-\frac{1}{12}})\right\}\right] \tag{5.34}$$

$$\leq\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\gamma_n\left[1+\frac{nP'_n\sum_{i\in S}|H_i|^2}{n'\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right]\leq 1+\delta_1+O(n^{-\frac{1}{12}})\right\}\right] \tag{5.35}$$

$$\leq\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\gamma_n\left[1+\frac{P'_n\sum_{i\in S}|H_i|^2}{\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right]\leq 1+\delta_1+O(n^{-\frac{1}{12}})\right\}\right] \tag{5.36}$$

$$=\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\log\left[1+\frac{P'_n\sum_{i\in S}|H_i|^2}{\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right]\right.\right.$$

$$\left.\left.\leq-\log(\gamma_n)+\log(1+\delta_1+O(n^{-1/12}))\right\}\right] \tag{5.37}$$

$$=\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\log\left[1+\frac{P'_n\sum_{i\in S}|H_i|^2}{\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right]\leq\tilde{\gamma}_n+\log(1+\delta_1+O(n^{-1/12}))\right\}\right] \tag{5.38}$$

$$=\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\log\left[1+\frac{P'_n\sum_{i\in S}|H_i|^2}{\left(1+P'_n\sum_{i\in D^c}|H_i|^2\right)}\right]\leq\gamma'_n\right\}\right] \tag{5.39}$$

where $\gamma'_n=\tilde{\gamma}_n+\log(1+\delta_1+O(n^{-1/12}))$.

Let $\delta_n=\log(1+\delta_1+O(n^{-1/12}))$. We have $\frac{\log(M_S)}{n-K}=\left(\sum_{i\in S}(R_i-\eta_i)\right)(1+o(1))$. There for sufficiently large n and sufficiently small δ and δ_1, we have $\gamma'_n\leq\sum_{i\in S}R_i$ *a.s.*

Hence

$$\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\log\left[1+\frac{P_n'\sum_{i\in S}|H_i|^2}{\left(1+P_n'\sum_{i\in D^c}|H_i|^2\right)}\right]\le\gamma_n'\right\}\right] \tag{5.40}$$

$$\le\mathbb{P}\left[\bigcup_{t\in[K_1]}\bigcup_{\substack{S\subset D\\|S|=t}}\left\{\log\left[1+\frac{P_n'\sum_{i\in S}|H_i|^2}{\left(1+P_n'\sum_{i\in D^c}|H_i|^2\right)}\right]\le\sum_{i\in S}R_i\right\}\right] \tag{5.41}$$

But we know that $P_n'\to P$, and on D, from (5.1) we have

$$\sum_{i\in S}R_i<\log\left(1+\frac{P\sum_{i\in S}|H_i|^2}{1+P\sum_{i\in D^c}|H_i|^2}\right)\quad a.s. \tag{5.42}$$

Hence the probability in (5.41) goes to 0 as $n\to\infty$.

So combining everything from (5.17), (5.26), (5.30), (5.31), (5.39), (5.41) and (5.42), we get $p_2\to 0$ as $n\to\infty$. Therefor $p_1\to 1-\frac{\mathbb{E}[D]}{K}$ as $n\to\infty$. Hence we have

$$\epsilon_n=P_e\to 1-\frac{\mathbb{E}\left[D\right]}{K}<\epsilon. \tag{5.43}$$

Hence $\limsup_{n\to\infty}\epsilon_n\le\epsilon$. Further, since $\eta_i>0$ were arbitrary, we can ensure $\liminf_{n\to\infty}\frac{1}{n}M_i^{(n)}\ge R_i,\forall i\in[K]$. □

Chapter 6

Numerical results and discussion

This chapter is devoted to presenting some numerical results of our bound for the K_a–MAC. As mentioned in the introduction, our performance metric is the minimum energy-per-bit ($E_b/N_0 = \frac{nP}{k}$) required to achieve a target probability of error ϵ for various values of K_a.

For our simulations, the parameters used are payload size $k = 100$ bits, blocklength $n = 30000$ and target probability of error $\epsilon = 0.1$. Our benchmark performance is the Shamai-Bettesh asymptotic bound [5]. Although this is asymptotic in n and for large K_a, it gives the functional dependence of probability of error (per-user) on spectral efficiency (which is $\frac{kK_a}{n}$ in our case) and the total power (of all users). Hence we can use this to compute the minimum E_b/N_0 required. Against this benchmark, we plot the results of the union bound (3.12), T–fold ALOHA (3.20) for $T = 1, 2, 3, 4$, the converse bound (3.35)(3.36) and two approximations of (3.11). We briefly describe these below.

6.1 Computing the bounds

6.1.1 T–fold ALOHA

We compute the T–fold ALOHA bound for $T = 1, 2, 3, 4$. As seen from(3.20), we need compute $P_e(M, n_1, t, LP)$ for each $t = 1, 2, .., T$. But first we need to optimize the number of slots for each K_a. Since direct optimization of the bound itself is computationally slow, we use a 2^{nd} order capacity approximation

$$P_e(M, n_1, t, P') = \mathbb{E}\left[\frac{n_1 C\left(P' \sum_{i=1}^{t} |H_i|^2\right) - \log(M)}{\sqrt{n_1 V\left(P' \sum_{i=1}^{t} |H_i|^2\right)}}\right] \tag{6.1}$$

where $C(x) = \log(1+x)$ and $V(x) = 1 - \frac{1}{(1+x)^2}$ are the capacity and dispersion of the standard AWGN channel [28], to find the (approximately) optimum L and set $n_1 = \lfloor n/L \rfloor$. Now using this n_1 and L we compute $P_e(M, n_1, t, LP)$ using a Monte-Carlo simulation of (3.3) as describe below.

To perform the Monte-Carlo simulation of (3.3), consider the statistic $g_1(Y, c_{[K_a]}, t) = \max_{S_0} g(Y, c_{[K_a]}, S_0)$. Since this doesn't depend on δ, we can construct a Gaussian kernel approximation of its empirical cumulative distribution function (CDF) using Monte-Carlo simulations and use it to optimize over δ in (3.3). We sample $\approx 10^3$ points and use the inbuilt kernel density estimation function is MATLAB® to approximate the empirical CDF.

6.1.2 Union bound

To compute the union bound (3.12), we need to optimize over the relevant parameters r, δ, γ and δ_1. To this end, we first set up an optimization problem to minimize one of the exponents in (3.12) subject to all the exponents being equal. Using the result of this optimization as the starting point, we optimize (3.12). We use the inbuilt optimization functions in MATLAB® to perform both tasks.

6.1.3 Converse bound

The converse bound (3.35)(3.36) was evaluated using a suitable modified version of the code in [10].

6.1.4 Approximation to the achievability bound (3.3)

Since it is not straightforward to evaluate the bound (3.3) (or (3.11)) for $K_a > 10$, we can approximate it based on the following observations. First of all, we can approximate the noise to be orthogonal to the code space. Hence we set $Z_2 = 0$ in (3.11). Next, we note that if the codewords were truly orthogonal, then the minimum in (3.11) would be achieved by the set S_0 which contains the t smallest fading coefficients (in absolute value). Since we are using Gaussian or spherical codewords in a high dimension ($n = 30000$), they are almost orthogonal. Hence we approximate the minimum by choosing the set corresponding to the smallest fading coefficients. Assuming $Z_2 = 0$ we can make the second approximation rigorous as described below. We call this the *orthogonal noise approximation.*

Orthogonal noise approximation

Consider the following generic problem. Let X be an $n \times K_a$ random matrix with each column iid $Unif\left(\sqrt{nP}(\mathcal{CS})^{n-1}\right)$. Let $S \subset [K_a]$ with $|S| = t$ where $0 \leq t < K_a$. Let $P_{X_{[S]}}$ denote the orthogonal projection operator onto the space spanned by the set of columns $X_S = \{X_i : i \in S\}$ of X. Let $H \in \mathbb{C}^{K_a}$ be a fixed vector. Let $H_S \in \mathbb{C}^t$ denote the subvector corresponding to S. We need to lower bound $\min_S \left\| P^{\perp}_{X_{[S]}} X_{S^c} H_{S^c} \right\|^2$ where minimum is over all t-sized subsets of $[K_a]$. The intuition is the following. If the matrix X were orthogonal, then the minimum occurs at that set which corresponds to the top t absolute values of H. Although X is not orthogonal, it is *almost orthogonal* (because it is approximately a random Gaussian matrix). Next, we formalize this intuition.

Fix $\gamma > 0$ and $0 < \gamma_1 < 1$. Let $B_1 = \cap_{i<j} \left\{ \frac{|\langle X_i, X_j \rangle|}{\|X_i\| \|X_j\|} < \gamma \right\}$ and $B_2 = \{\sigma_{\min}(X) \geq \sqrt{nP}(1 - \sqrt{\frac{K_a}{n}} - \gamma_1)\}$ where $\sigma_{\min}(X)$ is the smallest singular value of X. On the events B_1 and B_2, we have

$$\begin{aligned}
\left\|P_{X_{[S]}} X_{S^c} H_{S^c}\right\| &= \left\|X_S(X_S^{\dagger}X_S)^{-1}X_S^{\dagger}X_{S^c}H_{S^c}\right\| \leq \left\|X_S(X_S^{\dagger}X_S)^{-1}\right\| \left\|X_S^{\dagger}X_{S^c}H_{S^c}\right\| \\
&\leq \frac{1}{\sigma_{\min(X_s)}} nP\gamma\sqrt{t(K_a-t)}\left\|H_{S^c}\right\| \\
&\leq \frac{1}{\sigma_{\min(X)}} nP\gamma\sqrt{t(K_a-t)}\left\|H_{S^c}\right\| \\
&\leq \frac{1}{\sqrt{nP}(1-\sqrt{\frac{K_a}{n}}-\gamma_1)} nP\gamma\sqrt{t(K_a-t)}\left\|H_{S^c}\right\| \\
&= \frac{\sqrt{nP}\sqrt{t(K_a-t)}\gamma}{1-\sqrt{\frac{K_a}{n}}-\gamma_1}\left\|H_{S^c}\right\|. \qquad (6.2)
\end{aligned}$$

Similarly,

$$\left\|X_{S^c}H_{S^c}\right\| \geq \sigma_{\min}(X)\left\|H_{S^c}\right\| \geq \sqrt{nP}\left(1-\sqrt{\frac{K_a}{n}}-\gamma_1\right)\left\|H_{S^C}\right\|. \qquad (6.3)$$

Hence on the events B_1 and B_2, we have

$$\begin{aligned}
&\left\|P_{X_{[S]}}^{\perp} X_{S^c}H_{S^c}\right\|^2 = \left\|X_{S^c}H_{S^c}\right\|^2 - \left\|P_{X_{[S]}}X_{S^c}H_{S^c}\right\|^2 \\
&\geq nP\left\|H_{S^c}\right\|^2\left(\left(1-\sqrt{\frac{K_a}{n}}-\gamma_1\right)^2 - \frac{\gamma^2 t(K_a-t)}{\left(1-\sqrt{\frac{K_a}{n}}-\gamma_1\right)^2}\right) \\
&= h(n, K_a-t, K_a, \gamma, \gamma_1)nP\left\|H_{S^c}\right\|^2.
\end{aligned}$$

So if $\gamma < \frac{\left(1-\sqrt{\frac{K_a}{n}}-\gamma_1\right)^2}{\sqrt{t(K_a-t)}}$ then this lower bound is minimized at the set S which corresponds to the t highest absolute values of the vector H.

Therefore we have,

$$\begin{aligned}
\mathbb{P}[F_t] \lessapprox \inf_{\delta>0} \Bigg[& e^{R_2-(n-K_a)\delta} + \mathbb{P}[B_1^c] + \mathbb{P}[B_2^c] \\
&+\mathbb{P}\left[\frac{\left\|Z_3\right\|^2}{\left\|Z_3\right\|^2 + h(n,t,K_a,\gamma,\gamma_1)nP\left\|H_{S_0^*}\right\|^2} \geq 1-\epsilon\right]\Bigg] \qquad (6.4)
\end{aligned}$$

where $\gamma < \frac{\left(1-\sqrt{\frac{K_a}{n}}-\gamma_1\right)^2}{\sqrt{t(K_a-t)}}$ and S_0^* is the set corresponding to t smallest fading coefficients (i.e. smallest absolute values of H). Further we have from (4.36) and [12, Theorem 9.26]

$$\mathbb{P}\left[B_1^c\right] \leq \binom{Ka}{2} 4e^{-n\gamma^2/2} \text{ from eq.(4.36)}$$
$$\mathbb{P}\left[B_2^c\right] \lessapprox e^{-n\gamma_1^2/2}$$

(the second inequality is approximate since the matrix we have is not truly Gaussian but is very well approximated by a Gaussian matrix).

Hence, for a good choice of γ and γ_1, we can perform a Monte-Carlo simulation of (3.12) as described in the previous sub-section.

But note that for large K_a, there may not exist a good γ such that the probability of error is less than our target.

6.2 Plots

In this section, we present the plots of E_b/N_0 vs K_a with $n = 30000$, $k = 100$ bits and $\epsilon = 0.1$. In fig. 6-1, we have plotted the minimum E_b/N_0 required for $\epsilon \leq 0.1$ as a function of the number of active users K_a for the FBL bound approximations, the union bound, converse bound, T-fold ALOHA and the Shamai-Bettesh asymptotic bound [5]. We have also plotted the result of using 4-fold ALOHA on the LDPC code developed in [17]. In fig. 6-2, we plot the probability of error (per-user) as a function of SNR(dB) with parameters $n = 300$, $k = 100$ and $K_a = 2$ for our FBL bound, treat interference as noise (TIN), TIN with SIC and the joint asymtotic bound from [5], and the $(300, 100)$ LDPC code from [17]. It is interesting and intriguing to note that even at that short blocklength of $n = 300$, the LDPC code performs pretty well compared to both the FBL bound and the asymptotic bound. The reason is probably due to the fact that (as shown in [36]) in quasi-static scenario finite-blocklength performance is not very sensitive to quality of the code, since the probability of error predominantly governed by the fading coefficient realization.

We can observe from fig. 6-2 that the FBL bound is very close to the Shamai-Bettesh asymptotic bound in terms of probability of error vs SNR(dB). Further from fig. 6-1, we see that the approximation to the FBL bound is far off from the Shamai-Bettesh asymptotic bound for large K_a but using 4-fold ALOHA, we can come quite close to it. Therefore the projection decoder doesn't give good performance when tried to decode all users. Intuitively, this is what we can expect: it is quite difficult the user with the smallest fading coefficient since the expectation of the smallest of $\{|H_i|^2 : 1 \leq i \leq K_a\}$ is $\frac{1}{K_a}$.

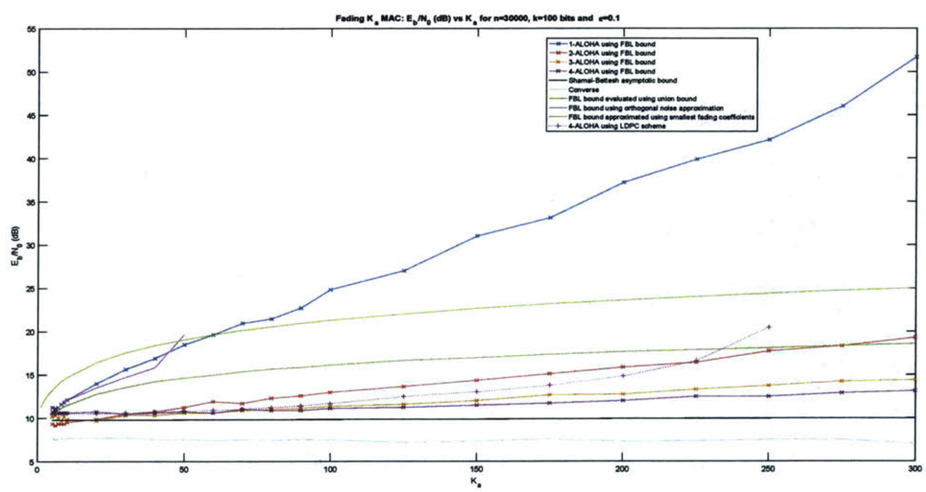

Figure 6-1: E_b/N_0 vs K_a for $\epsilon \leq 0.1$, $n = 30000$, $k = 100$ bits

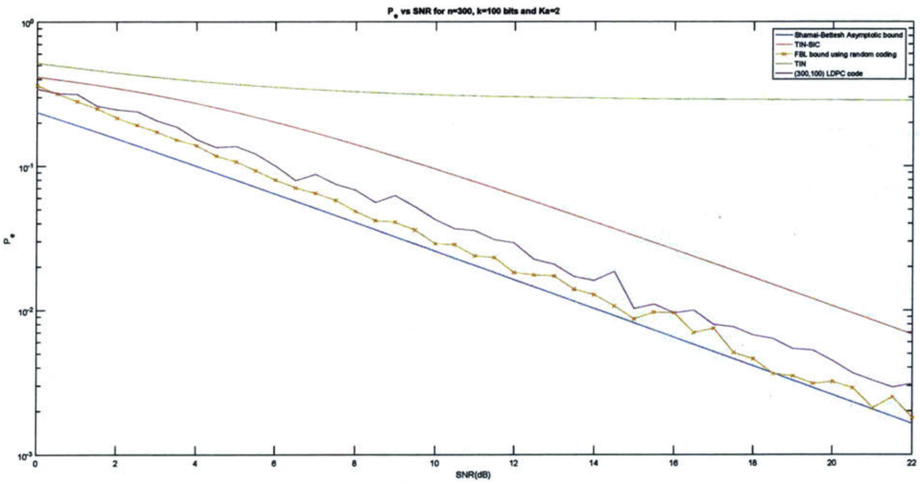

Figure 6-2: Probability of error (per-user) P_e vs SNR for $n = 300$, $k = 100$ bits and $K_a = 2$

Chapter 7

Conclusion

In this thesis, motivated by massive machine type communications (mMTC) and internet of things (IoT), we considered the problem of energy-efficient random access for a quasi-static Rayleigh fading model. Using tools from finite-blocklength information theory, random coding and a subspace projection based decoder we developed upper bounds for the probability of error per user for our model. We used this bound along with T–fold ALOHA to discuss the trade-off between the number of active users and the minimum energy-per-bit to achieve a desired probability of error (per-user). Further we provided some approximations to evaluate the bound for moderately large number of users. We demonstrated that attempting to decode all active users is not a good idea, and showed that T-fold ALOHA method with random coding achieves a much better trade-off and it is off from the asymptotic bound of [5] by at most $4dB$ even at $K_a = 300$. We also developed a simple converse bound for our model by generalizing the meta converse of [28] to list-decoding. In terms of the rationale behind using the projection decoder, we proved that it achieves the ϵ-capacity of the K-user quasi static Rayleigh fading MAC under the classical probability of error, and also achieves the the asymptotic bound from [5] under per-user error.

In terms of future work, as mentioned in the previous paragraph, the 4-fold ALOHA bound is pretty close to the asymptotic bound. But in the AWGN setup, there is a significant gap between 4-fold ALOHA and the random coding bounds [24, 33]. This could be because of the lack of power control at the transmitter. Since in a practical situation the transmitters estimate the channel and adjust their powers accordingly, the effective fading is just over phase. So it will be interesting to see how the trade-off changes under uniform phase fading.

Further, resolving higher order collisions in coded slotted ALOHA (CSA) to get a coded-slotted version of T-fold ALOHA might seem to give improvements. But already the 4-fold ALOHA is close to the asymptotic bound. For higher values of T, we might get even closer trade-offs. So it is not clear if the coded-slotted version would lead to significant improvements since we also incur a penalty on power by repeating packets but it remains to be seen. Finally, we have assumed a quasi-static model of fading, but in reality, the channel conditions might vary over the blocklength. So analyzing a *block-fading* version of model seems like an interesting direction.

Appendix A

Auxiliary results

We have the following concentration result for non-central chi-squared distribution from [7, Lemma 8.1] which we use extensively.

Lemma A.0.1 ([7]). *Let χ be a non-central chi-squared distributed variable with d degrees of freedom and non-centrality parameter λ. Then $\forall x > 0$*

$$\begin{aligned} \mathbb{P}\left[\chi - (d+\lambda) \geq 2\sqrt{(d+2\lambda)x} + 2x\right] &\leq e^{-x} \\ \mathbb{P}\left[\chi - (d+\lambda) \leq -2\sqrt{(d+2\lambda)x}\right] &\leq e^{-x} \end{aligned} \tag{A.1}$$

Hence, for $x > 0$, we have

$$\mathbb{P}\left[\chi - (d+\lambda) \geq x\right] \leq e^{-\frac{1}{2}\left(x+d+2\lambda-\sqrt{d+2\lambda}\sqrt{2x+d+2\lambda}\right)}. \tag{A.2}$$

and for $x < (d+\lambda)$, we have

$$\mathbb{P}\left[\chi \leq x\right] \leq e^{-\frac{1}{4}\frac{(d+\lambda-x)^2}{d+2\lambda}}. \tag{A.3}$$

Observe that, in (A.2), the exponent is always negative for $x > 0$ and finite λ due to AM-GM inequality

Appendix B

Proofs of certain claims

Proof of Claim 4. We have $Y = \sum_{i\in[K]} H_i c_i + Z$. Further $Y = P_{c[[K]]}Y + P^{\perp}_{c[[K]]}Y$. But $P^{\perp}_{c[[K]]}Y = P^{\perp}_{c[[K]]}Z$. Hence $\|Y\|^2 - \left\|P_{c[[K]]}Y\right\|^2 = \left\|P^{\perp}_{c[[K]]}Z\right\|^2 = \|Z\|^2 - \left\|P_{c[[K]]}Z\right\|^2 \leq \|Z\|^2 - \left\|P_{c[S^c]}Z\right\|^2 = \left\|P^{\perp}_{c[S^c]}Z\right\|^2$.

Also $\|Y\|^2 - \left\|P_{c[S^c]}Y\right\|^2 = \left\|P^{\perp}_{c[S^c]}Y\right\|^2 = \left\|P^{\perp}_{c[S^c]}\sum_{i\in S} H_i c_i + P^{\perp}_{c[S^c]}Z\right\|^2$. Hence we have

$$
\begin{aligned}
&\mathbb{P}\left[\frac{\|Y\|^2 - \left\|P_{c[[K]]}Y\right\|^2}{\|Y\|^2 - \left\|P_{c[S^c]}Y\right\|^2} \geq \gamma_n\right] \\
&= \mathbb{P}\left[\|Z\|^2 - \left\|P_{c[[K]]}Z\right\|^2 \geq \gamma_n \left\|P^{\perp}_{c[S^c]}\sum_{i\in S} H_i c_i + P^{\perp}_{c[S^c]}Z\right\|^2\right] \\
&\leq \mathbb{P}\left[\left\|P^{\perp}_{c[S^c]}Z\right\|^2 \geq \gamma_n \left\|P^{\perp}_{c[S^c]}\sum_{i\in S} H_i c_i + P^{\perp}_{c[S^c]}Z\right\|^2\right] \\
&= \mathbb{P}\left[(1-\gamma_n)\left\|P^{\perp}_{c[S^c]}Z\right\|^2 - 2\gamma_n Re\left\langle P^{\perp}_{c[S^c]}Z, P^{\perp}_{c[S^c]}\sum_{i\in S} H_i c_i\right\rangle \geq \gamma_n \left\|P^{\perp}_{c[S^c]}\sum_{i\in S} H_i c_i\right\|^2\right] \\
&= \mathbb{P}\left[(1-\gamma_n)^2\left\|P^{\perp}_{c[S^c]}Z\right\|^2 - 2\gamma_n(1-\gamma_n) Re\left\langle P^{\perp}_{c[S^c]}Z, P^{\perp}_{c[S^c]}\sum_{i\in S} H_i c_i\right\rangle\right. \\
&\qquad \left.\geq \gamma_n(1-\gamma_n)\left\|P^{\perp}_{c[S^c]}\sum_{i\in S} H_i c_i\right\|^2\right]
\end{aligned}
$$

$$= \mathbb{P}\left[\left\|(1-\gamma_n)P^{\perp}_{c_{[S^c]}}Z - \gamma_n P^{\perp}_{c_{[S^c]}}\sum_{i\in S} H_i c_i\right\|^2 \geq \gamma_n \left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S} H_i c_i\right\|^2\right]$$

$$= \mathbb{P}\left[\left\|P^{\perp}_{c_{[S^c]}}Z - \frac{\gamma_n}{(1-\gamma_n)} P^{\perp}_{c_{[S^c]}}\sum_{i\in S} H_i c_i\right\|^2 \geq \frac{\gamma_n}{(1-\gamma_n)^2} \left\|P^{\perp}_{c_{[S^c]}}\sum_{i\in S} H_i c_i\right\|^2\right]. \tag{B.1}$$

□

Proof of Lemma 4.2.2. First of all, rank of $P^{\perp}_{c_{[S^c]}}$ is $n-K+t$ because the vectors in $c_{[S^c]}$ are linearly independent almost surely. Let U be a unitary change of basis matrix that rotates the range space of $P^{\perp}_{c_{[S^c]}}$ to the space corresponding to first $(n-K+t)$ coordinates. Then

$$\begin{aligned}
&\left\|\mathcal{CN}(-\frac{\gamma_n}{1-\gamma_n} P^{\perp}_{c_{[S^c]}}\sum_{i\in S} H_i c_i, P^{\perp}_{c_{[S^c]}})\right\|^2 \\
&= \left\|U\left(\mathcal{CN}(-\frac{\gamma_n}{1-\gamma_n} P^{\perp}_{c_{[S^c]}}\sum_{i\in S} H_i c_i, P^{\perp}_{c_{[S^c]}})\right)\right\|^2 \\
&= \left\|\mathcal{CN}(-\frac{\gamma_n}{1-\gamma_n} U P^{\perp}_{c_{[S^c]}}\sum_{i\in S} H_i c_i, U P^{\perp}_{c_{[S^c]}} U^*)\right\|^2.
\end{aligned} \tag{B.2}$$

Now $UP^{\perp}_{c_{[S^c]}}U^*$ is a diagonal matrix with first $(n-K+t)$ diagonal entries being ones and rest all 0. The definition of non-cental chi-squared distribution $\chi_2'(\lambda, d)$ is the sum of d squares of independent Gaussians with sum of squares of their means being λ. Using this and (4.18) we have proved the lemma. □

Proof of Claim 5. We have

$$\begin{aligned}
f_n(x) &= x + 1 + \frac{2\gamma_n}{1-\gamma_n}(1+x) \\
&\quad - \sqrt{1 + \frac{2\gamma_n}{1-\gamma_n}(1+x)}\sqrt{2x + 1 + \frac{2\gamma_n}{1-\gamma_n}(1+x)} \\
&= \frac{1}{1-\gamma_n}\left[(1+\gamma_n)(x+1)) - 2\sqrt{\gamma_n}\sqrt{\left(x + \frac{(1+\gamma_n)^2}{4\gamma_n}\right)^2 - \frac{(1-\gamma_n^2)^2}{16\gamma_n^2}}\right]
\end{aligned} \tag{B.3}$$

Hence

$$f'(x) = \frac{1}{1-\gamma_n}\left[1+\gamma_n - 2\sqrt{\gamma_n}\frac{a}{\sqrt{a^2-b^2}}\right]$$
$$= \frac{1}{1-\gamma_n}\left(\sqrt{\gamma_n} - \sqrt{\frac{a+b}{a-b}}\right)\left(\sqrt{\gamma_n} - \sqrt{\frac{a-b}{a+b}}\right) \tag{B.4}$$

where $a = \left(x + \frac{(1+\gamma_n)^2}{4\gamma_n}\right)$ and $b = \frac{1-\gamma_n^2}{4\gamma_n}$. Also $a > 0$ and $b > 0$. Further $a + b > a - b$ and

$$\sqrt{\gamma_n} < \sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{\gamma_n(1+\gamma_n+2x)}{1+\gamma_n+2\gamma_n x}}$$
$$\iff 2\gamma_n x + 1 + \gamma_n < 2x + 1 + \gamma_n$$
$$\iff 0 < \gamma_n < 1$$

which is true. Hence both the factors in (B.4) are negative. Therefore $f'(x) > 0$. $\square$

Bibliography

[1] 3GPP R1-164688. RSMA. Technical report, Qualcomm Inc.

[2] 3GPP R1-164689. RSMA. Technical report, Qualcomm Inc.

[3] Md Shipon Ali, Ekram Hossain, and Dong In Kim. Lte/lte-a random access for massive machine-type communications in smart cities. *IEEE Communications Magazine*, 55(1):76–83, 2017.

[4] Amin Azari, Guowang Miao, Cedomir Stefanovic, and Petar Popovski. Grant-free radio access for short-packet communications over 5g networks. In *IEEE Globecom 2017*, 2017.

[5] I Bettesh and S Shamai. Outages, expected rates and delays in multiple-users fading channels. In *Proceedings of the 2000 Conference on Information Science and Systems*, volume 1, 2000.

[6] Ezio Biglieri, John Proakis, and Shlomo Shamai. Fading channels: Information-theoretic and communications aspects. *IEEE Transactions on Information Theory*, 44(6):2619–2692, 1998.

[7] Lucien Birgé. An alternative point of view on lepski's method. *Lecture Notes-Monograph Series*, pages 113–133, 2001.

[8] Federico Boccardi, Robert W Heath, Angel Lozano, Thomas L Marzetta, and Petar Popovski. Five disruptive technology directions for 5g. *IEEE Communications Magazine*, 52(2):74–80, 2014.

[9] Enrico Casini, Riccardo De Gaudenzi, and Oscar Del Rio Herrero. Contention resolution diversity slotted aloha (crdsa): An enhanced random access scheme for satellite access packet networks. *IEEE Transactions on Wireless Communications*, 6(4), 2007.

[10] A Collins, G Durisi, T Erseghe, V Kostina, J Östman, Y Polyanskiy, I Tal, and Wei Yang. Spectre toolbox. `https://github.com/yp-mit/spectre`, 2016.

[11] Riccardo De Gaudenzi, Oscar del Rio Herrero, Guray Acar, and Eloi Garrido Barrabes. Asynchronous contention resolution diversity aloha: Making crdsa truly asynchronous. *IEEE Transactions on Wireless Communications*, 13(11):6193–6206, 2014.

[12] Simon Foucart and Holger Rauhut. *A mathematical introduction to compressive sensing*, volume 1. Birkhäuser Basel, 2013.

[13] Robert G. Gallager. *Low-Density Parity-Check Codes.* 1963.

[14] Sylvie Ghez, Sergio Verdu, and Stuart C Schwartz. Stability properties of slotted Aloha with multipacket reception capability. 33(7):640–649, 1988.

[15] Te Sun Han. An information-spectrum approach to capacity theorems for the general multiple-access channel. *IEEE Transactions on Information Theory*, 44(7):2773–2795, 1998.

[16] Reza Hoshyar, Ferry P Wathan, and Rahim Tafazolli. Novel low-density signature for synchronous cdma systems over awgn channel. *IEEE Transactions on Signal Processing*, 56(4):1616–1626, 2008.

[17] S Kowshik, A Frolov, and Y Polyanskiy. Energy-efficient coded random access for the wireless uplink. Submitted to IEEE Globecom 2018, 2018.

[18] Frank R Kschischang, Brendan J Frey, and H-A Loeliger. Factor graphs and the sum-product algorithm. *IEEE Transactions on information theory*, 47(2):498–519, 2001.

[19] Michel Ledoux. *The concentration of measure phenomenon.* Number 89. American Mathematical Soc., 2001.

[20] Gianluigi Liva. Graph-based analysis and optimization of contention resolution diversity slotted aloha. *IEEE Transactions on Communications*, 59(2):477–487, 2011.

[21] Luiza Medova, Anton Glebov, Pavel Rybin, and Alexey Frolov. On ldpc code based massive random-access scheme for the gaussian multiple access channel. *arXiv preprint arXiv:1803.08377*, 2018.

[22] Guowang Miao, Amin Azari, and Taewon Hwang. E^2-mac: Energy efficient medium access for massive m2m communications. *IEEE Transactions on communications*, 64(11):4720–4735, 2016.

[23] Hosein Nikopour and Hadi Baligh. Sparse code multiple access. In *Personal Indoor and Mobile Radio Communications (PIMRC), 2013 IEEE 24th International Symposium on*, pages 332–336. IEEE, 2013.

[24] Or Ordentlich and Yury Polyanskiy. Low complexity schemes for the random access gaussian channel. In *Information Theory (ISIT), 2017 IEEE International Symposium on*, pages 2528–2532. IEEE, 2017.

[25] Maria Rita Palattella, Mischa Dohler, Alfredo Grieco, Gianluca Rizzo, Johan Torsner, Thomas Engel, and Latif Ladid. Internet of things in the 5g era: Enablers, architecture, and business models. *IEEE Journal on Selected Areas in Communications*, 34(3):510–527, 2016.

[26] Yury Polyanskiy. *Channel coding: non-asymptotic fundamental limits*. Princeton University, 2010.

[27] Yury Polyanskiy. A perspective on massive random-access. In *Information Theory (ISIT), 2017 IEEE International Symposium on*, pages 2523–2527. IEEE, 2017.

[28] Yury Polyanskiy, H Vincent Poor, and Sergio Verdú. Channel coding rate in the finite blocklength regime. *IEEE Transactions on Information Theory*, 56(5):2307–2359, 2010.

[29] Tom Richardson and Ruediger Urbanke. *Modern coding theory*. Cambridge university press, 2008.

[30] Lawrence G Roberts. Aloha packet system with and without slots and capture. *ACM SIGCOMM Computer Communication Review*, 5(2):28–42, 1975.

[31] R Tanner. A recursive approach to low complexity codes. *IEEE Transactions on information theory*, 27(5):533–547, 1981.

[32] Monirosharieh Vameghestahbanati, Ian Marsland, Ramy H Gohary, and Halim Yanikomeroglu. Multidimensional constellations for uplink scma systems—a comparative study. *arXiv preprint arXiv:1804.05814*, 2018.

[33] Avinash Vem, Krishna R Narayanan, Jun Cheng, and Jean-Francois Chamberland. A user-independent serial interference cancellation based coding scheme for the unsourced random access gaussian channel. In *Information Theory Workshop (ITW), 2017 IEEE*, pages 121–125. IEEE, 2017.

[34] Roman Vershynin. High dimensional probability, 2016.

[35] Fan Wei and Wen Chen. Low complexity iterative receiver design for sparse code multiple access. *IEEE Transactions on Communications*, 65(2):621–634, 2017.

[36] Wei Yang, Giuseppe Durisi, Tobias Koch, and Yury Polyanskiy. Quasi-static MIMO fading channels at finite blocklength. *CoRR*, abs/1311.2012, 2013.

[37] Wei Yang, Giuseppe Durisi, Tobias Koch, and Yury Polyanskiy. Quasi-static simo fading channels at finite blocklength. In *Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on*, pages 1531–1535. IEEE, 2013.

[38] Yifei Yuan, Zhifeng Yuan, Guanghui Yu, Chien-hwa Hwang, Pei-kai Liao, Anxin Li, and Kazuaki Takeda. Non-orthogonal transmission technology in lte evolution. *IEEE Communications Magazine*, 54(7):68–74, 2016.