Sampling of the Wiener Process for Remote Estimation over a Channel with Random Delay

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Abstract

In this paper, we consider a problem of sampling a Wiener process, with samples forwarded to a remote estimator over a channel that is modeled as a queue. The estimator reconstructs an estimate of the *real-time* signal value from causally received samples. We study the optimal *online* sampling strategy that minimizes the mean square estimation error subject to a sampling rate constraint. We prove that the optimal sampling strategy is a threshold policy, and find the optimal threshold. This threshold is determined by how much the Wiener process varies during the random service time and the maximum allowed sampling rate. Further, if the sampling times are independent of the observed Wiener process, the optimal sampling problem reduces to an age of information optimization problem that has been recently solved. Our comparisons show that the estimation error of the optimal sampling policy can be much smaller than those of age-optimal sampling, zero-wait sampling, and classic periodic sampling.

Index Terms

Sampling, remote estimation, age of information, Wiener process, queueing system.

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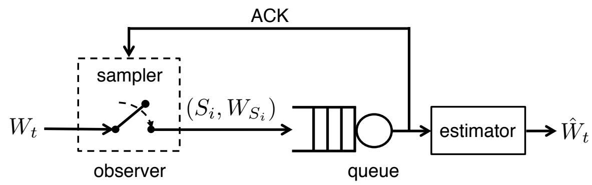


Fig. 1: System model.

I. Introduction

In many networked control and monitoring systems (e.g., airplane/vehicular control, smart grid, stock trading, robotics, etc.), timely updates of the system state are critical for making decisions. Recently, the *age of information*, or simply the *age*, has been proposed as a metric for characterizing the timeliness of information updates. Suppose that the i-th update is generated at time S_i and delivered at time D_i. At time t, the freshest update delivered to the destination was generated at time $U(t) = \max\{S_i : D_i \leq t\}$. The age $\Delta(t)$ at time t is defined as [1], [2]

$$\Delta(t) = t - U(t) = t - \max\{S_i : D_i \leq t\}, \tag{1}$$

which is the time difference between the generation time $U(t)$ of the freshest received update and the current time t.

The age of information, as well as the more general non-linear age penalty models in [3]–[6], are useful for measuring the timeliness of *message updates*, such as news, fire alarm, email notifications, and social updates. However, the state of many systems is in the form of a continuous-time signal s_t, such as the orientation and location of a vehicle, the wind speed of a hurricane, and the price chart of a stock. These signals may change slowly at some time and vary more dynamically later. Hence, the time difference between the source and destination, described by the age $\Delta(t) = t - U(t)$, cannot fully determine the amount of change $s_t - s_{U(t)}$ in the signal value. This motivated us to investigate timely updates of *signal samples* and try to understand the connection between the age of information concept and online signal sampling and reconstruction.

Let us consider a status update system with two terminals (see Fig. 1): An observer measuring a continuous-time signal that is modeled as a Wiener process W_t,[1] and an estimator, whose goal is to provide the best-guess $\hat{W}_t$ for the real-time signal value W_t at all time t. These two terminals are connected by a channel that transmits time-stamped samples of the form (S_i, W_{S_i}), where the sampling

[1]Other signal models will be considered in our future work.

times S_i satisfy $0 \leq S_1 \leq S_2 \leq \ldots$ The channel is modeled as a FIFO queue with random *i.i.d.* service time Y_i, where $Y_i \geq 0$ represents the transmission time of sample i from the observer to the estimator.

Unless it arrives at an empty system, sample i needs to wait in the queue until its service starts. Let G_i be the service starting time of sample i such that $S_i \leq G_i$. The delivery time of sample i is $D_i = G_i + Y_i$. The initial value $W_0 = 0$ is known by the estimator for free, which is represented by $S_0 = D_0 = 0$. At any time t, the estimator forms an estimate $\hat{W}_t$ using the samples received up to time t, i.e., $\{(S_i, W_{S_i}) : D_i \leq t\}$. Similar with [7], we assume that the estimator neglects the implied knowledge when no sample was delivered. The quality of remote estimation via the time-average mean-square error (MSE) between W_t and $\hat{W}_t$:

$$\mathsf{mse} = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (W_t - \hat{W}_t)^2 dt\right]. \tag{2}$$

Our goal is to find the optimal sampling strategy that minimizes mse by choosing the sampling times S_i *causally* subject to a sampling rate constraint

$$\liminf_{n\to\infty} \frac{1}{n}\mathbb{E}[S_n] \geq \frac{1}{f_{\max}}, \tag{3}$$

where $f_{\max}$ is the maximum allowed sampling rate. In practice, the sampling rate constraint (3) is imposed when there is a need to reduce the cost (e.g., energy consumption) for the transmission, storage, and processing of the samples. Later on in the paper, the unconstrained problem with $f_{\max} = \infty$ will also be solved.

A. Contributions

The contributions of this paper are summarized as follows:

- We formulate the optimal sampling problem as a constrained continuous-time Markov decision problem with a continuous state space, and solve it exactly. We prove that the optimal sampling strategy is a threshold policy, and find the optimal threshold. Let Y be a random variable with the same distribution as Y_i. The optimal threshold is determined by $f_{\max}$ and W_Y, where W_Y is a random variable that has the same distribution with the amount of signal variation $(W_{t+Y} - W_t)$ during the random service time Y. The random variable W_Y illustrates a tight *coupling* between the source process W_t and the service time Y in the optimal sampling policy.
- Our threshold-based optimal sampling policy has an important difference from the previous threshold-based sampling policies studied in, e.g., [8]–[27]: We have proven that, because of the queueing system model and the strong Markov property of the Wiener process, it is suboptimal to take a new sample when the server is busy. Consequently, the threshold should be *disabled* when the server is

busy and *reactivated* once the server becomes available again. On the other hand, sampling policies that ignore the idle/busy state of the server, such as periodic sampling, can have a large estimation error.

- In the absence of the sampling rate constraint (i.e., $f_{\max} = \infty$), the optimal sampling strategy is *not* zero-wait sampling in which a new sample is generated once the previous sample is delivered; rather, it is optimal to wait for a certain amount of time after the previous sample is delivered, and then take the next sample.
- If the sampling times S_i are independent of the Wiener process (i.e., the sampling times are chosen without using information about the source process W_t), the optimal sampling problem reduces to an age of information optimization problem that has been solved recently [3], [4]. The asymptotics of the MSE-optimal and age-optimal sampling policies at long/short service time or low/high sampling rates are also studied.
- Our theoretical and numerical comparisons show that the MSE of the optimal sampling policy can be much smaller than those of age-optimal sampling, periodic sampling, and the zero-wait sampling policy described in (7) below. In particular, periodic sampling is far from optimal if the sampling rate is low or high; age-optimal sampling is far from optimal if the sampling rate is low; periodic sampling, age-optimal sampling, and zero-wait sampling policies are all far from optimal if the service times are highly random.

B. Paper Structure

The rest of this paper is organized as follows. In Section II, we discuss some related work. In Section III, we describe the system model and the formulation of the optimal sampling and remote estimation problem. In Section IV, we present the optimal sampling and estimation solution to this problem and compare it with some other sampling policies. In Section V, we describe the proof of this optimal solution. Some simulation results are provided in Section VI.

II. Related Work

A. Sampling and Source Coding of the Wiener Process

In [29], Berger calculated the rate-distortion function of the Wiener process. In source coding theory, rate-distortion function represents the optimal tradeoff between the bitrate of source coding and the distortion (e.g., MSE) in the recovery of the process from its coded version. To achieve the rate-distortion function, Berger used Karhunen-Loève (KL) transform to map each realization of the Wiener process over a long time interval to a sequence of discrete coefficients, and then applied the optimal source coding to encode the KL coefficients. Based on this, a lossy source coding theorem was established in [29].

In [30], Kipnis et. al. considered a source coding problem, and derived the minimal distortion in recovering the continuous-time Wiener process from a coded version of its periodic samples. In this setting, the authors showed that the rate-distortion function is attained in three steps: First, obtain the MMSE estimate of the Wiener process from its periodic samples, which is given by linear interpolation between neighboring samples. The obtained estimate is a continuous-time stochastic process. Second, compute the KL coefficients of the continuous-time estimate process. Third, apply the optimal source coding to encode the KL coefficients. The KL transform in [29], [30] require batch processing of the realization of the Wiener process during a sufficiently long time interval. Hence, the source coding schemes therein have a long encoding delay and can be considered as offline solutions for signal reconstructions. In recent years, non-asymptotic source coding schemes were investigated in, e.g., [31]–[34], where the source is reconstructed within a short delay. These studies are related to our work; one difference is that our goal is to estimate the real-time value of the source, instead of reconstructing the source that was encoded a short time ago.

B. Age of Information Optimization

The results in this paper are closely related to recent studies on the age of information in, e.g., [2]–[6], [35]–[43]. In [35], [36], Yates and Kaul provided a simple example about an status updating system, where samples of a Wiener process W_t are forwarded to a remote estimator: At any time t, if the freshest sample delivered to the estimator was generated at time $U(t)$, then the age of the delivered samples is $\Delta(t) = t - U(t)$. Furthermore, the MMSE estimate of W_t is $\hat{W}_t = W_{t-\Delta(t)}$ and the variance of this estimator is $\mathbb{E}[(W_t - \hat{W}_t)^2] = \Delta(t)$. Similarly, in Section IV-A, we use the strong Markov property of the Wiener process to show that if the sampling times $(S_0, S_1, \ldots)$ are determined without using any knowledge of the observed Wiener process $\{W_t, t \geq 0\}$, then minimizing the MSE $\limsup_{T\rightarrow\infty} \frac{1}{T}\mathbb{E}[\int_0^T (W_t - \hat{W}_t)^2 dt]$ is equivalent to minimizing the time-average age $\limsup_{T\rightarrow\infty} \frac{1}{T}\mathbb{E}[\int_0^T \Delta(t) dt]$.

However, if the sampler has access to the history observations of W_t, it can use this knowledge to find better sampling times and achieve a smaller MSE than age of information optimization. Hence, the age of information optimization problem solved in [3], [4] is a degenerated case of the MSE minimization problem solved in this paper. One observation of this work is that the behavior of the optimal sampling policy for minimizing the MSE is quite different from the age-optimal sampling policy obtained in [3], [4] (see Section IV): In the MSE-optimal sampling policy, a new sample is taken when *the difference* *between the observed signal values* at the two terminals has reached a threshold; while in the age-optimal policy, a new sample is taken when the *time difference* between the two terminals has reached a threshold.

C. Remote Estimation

The paper can also be considered as a contribution to the rich literature on remote estimation, e.g., [8]–[27], [44], [45], by adding a single-server queue between the sampler and estimator. In [8], Åström and Bernhardsson considered an infinite-horizon state estimation problem in first-order stochastic systems with instantaneously received samples. They provided the first analysis showing that a threshold-based sampling method can achieve a smaller estimation error than the classic periodic sampling method.

In [9], [10], Hajek et. al. investigated the joint optimization of paging and registration policies in cellular networks, which is equivalent with a joint sampling and estimation optimization problem with an indicator-type cost function and an infinite time horizon. They used majorization theory and Riesz's rearrangement inequality to show that, if the state process is modeled as a symmetric or Gaussian random walk, a threshold-based sampler and a nearest distance estimator are jointly optimal. In [11], [12], Lipsa and Martins considered remote state estimation in first-order linear time-invariant (LTI) discrete time systems with a quadratic cost function and finite time horizon. They showed that a time-dependent threshold-based sampler and Kalman-like estimator are jointly optimal. In [13], Nayyar et. al. considered a remote estimation problem with an energy-harvesting sensor and a remote estimator, where the sampling decision at the sensor is constrained by the energy level of the battery. They proved that an energy-level dependent threshold-based sampler and a Kalman-like estimator are jointly optimal. The proof techniques in [11]–[13] are related to those developed in [9], [10]. In [14], the authors provided an alternative way to prove the main results of [11], [12]. In these studies, dynamic programming based iterative algorithms were used to find the optimal threshold. Recently, more efficient approaches are provided in [15] to find the optimal threshold.

In [16], Imer and Başar studied the optimal sampling and remote estimation of an *i.i.d.* state process over a finite time horizon. In this study, the sampler has a hard upper bound on the number of allowed samples (i.e., a hard constraint). It was shown that there exists a unique optimal sampling policy within the class of threshold-based sampling policies. In [17], the authors derived the exact MMSE estimator for a class of threshold-based sampling policies. The jointly optimal design was not proven in [16], [17].

There exist a few studies on the sampling and remote estimation of continuous-time processes. In [18], Rabi et. al. considered a continuous-time version of [16], again with a hard constraint on the number of samples. In [19], Nar and Başar studied the optimal sampling of multidimensional Wiener processes subject to a time-average sampling rate constraint (i.e., a soft constraint), where the samples are immediately available to the estimator (i.e., $Y_i = 0$). In both [18] and [19], it was shown that a threshold-based sampling policy is optimal among all causal sampling policies of the Wiener process, and the optimal threshold was obtained exactly. Following [9], [10] for discrete-time random walks, the authors of [18] conjectured that, when the state is a continuous-time linear diffusion process, the MMSE estimator under deterministic sampling is the optimal sampling policy.

In the studies mentioned above, it was assumed that the samples are transmitted from the sampler to the estimator over a perfect channel that is error and noise free. There exists some recent studies with explicit channel models. In [20]–[22], Gao et. al. considered optimal communication scheduling and remote estimation over an additive noise channel. Because of the noise, the transmitter needs to encode its message before transmission. In [20], [21], it was shown that if the transmission scheduling policy is threshold-based, then the optimal encoder and decoder are piecewise affine. In [22], it was shown that if (i) the encoder and decoder (i.e., estimator) are piecewise affine, and (ii) the transmission scheduler satisfies some technical assumption, the optimal transmission scheduling policy is threshold-based. Some extensions of this research were reported in [23]–[25]. In [26], [27], [44], Chakravorty and Mahajan considered optimal communication scheduling and remote estimation over a few channel models, where it was proved that a threshold-based transmitter and a Kalman-like estimator are jointly optimal. In [45], Mahajan and Teneketzis provided a dynamic programming based numerical method to find the optimal transmission and estimation strategies over a channel with a constant transmission time but not over a queue with random service time.

One key difference between this paper and previous studies on remote estimation, e.g., [8]–[27], [44], [45], is that the channel between the sampler and estimator is modeled as a queue with random service times. As we will see later, this queueing model affects the structure of the optimal sampler. Specifically, we prove that a threshold-based sampler is optimal, where the sampler needs to disable the threshold when there is a packet in service and reactivate the threshold after all previous packets are delivered. A novel proof procedure is developed to establish the optimality of the proposed sampler.

One interesting future direction is to generalize our results and show that the threshold-based sampler and the MMSE estimator are jointly optimal, which needs to employ the majorization arguments in [9]–[13], [26], [27], [44]. In fact, it was commented in [10, p. 619] that the joint optimality results in [10] can be generalized to the case of continuous-time processes.

III. System Model and Problem Formulation

A. Sampling Policies

Let $I_t \in \{0,1\}$ denote the idle/busy state of the server at time t. As shown in Fig. 1, the server state I_t is known by the sampler through acknowledgements (ACKs). We assume that once a sample is delivered to the estimator, an ACK is fed back to the sampler with zero delay. Hence, the information that is available to the sampler at time t can be expressed as $\{W_s, I_s : 0 \leq s \leq t\}$.

In online sampling policies, each sampling time S_i is chosen causally using the information available at the sampler. To characterize this statement precisely, we define the σ-fields

$$\mathcal{N}_t = \sigma(W_s, I_s : 0 \leq s \leq t),\ \mathcal{N}_t^+ = \cap_{s>t}\mathcal{N}_s.$$

Then, $\{\mathcal{N}_t^+, t \geq 0\}$ is a *filtration* (i.e., a non-decreasing and right-continuous family of σ-fields) of the information available at the sampler. Each sampling time S_i is a *stopping time* with respect to the filtration $\{\mathcal{N}_t^+, t \geq 0\}$, i.e.,

$$\{S_i \leq t\} \in \mathcal{N}_t^+,\ \forall t \geq 0. \tag{4}$$

Let $\pi = (S_0, S_1, \ldots)$ denote a sampling policy and let Π denote a set of *online* (or *causal*) sampling policies satisfying the following two conditions: (i) Each sampling policy $\pi \in \Pi$ satisfies (4) for all $i = 0, 1, \ldots$ (ii) The inter-sampling times $\{T_i = S_{i+1} - S_i, i = 0, 1, \ldots\}$ form a *regenerative process* [46, Section 6.1]: There exist integers $0 \leq k_1 < k_2 < \ldots$ such that the post-k_j process $\{T_{k_j+i}, i = 0, 1, \ldots\}$ has the same distribution as the post-k_1 process $\{T_{k_1+i}, i = 0, 1, \ldots\}$ and is independent of the pre-k_j process $\{T_i, i = 0, 1, \ldots, k_j - 1\}$; in addition, $\mathbb{E}[S_{k_1}^2] < \infty$ and $0 < \mathbb{E}[(S_{k_{j+1}} - S_{k_j})^2] < \infty$ for $j = 1, 2, \ldots$[2] By Condition (ii), we can obtain that, almost surely,

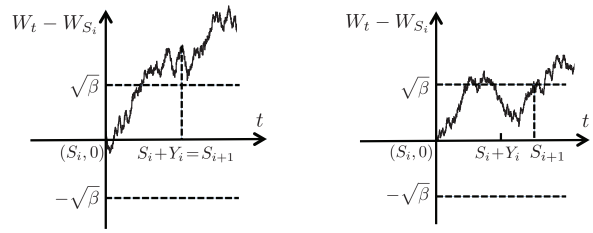
$$\lim_{i\to\infty} S_i = \infty,\ \ \lim_{i\to\infty} D_i = \infty. \tag{5}$$

Some examples of the sampling policies in Π are:

1. *Periodic sampling* [47], [48]: The inter-sampling times are constant, such that for some $\beta \geq 0$,

$$S_{i+1} = S_i + \beta. \tag{6}$$

[2] We assume that T_i is a regenerative process because we analyze the time-average MMSE in (11), but operationally a nicer definition is $\limsup_{n\to\infty} \mathbb{E}[\int_0^{D_n} (W_t - \hat{W}_t)^2 dt]/\mathbb{E}[D_n]$. These two definitions are equivalent when T_i is a regenerative process.



(a) If $|W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}$, sample $i+1$ is taken at time $S_{i+1} = S_i + Y_i$.

(b) If $|W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}$, sample $i+1$ is taken at time t that satisfies $t \geq S_i + Y_i$ and $|W_t - W_{S_i}| = \sqrt{\beta}$.

Fig. 2: Illustration of the threshold-based sampling policy (9), where no sample is taken during $[S_i, S_i + Y_i)$.

2. *Zero-wait sampling* [2]–[4], [37]: A new sample is generated once the previous sample is delivered, i.e.,

$$S_{i+1} = S_i + Y_i. \tag{7}$$

3. *Threshold policy in time difference* [3], [4], [37]: The sampling times are given by

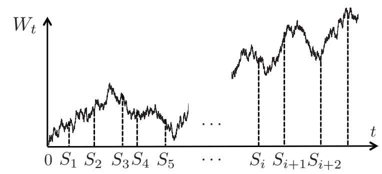
$$S_{i+1} = \inf \{t \geq S_i + Y_i : t - S_i \geq \beta\}. \tag{8}$$

4. *Threshold policy in signal difference*: The sampling times are given by

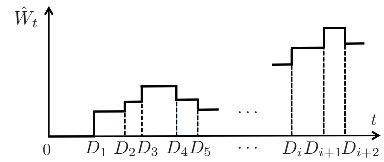
$$S_{i+1} = \inf \left\{t \geq S_i + Y_i : |W_t - W_{S_i}| \geq \sqrt{\beta}\right\}, \tag{9}$$

which can be understood as follows: As illustrated in Fig. 2, if $|W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}$, sample $i+1$ is generated at the time $S_{i+1} = S_i + Y_i$ when sample i is delivered; otherwise, if $|W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}$, sample $i+1$ is generated at the earliest time t such that $t \geq S_i + Y_i$ and $|W_t - W_{S_i}|$ reaches the threshold $\sqrt{\beta}$. It is worthwhile to emphasize that even if there exists time $t \in [S_i, S_i + Y_i)$ such that $|W_t - W_{S_i}| \geq \sqrt{\beta}$, no sample is taken at such time t, as depicted in both cases of Fig. 2. In other words, the threshold-based control is *disabled* during $[S_i, S_i + Y_i)$ and is *reactivated* at time $S_i + Y_i$.

A sampling policy $\pi \in \Pi$ is said to be *signal-ignorant* (*signal-aware*), if π is (not) independent of the Wiener process $\{W_t, t \geq 0\}$. The sampling policies (6), (7), and (8) are signal-ignorant, and the sampling policy (9) is signal-aware.



(a) Wiener process W_t and its samples.



(b) Estimate process $\hat{W}_t$ using causally received samples.

Fig. 3: Illustration of the MMSE estimation policy (10).

B. MMSE Estimation Policy

At time t, the information available to the estimator contains two part: (i) $M_t = \{(S_i, W_{S_i}, D_i) : D_i \leq t\}$, which contains the sampling time S_i, sample value W_{S_i}, and delivery time D_i of the samples delivered by time t and (ii) the facts that no sample has been received after the latest sample delivery time $\max\{D_i : D_i \leq t\}$. Similar with [7], we assume that the estimator neglects the implied knowledge when no sample was delivered. In this case, the minimum mean-square error (MMSE) estimation policy [49] is given by (see Appendix A for its derivation)

$$\begin{aligned}\hat{W}_t =& \mathbb{E}[W_t | M_t] \\ =& W_{S_i}, \text{ if } t \in [D_i, D_{i+1}), \; i = 0, 1, 2, \ldots,\end{aligned} \tag{10}$$

which is illustrated in Fig. 3(b).

C. Optimal Sampling Problem

We assume that the Wiener process $\{W_t, t \geq 0\}$ and the service times $\{Y_i, i = 1, 2, \ldots\}$ are determined by two external processes, which are *mutually independent* and do not change according to the sampling policy. In addition, we assume that the Y_i's are *i.i.d.* with $\mathbb{E}[Y_i^2] < \infty$. The optimal sampling problem for minimizing the MSE subject to a sampling rate constraint is formulated as

$$\mathsf{mse}_{\text{opt}} \triangleq \inf_{\pi \in \Pi} \; \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T (W_t - \hat{W}_t)^2 dt\right] \tag{11}$$

$$\text{s.t.} \;\; \liminf_{n \to \infty} \frac{1}{n} \mathbb{E}[S_n] \geq \frac{1}{f_{\max}}, \tag{12}$$

where $\mathsf{mse}_{\text{opt}}$ denotes the optimal value of (11).

IV. Main Result

Problem (11) is a constrained continuous-time Markov decision problem with a continuous state space. Such problems are often lack of closed-form solutions, however we were able to solve (11) exactly:

Theorem 1. *If the service times Y_i's are i.i.d. with $\mathbb{E}[Y_i^2] < \infty$, then there exists $\beta \geq 0$ such that the sampling policy* (9) *is an optimal solution of* (11)*, and the optimal β is determined by solving*[3]

$$\mathbb{E}[\max(\beta, W_Y^2)] = \max\left(\frac{1}{f_{\max}}, \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta}\right), \tag{13}$$

where Y is a random variable with the same distribution as Y_i. The optimal value of (11) *is then given by*

$$\mathsf{mse}_{\text{opt}} = \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{6\mathbb{E}[\max(\beta, W_Y^2)]} + \mathbb{E}[Y]. \tag{14}$$

Proof. See Section V. □

The sampling policy in (9) and (13) is called the "MSE-optimal" sampling policy. The equation (13) can be solved by using the bisection method or other one-dimensional search methods. Hence, Problem (11) does not suffer from the "curse of dimensionality" issue encountered in many Markov decision problems.

According to (13), the threshold $\sqrt{\beta}$ is determined by the maximum allowed sampling rate $f_{\max}$ and W_Y, where W_Y is a random variable that has the same distribution with the amount of signal variation $(W_{t+Y} - W_t)$ during the random service time Y. This illustrates a tight *coupling* between the source process W_t and the service time Y in the optimal sampling policy. Finally, we note that the optimal sampling policy in (9) and (13) is quite general in the sense that it holds for any service time distribution satisfying $\mathbb{E}[Y^2] < \infty$.

A. Signal-Ignorant Sampling and the Age of Information

Let $\Pi_{\text{signal-ignorant}} \subset \Pi$ denote the set of signal-ignorant sampling policies, defined as

$$\Pi_{\text{signal-ignorant}} = \{\pi \in \Pi : \pi \text{ is independent of } \{W_t, t \geq 0\}\}.$$

[3] If $\beta \to 0$, the last terms in (13) and (19) are determined by L'Hospital's rule.

In these policies, the sampling decisions depend only on the service time $\{Y_i, i = 1, 2, \ldots\}$ but not the source process $\{W_t, t \geq 0\}$. For each $\pi \in \Pi_{\text{signal-ignorant}}$, the objective function in (11) can be written as (see Appendix B for the proof)

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Delta(t)dt\right], \tag{15}$$

where

$$\Delta(t) = t - S_i,\ t \in [D_i, D_{i+1}),\ i = 0, 1, 2, \ldots, \tag{16}$$

is the *age of information* [2], that is, the time difference between the generation time of the freshest received sample and the current time t. In this case, (11) reduces to the following age of information optimization problem [3], [4]:[4]

$$\begin{aligned} \mathsf{mse}_{\text{age-opt}} \triangleq \inf_{\pi\in\Pi_{\text{signal-ignorant}}} &\ \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Delta(t)dt\right] \\ \text{s.t.} &\ \liminf_{n\to\infty} \frac{1}{n}\mathbb{E}[S_n] \geq \frac{1}{f_{\max}}, \end{aligned} \tag{17}$$

where $\mathsf{mse}_{\text{age-opt}}$ denotes the optimal value of (17). Because $\Pi_{\text{signal-ignorant}} \subset \Pi$,

$$\mathsf{mse}_{\text{opt}} \leq \mathsf{mse}_{\text{age-opt}}. \tag{18}$$

Theorem 2. *[3], [4] If the service times Y_i's are i.i.d. with $\mathbb{E}[Y_i^2] < \infty$, then there exists $\beta \geq 0$ such that the sampling policy* (8) *is an optimal solution of* (17)*, and the optimal β is determined by solving*

$$\mathbb{E}[\max(\beta, Y)] = \max\left(\frac{1}{f_{\max}}, \frac{\mathbb{E}[\max(\beta^2, Y^2)]}{2\beta}\right), \tag{19}$$

where Y is a random variable with the same distribution as Y_i. The optimal value of (17) *is then given by*

$$\mathsf{mse}_{\text{age-opt}} \triangleq \frac{\mathbb{E}[\max(\beta^2, Y^2)]}{2\mathbb{E}[\max(\beta, Y)]} + \mathbb{E}[Y]. \tag{20}$$

The sampling policy in (8) and (19) is referred to as the "age-optimal" sampling policy. In the following, the asymptotics of the MSE-optimal and age-optimal sampling policies at low/high service time or low/high sampling frequencies are studied.

[4]Problem (11) is significantly more challenging than (17), because in (11) the sampler needs to make decisions based on the signal process W_t. More powerful techniques than those in [3], [4] are developed in Section V to solve (11).

B. Short Service Time or Low Sampling Rate

Let

$$Y_i = \alpha X_i \tag{21}$$

represent the scaling of the service time Y_i with α, where $\alpha \geq 0$ and the X_i's are *i.i.d.* positive random variables. If $\alpha \to 0$ or $f_{\max} \to 0$, we can obtain from (13) that (see Appendix C for the proof)

$$\beta = \frac{1}{f_{\max}} + o\left(\frac{1}{f_{\max}}\right), \tag{22}$$

where $f(x) = o(g(x))$ as $x \to a$ means that $\lim_{x\to a} f(x)/g(x) = 0$. In this case, the MSE-optimal sampling policy in (9) and (13) becomes

$$S_{i+1} = \inf\left\{t \geq S_i : |W_t - W_{S_i}| \geq \sqrt{\frac{1}{f_{\max}}}\right\}, \tag{23}$$

and as shown in Appendix C, the optimal value of (11) becomes

$$\mathsf{mse}_{\text{opt}} = \frac{1}{6f_{\max}} + o\left(\frac{1}{f_{\max}}\right). \tag{24}$$

The sampling policy (23) was also obtained in [19] for the case that $Y_i = 0$ for all i.

Similarly, if $\alpha \to 0$ or $f_{\max} \to 0$, the age-optimal sampling policy in (8) and (19) becomes periodic sampling (6) with $\beta = 1/f_{\max} + o(1/f_{\max})$, and the optimal value of (17) is $\mathsf{mse}_{\text{age-opt}} = 1/(2f_{\max}) + o(1/f_{\max})$. Therefore,

$$\lim_{\alpha\to 0} \frac{\mathsf{mse}_{\text{opt}}}{\mathsf{mse}_{\text{age-opt}}} = \lim_{f_{\max}\to 0} \frac{\mathsf{mse}_{\text{opt}}}{\mathsf{mse}_{\text{age-opt}}} = \frac{1}{3}. \tag{25}$$

C. Long Service Time or Unbounded Sampling Rate

If $\alpha \to \infty$ or $f_{\max} \to \infty$, as shown in Appendix D, the MSE-optimal sampling policy for solving (11) is given by (9) where β is determined by solving

$$2\beta\mathbb{E}[\max(\beta, W_Y^2)] = \mathbb{E}[\max(\beta^2, W_Y^4)]. \tag{26}$$

Similarly, if $\alpha \to \infty$ or $f_{\max} \to \infty$, the age-optimal sampling policy for solving (17) is given by (8) where β is determined by solving

$$2\beta\mathbb{E}[\max(\beta, Y)] = \mathbb{E}[\max(\beta^2, Y^2)]. \tag{27}$$

In these limits, the ratio between $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$ depends on the distribution of Y.

When the sampling rate is unbounded, i.e., $f_{\max} = \infty$, one logically reasonable policy is the zero-wait sampling policy in (7) [2]–[4], [37]. This zero-wait sampling policy achieves the maximum throughput and the minimum queueing delay. Surprisingly, this zero-wait sampling policy *does not always* minimize the age of information in (17) and *almost never* minimizes the MSE in (11), as stated below:

Theorem 3. *When $f_{\max} = \infty$, the zero-wait sampling policy* (7) *is optimal for solving* (11) *if and only if $Y = 0$ with probability one.*

Proof. See Appendix E. □

Theorem 4. *[4] When $f_{\max} = \infty$, the zero-wait sampling policy* (7) *is optimal for solving* (17) *if and only if*

$$\mathbb{E}[Y^2] \leq 2 \operatorname{ess\,inf} Y \ \mathbb{E}[Y], \tag{28}$$

where $\operatorname{ess\,inf} Y = \sup\{y \in [0, \infty) : \Pr[Y < y] = 0\}$ can be considered as the minimum possible value of Y.

Proof. See Appendix E. □

V. Proof of the Main Result

We prove Theorem 1 in four steps: First, we show that no sample should be generated when the server is busy, which simplifies the optimal online sampling problem. Second, we study the Lagrangian dual problem of the simplified problem, and decompose the Lagrangian dual problem into a series of *mutually independent* per-sample control problems. Each of these per-sample control problems is a continuous-time Markov decision problem. Further, we utilize optimal stopping theory [50] to solve the per-sample control problems. Finally, we show that the Lagrangian duality gap of our Markov decision problem is zero. By this, Problem (11) is solved. The details are as follows.

A. *Simplification of Problem* (11)

The following lemma is useful for simplifying (11).

Lemma 1. *In the optimal sampling problem* (11)*, it is suboptimal to take a new sample before the previous sample is delivered.*

Proof. See Appendix F. □

In recent studies on age of information [3], [4], Lemma 1 obviously holds and hence was used without a proof: Any sample taken while the server is busy is strictly worse than a sample taken just when the server becomes idle, in terms of the age. However, this lemma needs to be proven for the MSE minimization problem (11), in which we used the strong Markov property of the Wiener process and the orthogonality principle of MMSE estimation.

By Lemma 1, we only need to consider a sub-class of sampling policies $\Pi_1 \subset \Pi$ such that each sample is generated and submitted to the server after the previous sample is delivered, i.e.,

$$\Pi_1 = \{\pi \in \Pi : S_i \geq D_{i-1} \text{ for all } i\}. \tag{29}$$

This completely eliminates the waiting time wasted in the queue, and hence the queue is always kept empty. The *information* that is available for determining S_i includes the history of signal values $(W_t, t \in [0, S_i])$ and the service times $(Y_1, \ldots, Y_{i-1})$ of the previous samples.[5] To characterize this statement precisely, let us define the σ-fields $\mathcal{F}_t = \sigma(W_s : s \in [0, t])$ and $\mathcal{F}_t^+ = \cap_{s>t}\mathcal{F}_s$. Then, $\{\mathcal{F}_t^+, t \geq 0\}$ is the filtration (i.e., a non-decreasing and right-continuous family of σ-fields) of the Wiener process W_t. Given the service times $(Y_1, \ldots, Y_{i-1})$ of previous samples, S_i is a *stopping time* with respect to the filtration $\{\mathcal{F}_t^+, t \geq 0\}$ of the Wiener process W_t, that is

$$[\{S_i \leq t\}|Y_1, \ldots, Y_{i-1}] \in \mathcal{F}_t^+, \ \forall\ t \geq 0. \tag{30}$$

Then, the policy space Π_1 can be alternatively expressed as[6]

$$\begin{aligned} \Pi_1 =\{S_i :\ & [\{S_i \leq t\}|Y_1, \ldots, Y_{i-1}] \in \mathcal{F}_t^+, \ \forall\ t \geq 0, \\ & S_i \geq D_{i-1} \text{ for all } i, \\ & T_i = S_{i+1} - S_i \text{ is a regenerative process}\}. \end{aligned} \tag{31}$$

Let $Z_i = S_{i+1} - D_i \geq 0$ represent the *waiting time* between the delivery time D_i of sample i and the generation time S_{i+1} of sample $i+1$. Then, $S_i = Z_0 + \sum_{j=1}^{i-1}(Y_j + Z_j)$ and $D_i = \sum_{j=0}^{i-1}(Z_j + Y_{j+1})$. If $(Y_1, Y_2, \ldots)$ is given, $(S_0, S_1, \ldots)$ is uniquely determined by $(Z_0, Z_1, \ldots)$. Hence, one can also use $\pi = (Z_0, Z_1, \ldots)$ to represent a sampling policy.

Because T_i is a regenerative process, using the renewal theory ([51] and [46, Section 6.1]), one can

[5]Note that the generation times $(S_1, \ldots, S_{i-1})$ of previous samples are also included in this information.

[6]Recall that any policy in Π satisfies "$T_i = S_{i+1} - S_i$ is a regenerative process".

show that in Problem (11), $\frac{1}{n}\mathbb{E}[S_n]$ is a convergent sequence and

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T (W_t-\hat{W}_t)^2dt\right]\\=&\lim_{n\to\infty}\frac{\mathbb{E}\left[\int_0^{D_n}(W_t-\hat{W}_t)^2dt\right]}{\mathbb{E}[D_n]}\\=&\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t-W_{S_i})^2dt\right]}{\sum_{i=0}^{n-1}\mathbb{E}\left[Y_i+Z_i\right]},\end{aligned}$$

where in the last step we have used $\mathbb{E}\left[D_n\right]=\mathbb{E}[\sum_{i=0}^{n-1}(Z_i+Y_{i+1})]=\mathbb{E}[\sum_{i=0}^{n-1}(Y_i+Z_i)]$. Hence, (11) can be rewritten as the following Markov decision problem:

$$\mathsf{mse}_{\text{opt}}\triangleq\inf_{\pi\in\Pi_1}\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t-W_{S_i})^2dt\right]}{\sum_{i=0}^{n-1}\mathbb{E}\left[Y_i+Z_i\right]}\tag{32}$$

$$\text{s.t.}\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}\left[Y_i+Z_i\right]\geq\frac{1}{f_{\max}},\tag{33}$$

where $\mathsf{mse}_{\text{opt}}$ is the optimal value of (32).

In order to solve (32), let us consider the following Markov decision problem with a parameter $c\geq 0$:

$$p(c)\triangleq\inf_{\pi\in\Pi_1}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t-W_{S_i})^2dt-c(Y_i+Z_i)\right]\tag{34}$$

$$\text{s.t.}\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}\left[Y_i+Z_i\right]\geq\frac{1}{f_{\max}},$$

where $p(c)$ is the optimum value of (34). Similar with Dinkelbach's method [52] for nonlinear fractional programming, we can obtain the following lemma for our Markov decision problem:

Lemma 2. *The following assertions are true:*

(a). $\mathsf{mse}_{\text{opt}}\gtreqless c$ *if and only if* $p(c)\gtreqless 0$.

(b). *If* $p(c)=0$*, the solutions to* (32) *and* (34) *are identical.*

Proof. See Appendix G. □

Hence, the solution to (32) can be obtained by solving (34) and seeking a $c_{\text{opt}}\geq 0$ such that

$$p(c_{\text{opt}})=0.\tag{35}$$

B. Lagrangian Dual Problem of (34) *when* $c = c_{\text{opt}}$

Although (34) is a continuous-time Markov decision problem with a continuous state space, rather than a convex optimization problem, it is possible to use the Lagrangian dual approach to solve (34) and show that it admits no duality gap.

When $c = c_{\text{opt}}$, define the following Lagrangian

$$\begin{aligned} & L(\pi; \lambda) \\ = & \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - (c_{\text{opt}} + \lambda)(Y_i + Z_i)\right] \\ & + \frac{\lambda}{f_{\max}}, \end{aligned} \tag{36}$$

where $\lambda \geq 0$ is the dual variable. Let

$$g(\lambda) \triangleq \inf_{\pi\in\Pi_1} L(\pi; \lambda). \tag{37}$$

Then, the Lagrangian dual problem of (34) is defined by

$$d \triangleq \max_{\lambda \geq 0} g(\lambda), \tag{38}$$

where d is the optimum value of (38). Weak duality [53], [54] implies that $d \leq p(c_{\text{opt}})$. In Section V-D, we will establish strong duality, i.e., $d = p(c_{\text{opt}})$.

In the sequel, we solve (37). Using the stopping times and martingale theory of the Wiener process, we can obtain the following lemma:

Lemma 3. *Let* $\tau \geq 0$ *be a stopping time of the Wiener process* W_t *with* $\mathbb{E}[\tau^2] < \infty$*, then*

$$\mathbb{E}\left[\int_0^\tau W_t^2 dt\right] = \frac{1}{6}\mathbb{E}\left[W_\tau^4\right]. \tag{39}$$

Proof. See Appendix H. □

By using Lemma 3 and the sufficient statistics of (37), we can show that for every $i = 1, 2, \ldots$,

$$\begin{aligned} & \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt\right] \\ = & \frac{1}{6}\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right] + \mathbb{E}\left[Y_i + Z_i\right]\mathbb{E}\left[Y_i\right], \end{aligned} \tag{40}$$

which is proven in Appendix I.

For any $s \geq 0$, define the σ-fields $\mathcal{F}_t^s = \sigma(W_{s+v} - W_s : v \in [0, t])$ and $\mathcal{F}_t^{s+} = \cap_{v>t}\mathcal{F}_v^s$, as well as the filtration $\{\mathcal{F}_t^{s+}, t \geq 0\}$ of the time-shifted Wiener process $\{W_{s+t} - W_s, t \in [0, \infty)\}$. Define $\mathfrak{M}_s$ as the set of square-integrable stopping times of $\{W_{s+t} - W_s, t \in [0, \infty)\}$, i.e.,

$$\mathfrak{M}_s = \{\tau \geq 0 : \{\tau \leq t\} \in \mathcal{F}_t^{s+}, \mathbb{E}\left[\tau^2\right] < \infty\}.$$

By substituting (40) into (37) and using again the sufficient statistics of (37), we can obtain

Theorem 5. *An optimal solution* $(Z_0, Z_1, \ldots)$ *to* (37) *satisfies*

$$\begin{aligned} Z_i = \arg \inf_{\tau \in \mathfrak{M}_{S_i+Y_i}} \mathbb{E}\Bigg[&\frac{1}{2}(W_{S_i+Y_i+\tau} - W_{S_i})^4 \\ &-\beta(Y_i + \tau)\Bigg| W_{S_i+Y_i} - W_{S_i}, Y_i\Bigg], \end{aligned} \quad (41)$$

where β *is given by*

$$\beta = 3(c_{\text{opt}} + \lambda - \mathbb{E}\left[Y\right]) \geq 0. \quad (42)$$

Proof. See Appendix J. □

Note that because the Y_i's are *i.i.d.* and the strong Markov property of the Wiener process, the Z_i's as solutions of (41) are also *i.i.d.*

C. *Per-Sample Optimal Stopping Solution to* (41)

We use optimal stopping theory [50] to solve (41). Let us first pose (41) in the language of optimal stopping. A continuous-time two-dimensional Markov chain X_t on a probability space $(\mathbb{R}^2, \mathcal{F}, \mathbb{P})$ is defined as follows: Given the initial state $X_0 = x = (s, b)$, the state X_t at time t is

$$X_t = (s + t, b + W_t), \quad (43)$$

where $\{W_t, t \geq 0\}$ is a standard Wiener process. Define $\mathbb{P}_x(A) = \mathbb{P}(A|X_0 = x)$ and $\mathbb{E}_x Z = \mathbb{E}(Z|X_0 = x)$, respectively, as the conditional probability of event A and the conditional expectation of random variable Z for given initial state $X_0 = x$. Define the σ-fields $\mathcal{F}_t^X = \sigma(X_v : v \in [0, t])$ and $\mathcal{F}_t^{X+} = \cap_{v>t}\mathcal{F}_v^X$, as well as the filtration $\{\mathcal{F}_t^{X+}, t \geq 0\}$ of the Markov chain X_t. A random variable $\tau : \mathbb{R}^2 \to [0, \infty)$ is said to be a *stopping time* of X_t if $\{\tau \leq t\} \in \mathcal{F}_t^{X+}$ for all $t \geq 0$. Let $\mathfrak{M}$ be the set of square-integrable stopping times of X_t, i.e.,

$$\mathfrak{M} = \{\tau \geq 0 : \{\tau \leq t\} \in \mathcal{F}_t^{X+}, \mathbb{E}\left[\tau^2\right] < \infty\}.$$

Our goal is to solve the following optimal stopping problem:

$$\sup_{\tau\in\mathfrak{M}} \mathbb{E}_x g(X_\tau), \tag{44}$$

where $X_0 = x$ is the initial state of the Markov chain X_t, the function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$g(s, b) = \beta s - \frac{1}{2}b^4 \tag{45}$$

with parameter $\beta \geq 0$. Notice that (41) is a special case of (44) where the initial state is $x = (Y_i, W_{S_i+Y_i} - W_{S_i})$, and W_t is replaced by the time-shifted Wiener process $W_{S_i+Y_i+t} - W_{S_i}$.

Theorem 6. *For all $x = (s, b) \in \mathbb{R}^2$ and $\beta \geq 0$, an optimal stopping time for solving* (44) *is*

$$\tau^* = \inf\left\{t \geq 0 : |b + W_t| \geq \sqrt{\beta}\right\}. \tag{46}$$

In order to prove Theorem 6, let us define the function

$$u(x) = \mathbb{E}_x g(X_{\tau^*}) \tag{47}$$

and establish some properties of $u(x)$.

Lemma 4. *$u(x) \geq g(x)$ for all $x \in \mathbb{R}^2$, and*

$$u(s, b) = \begin{cases} \beta s - \frac{1}{2}b^4, & \text{if } b^2 \geq \beta; \\ \beta s + \frac{1}{2}\beta^2 - \beta b^2, & \text{if } b^2 < \beta. \end{cases} \tag{48}$$

Proof. See Appendix K. □

A function $f(x)$ is said to be *excessive* for the process X_t if [50]

$$\mathbb{E}_x f(X_t) \leq f(x), \text{ for all } t \geq 0, \ x \in \mathbb{R}^2. \tag{49}$$

By using the Itô-Tanaka-Meyer formula [55, Theorem 7.14 and Corollary 7.35] in stochastic calculus, we can obtain

Lemma 5. *The function $u(x)$ is excessive for the process X_t.*

Proof. See Appendix L. □

Now, we are ready to prove Theorem 6.

Proof of Theorem 6. In Lemma 4 and Lemma 5, we have shown that $u(x) = \mathbb{E}_x g(X_{\tau^*})$ is an excessive function and $u(x) \geq g(x)$. In addition, it is known that $\mathbb{P}_x(\tau^* < \infty) = 1$ for all $x \in \mathbb{R}^2$ [56, Theorem 8.5.3]. These conditions, together with the Corollary to Theorem 1 in [50, Section 3.3.1], imply that τ^* is an optimal stopping time of (44). This completes the proof. □

A consequence of Theorem 6 is

Corollary 1. *An optimal solution to* (41) *is*

$$Z_i = \inf\left\{t \geq 0 : |W_{S_i+Y_i+t} - W_{S_i}| \geq \sqrt{\beta}\right\}. \tag{50}$$

In addition, this solution satisfies

$$\mathbb{E}[Y_i + Z_i] = \mathbb{E}[\max(\beta, W_Y^2)], \tag{51}$$

$$\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4] = \mathbb{E}[\max(\beta^2, W_Y^4)]. \tag{52}$$

Proof. See Appendix M. □

D. Zero Duality Gap between (34) *and* (38)

Strong duality is established in the following theorem:

Theorem 7. *If* $c = c_{\text{opt}}$*, the following assertions are true:*

(a). *The duality gap between* (34) *and* (38) *is zero, i.e.,* $d = p(c_{\text{opt}})$.

(b). *A common optimal solution to* (11)*,* (32)*, and* (34) *is given by* (9) *and* (13)*. The optimal value of* (11) *is given by* (14).

Proof Sketch of Theorem 7. We use [53, Prop. 6.2.5] to find a *geometric multiplier* [53, Definition 6.1.1] for Problem (34). This tells us that the duality gap between (34) and (38) must be zero, because otherwise there is no geometric multiplier [53, Prop. 6.2.3(b)].[7] This result holds not only for convex optimization problem, but also for general non-convex optimization and Markov decision problems like (34). See Appendix N for the details. □

Hence, Theorem 1 follows from Theorem 7.

[7]Note that geometric multiplier is different from the traditional Lagrangian multiplier.

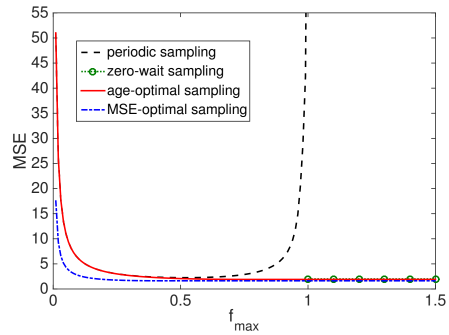


Fig. 4: MSE vs. $f_{\max}$ tradeoff for *i.i.d.* exponential service time and $\mathbb{E}[Y_i] = 1$, where zero-wait sampling is not feasible when $f_{\max} < 1/\mathbb{E}[Y_i]$ and hence is not plotted in that regime.

VI. Numerical Results

In this section, we evaluate the estimation performance achieved by the following four sampling policies:

1. *Periodic sampling:* The policy in (6) with $\beta = f_{\max}$.
2. *Zero-wait sampling [2]–[4], [37]:* The sampling policy in (7), which is feasible when $f_{\max} \geq 1/\mathbb{E}[Y_i]$.
3. *Age-optimal sampling [3], [4]:* The sampling policy in (8) and (19), which is the optimal solution to (17).
4. *MSE-optimal sampling:* The sampling policy in (9) and (13), which is the optimal solution to (11).

Let $\mathsf{mse}_{\text{periodic}}$, $\mathsf{mse}_{\text{zero-wait}}$, $\mathsf{mse}_{\text{age-opt}}$, and $\mathsf{mse}_{\text{opt}}$, be the MSEs of periodic sampling, zero-wait sampling, age-optimal sampling, MSE-optimal sampling, respectively. According to (18), as well as the facts that periodic sampling is feasible for (17) and zero-wait sampling is feasible for (17) when $f_{\max} \geq 1/\mathbb{E}[Y_i]$, we can obtain

$$\mathsf{mse}_{\text{opt}} \leq \mathsf{mse}_{\text{age-opt}} \leq \mathsf{mse}_{\text{periodic}},$$

$$\mathsf{mse}_{\text{opt}} \leq \mathsf{mse}_{\text{age-opt}} \leq \mathsf{mse}_{\text{zero-wait}}, \text{ when } f_{\max} \geq \frac{1}{\mathbb{E}[Y_i]},$$

which fit with our numerical results below.

Figure 4 depicts the tradeoff between MSE and $f_{\max}$ for *i.i.d.* exponential service time with mean $\mathbb{E}[Y_i] = 1/\mu = 1$. Hence, the maximum throughput of the queue is $\mu = 1$. In this setting, $\mathsf{mse}_{\text{periodic}}$ is characterized by eq. (25) of [2], which was obtained using a D/M/1 queueing model. For small values of $f_{\max}$, age-optimal sampling is similar with periodic sampling, and hence $\mathsf{mse}_{\text{age-opt}}$ and $\mathsf{mse}_{\text{periodic}}$ are of similar values. However, as $f_{\max}$ approaches the maximum throughput 1, $\mathsf{mse}_{\text{periodic}}$ blows up to

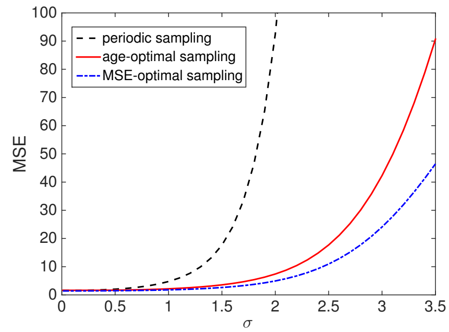


Fig. 5: MSE vs. the scale parameter σ of *i.i.d.* log-normal service time for $f_{\max} = 0.8$ and $\mathbb{E}[Y_i] = 1$, where zero-wait sampling is not feasible because $f_{\max} < 1/\mathbb{E}[Y_i]$.

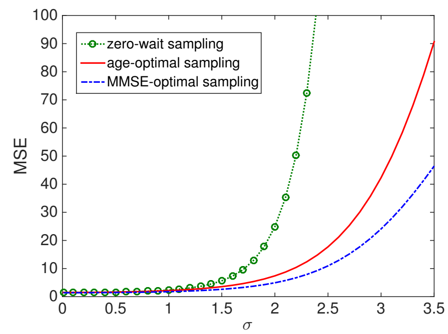


Fig. 6: MSE vs. the scale parameter σ of *i.i.d.* log-normal service time for $f_{\max} = 1.5$ and $\mathbb{E}[Y_i] = 1$, where $\mathsf{mse}_{\text{periodic}} = \infty$ due to queueing.

infinity. This is because the queue length in periodic sampling is large at high sampling frequencies, and the samples become stale during their long waiting times in the queue. On the other hand, $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$ decrease with respect to $f_{\max}$. The reason is that the set of feasible policies satisfying the constraint in (11) and (17) becomes larger as $f_{\max}$ grows, and hence the optimal values of (11) and (17) are decreasing in $f_{\max}$. Moreover, the gap between $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$ is large for small values of $f_{\max}$. The ratio $\mathsf{mse}_{\text{opt}}/\mathsf{mse}_{\text{age-opt}}$ tends to $1/3$ as $f_{\max} \to 0$, which is in accordance with (25). As we expected, $\mathsf{mse}_{\text{zero-wait}}$ is larger than $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$ when $f_{\max} \geq 1$. In summary, periodic sampling is far from optimal if the sampling rate is too low or sufficiently high; age-optimal sampling is far from optimal if the sampling rate is too low.

Figure 5 and Figure 6 illustrate the MSE of *i.i.d.* log-normal service time for $f_{\max} = 0.8$ and $f_{\max} = 1.5$, respectively, where $Y_i = e^{\sigma X_i}/\mathbb{E}[e^{\sigma X_i}]$, $\sigma > 0$ is the scale parameter of log-normal distribution, and $(X_1, X_2, \ldots)$ are *i.i.d.* Gaussian random variables with zero mean and unit variance. Because $\mathbb{E}[Y_i] = 1$, the maximum throughput of the queue is 1. In Fig. 5, since $f_{\max} < 1$, zero-wait sampling is not feasible and hence is not plotted. As the scale parameter σ grows, the tail of the log-normal distribution becomes heavier and heavier. We observe that $\mathsf{mse}_{\text{periodic}}$ grows quickly with respect to σ, much faster than $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$. In addition, the gap between $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$ increases as σ grows. In Fig. 6, because $f_{\max} > 1$, $\mathsf{mse}_{\text{periodic}}$ is infinite and hence is not plotted. We can find that $\mathsf{mse}_{\text{zero-wait}}$ grows quickly with respect to σ and is much larger than $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$. In summary, periodic sampling, age-optimal sampling, and zero-wait sampling policies are all far from optimal if the service times are highly random.

VII. CONCLUSIONS

In this paper, we have investigated optimal sampling of the Wiener process for remote estimation over a queue. The optimal sampling policy for minimizing the mean square estimation error subject to an average sampling rate constraint has been obtained. We prove that a threshold-based sampler is optimal and the optimal threshold is found exactly. Analytical and numerical comparisons with several important sampling policies, including age-optimal sampling, zero-wait sampling, and classic periodic sampling, have been provided. The results in this paper generalize recent research on age of information by adding a signal-based control model, and generalize existing studies on remote estimation by adding a queueing model with random service times.

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APPENDIX A
PROOF OF (10)

We use the calculus of variations to prove (10). Let us consider a functional h of the estimate $\hat{W}_t$, which is defined as

$$h(\hat{W}_t) = \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1}\wedge T} (\hat{W}_t - W_t)^2 dt \middle| (S_j, W_{S_j}, D_j)_{j\leq i} \right\}.$$

By using Lemma 4 in [4], it is not hard to show that $h(\hat{W}_t)$ is a convex functional of the estimate $\hat{W}_t$. In the sequent, we will find the optimal estimate that solves

$$\min_{\hat{W}_t} h(\hat{W}_t) \tag{53}$$

Let f_t and g_t be two estimates, which are functions of the information available at the estimator $\{S_i, W_{S_i}, D_i : D_i \leq t\}$. Similar to the one-sided sub-gradient in finite dimensional space, the one-sided Gâteaux derivative of the functional h in the direction of g at a point f is given by

$$\begin{aligned}
&\delta h(f;g) \\
=& \lim_{\epsilon \to 0^+} \frac{h(f_t + \epsilon g_t) - h(f_t)}{\epsilon} \\
=& \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (f_t + \epsilon g_t - W_t)^2 - (f_t - W_t)^2 dt \right. \\
& \left. \Bigg| (S_j, W_{S_j}, D_j)_{j \leq i} \right\}. \\
=& \lim_{\epsilon \to 0^+} \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} 2(f_t - W_t) g_t + \epsilon g_t^2 dt \right. \\
& \left. \Bigg| (S_j, W_{S_j}, D_j)_{j \leq i} \right\}
\end{aligned}$$

$$\begin{aligned}
=&\mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} 2(f_t - W_t) g_t dt \,\Bigg|\, (S_j, W_{S_j}, D_j)_{j \leq i} \right\} \\
=&\mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} 2\left(f_t - \mathbb{E}\left[W_t | (S_j, W_{S_j}, D_j)_{j \leq i}\right]\right) g_t dt \right. \\
& \left. \Bigg| (S_j, W_{S_j}, D_j)_{j \leq i} \right\},
\end{aligned} \tag{54}$$

where the last step follows from the iterated law of expectations. According to [57, p. 710], f_t is an optimal solution to (53) if and only if

$$\delta h(f;g) \geq 0, \quad \forall\, g.$$

By $\delta h(f;g) = -\delta h(f;-g)$, we get

$$\delta h(f;g) = 0, \quad \forall\, g. \tag{55}$$

Since g_t is arbitrary, by (54) and (55), the optimal solution to (53) is

$$\begin{aligned}
f_t =& \mathbb{E}[W_t | (S_j, W_{S_j}, D_j)_{j \leq i}], \\
=& W_{S_i} + \mathbb{E}[W_t - W_{S_i} | (S_j, W_{S_j}, D_j)_{j \leq i}] \\
& t \in [D_i \wedge T, D_{i+1} \wedge T).
\end{aligned}$$

Notice that under any online sampling policy π, $\{S_j, W_{S_j}, D_j,\ j \leq i\}$ are determined by the source $(W_t, t \in [0, S_i])$ and the service times $(Y_1, \ldots, Y_i)$. According to (i) the strong Markov property of the Wiener process [55, Theorem 2.16 and Remark 2.17] and (ii) the fact that the Y_i's are independent of the Wiener process W_t, we obtain that for any realization of $\{S_j, W_{S_j}, D_j,\ j \leq i\}$, $\{W_t - W_{S_i}, t \geq S_i\}$ is a Wiener process. Hence,

$$\mathbb{E}[W_t - W_{S_i} | (S_j, W_{S_j}, D_j)_{j \leq i}] = 0 \tag{56}$$

for all $t \geq S_i$. Therefore, the optimal solution to (53) is

$$f_t = W_{S_i},\ \text{ if } t \in [D_i \wedge T, D_{i+1} \wedge T), i = 1, 2, \ldots \tag{57}$$

Finally, we note that

$$\begin{aligned}
&\mathbb{E}\left\{\int_0^T (W_t - \hat{W}_t)^2 dt\right\} \\
\overset{(a)}{=}&\mathbb{E}\left\{\lim_{n\to\infty}\int_0^{D_n \wedge T} (W_t - \hat{W}_t)^2 dt\right\} \\
\overset{(b)}{=}&\lim_{n\to\infty}\mathbb{E}\left\{\int_0^{D_n \wedge T} (W_t - \hat{W}_t)^2 dt\right\} \\
=&\lim_{n\to\infty}\sum_{i=0}^{n}\mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_t)^2 dt\right\},
\end{aligned} \tag{58}$$

where Step (a) is due to $\lim_{n\to\infty} D_n = \infty$ almost surely (as in (5)) and in Step (b) we have used the monotonic convergence theorem. Let $T \to \infty$ in (57), we obtain that (10) is the MMSE estimator for minimizing the mse in (2). This completes the proof.

Appendix B
Proof of (15)

If π is independent of $\{W_t, t \in [0, \infty)\}$, the S_i's and D_i's are independent of $\{W_t, t \in [0, \infty)\}$. Define $x \wedge y = \min\{x, y\}$. For any $T > 0$, let us consider the term

$$\mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_t)^2 dt\right\}$$

in the following two cases:

Case 1: If $D_i \wedge T \geq S_i$, we can obtain

$$\begin{aligned}
&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t-\hat{W}_t)^2dt\right\}\\
=&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t-\hat{W}_{S_i})^2dt\right\}\\
\overset{(a)}{=}&\mathbb{E}\left\{\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t-W_{S_i})^2dt\middle|S_i,D_i,D_{i+1}\right\}\right\}\\
\overset{(b)}{=}&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}\mathbb{E}\left\{(W_t-W_{S_i})^2|S_i,D_i,D_{i+1}\right\}dt\right\}\\
\overset{(c)}{=}&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}\mathbb{E}\left[(W_t-W_{S_i})^2\right]dt\right\}\\
\overset{(d)}{=}&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(t-S_i)dt\right\}\\
\overset{(e)}{=}&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}\Delta(t)dt\right\},
\end{aligned}\tag{59}$$

where Step (a) is due to the law of iterated expectations, Step (b) is due to Fubini's theorem, Step (c) is because S_i, D_i, D_{i+1} are independent of the Wiener process, Step (d) is due to Wald's identity $\mathbb{E}[W_T^2] = T$ [55, Theorem 2.48] and the strong Markov property of the Wiener process [55, Theorem 2.16], and Step (e) is due to (16).

Case 2: If $D_i \wedge T < S_i$, then the fact $D_i \geq S_i$ implies that $T < S_i \leq D_i \leq D_{i+1}$. Hence, $D_i \wedge T = D_{i+1} \wedge T = T$ and

$$\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t-\hat{W}_t)^2dt\right\}=\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}\Delta(t)dt\right\}=0.$$

Therefore, (59) holds in both cases.

By using an argument similar to (58), we can obtain

$$\lim_{n\to\infty}\sum_{i=0}^{n}\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}\Delta(t)dt\right\}=\mathbb{E}\left\{\int_0^T\Delta(t)dt\right\}.\tag{60}$$

Combining (58)-(60), (15) is proven.

Appendix C
Proofs of (22) and (24)

If $f_{\max} \to 0$, (13) tells us that

$$\mathbb{E}[\max(\beta, W_Y^2)] = \frac{1}{f_{\max}},$$

which implies

$$\beta \leq \frac{1}{f_{\max}} \leq \beta + \mathbb{E}[W_Y^2] = \beta + \mathbb{E}[Y].$$

Hence,

$$\frac{1}{f_{\max}} - \mathbb{E}[Y] \leq \beta \leq \frac{1}{f_{\max}}.$$

If $f_{\max} \to 0$, (22) follows.

Because Y is independent of the Wiener process, using the law of iterated expectations and the Gaussian distribution of the Wiener process, we can obtain $\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^2]$ and $\mathbb{E}[W_Y^2] = 3\mathbb{E}[Y]$. Hence,

$$\beta \leq \mathbb{E}[\max(\beta, W_Y^2)] \leq \beta + \mathbb{E}[W_Y^2] = \beta + \mathbb{E}[Y],$$
$$\beta^2 \leq \mathbb{E}[\max(\beta^2, W_Y^4)] \leq \beta^2 + \mathbb{E}[W_Y^4] = \beta^2 + 3\mathbb{E}[Y^2].$$

Therefore,

$$\frac{\beta^2}{\beta + \mathbb{E}[Y]} \leq \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{\mathbb{E}[\max(\beta, W_Y^2)]} \leq \frac{\beta^2 + 3\mathbb{E}[Y^2]}{\beta}. \tag{61}$$

By combining (14), (22), and (61), (24) follows in the case of $f_{\max} \to 0$.

If $\alpha \to 0$, then $Y \to 0$ and $W_Y \to 0$ with probability one. Hence, $\mathbb{E}[\max(\beta, W_Y^2)] \to \beta$ and $\mathbb{E}[\max(\beta^2, W_Y^4)] \to \beta^2$. Substituting these into (13) and (61), yields

$$\lim_{\alpha \to 0} \beta = \frac{1}{f_{\max}}, \ \lim_{\alpha \to 0} \left\{ \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{6\mathbb{E}[\max(\beta, W_Y^2)]} + \mathbb{E}[Y] \right\} = \frac{1}{6 f_{\max}}.$$

By this, (22) and (24) are proven in the case of $\alpha \to 0$. This completes the proof.

APPENDIX D
PROOF OF (26)

If $f_{\max} \to \infty$, the sampling rate constraint in (11) can be removed. By (13), the optimal β is determined by (26).

If $\alpha \to \infty$, let us consider the equation

$$\mathbb{E}[\max(\beta, W_Y^2)] = \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta}. \tag{62}$$

If Y grows by α times, then β and $\mathbb{E}[\max(\beta, W_Y^2)]$ in (62) both should grow by α times, and $\mathbb{E}[\max(\beta^2, W_Y^4)]$

in (62) should grow by α^2 times. Hence, if $\alpha \to \infty$, it holds in (13) that

$$\frac{1}{f_{\max}} \leq \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta}$$

and the solution to (13) is given by (26). This completes the proof.

Appendix E
Proofs of Theorems 3 and 4

Proof of Theorem 3. The zero-wait policy can be expressed as (9) with $\beta = 0$. Because Y is independent of the Wiener process, using the law of iterated expectations and the Gaussian distribution of the Wiener process, we can obtain $\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^2]$. According to (26), $\beta = 0$ if and only if $\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^2] = 0$ which is equivalent to $Y = 0$ with probability one. This completes the proof. □

Proof of Theorem 4. In the one direction, the zero-wait policy can be expressed as (8) with $\beta \leq \operatorname{ess\,inf} Y$. If the zero-wait policy is optimal, then the solution to (27) must satisfy $\beta \leq \operatorname{ess\,inf} Y$, which further implies $\beta \leq Y$ with probability one. From this, we can get

$$2\operatorname{ess\,inf} Y \mathbb{E}[Y] \geq 2\beta\mathbb{E}[Y] = \mathbb{E}[Y^2],$$

By this, (28) follows.

In the other direction, if (28) holds, we will show that the zero-wait policy is age-optimal by considering the following two cases.

Case 1: $\mathbb{E}[Y] > 0$. By choosing

$$\beta = \frac{\mathbb{E}[Y^2]}{2\mathbb{E}[Y]}, \tag{63}$$

we can get $\beta \leq \operatorname{ess\,inf} Y$ from (28) and hence

$$\beta \leq Y \tag{64}$$

with probability one. According to (63) and (64), such a β is the solution to (27). Hence, the zero-wait policy expressed by (8) with $\beta \leq \operatorname{ess\,inf} Y$ is the age-optimal policy.

Case 2: $\mathbb{E}[Y] = 0$ and hence $Y = 0$ with probability one. In this case, $\beta = 0$ is the solution to (27). Hence, the zero-wait policy expressed by (8) with $\beta = 0$ is the age-optimal policy.

Combining these two cases, the proof is completed. □

Appendix F
Proof of Lemma 1

Suppose that in the sampling policy π, sample i is generated when the server is busy sending another sample, and hence sample i needs to wait for some time before being submitted to the server, i.e., $S_i < G_i$. Let us consider a *virtual* sampling policy $\pi' = \{S_0, \ldots, S_{i-1}, G_i, S_{i+1}, \ldots\}$ such that the generation time of sample i is postponed from S_i to G_i. We call policy π' a virtual policy because it may happen that $G_i > S_{i+1}$. However, this will not affect our proof below. We will show that the MSE of the sampling policy π' is smaller than that of the sampling policy $\pi = \{S_0, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots\}$.

Note that the Wiener process $\{W_t : t \in [0, \infty)\}$ does not change according to the sampling policy, and the sample delivery times $\{D_0, D_1, D_2, \ldots\}$ remain the same in policy π and policy π'. Hence, the only difference between policies π and π' is that *the generation time of sample i is postponed from S_i to G_i*. The MMSE estimator under policy π is given by (10) and the MMSE estimator under policy π' is given by

$$\begin{aligned}\hat{W}_t =&\mathbb{E}[W_t|(S_j, W_{S_j}, D_j)_{j\leq i-1}, (G_i, W_{G_i}, D_i)]\\ =&\begin{cases} 0, & t \in [0, D_1);\\ W_{G_i}, & t \in [D_i, D_{i+1});\\ W_{S_j}, & t \in [D_j, D_{j+1}),\ j \neq i, j \geq 1.\end{cases}\end{aligned} \tag{65}$$

Next, we consider a third virtual sampling policy π'' in which the samples (W_{G_i}, G_i) and (W_{S_i}, S_i) are both delivered to the estimator at time D_i. Clearly, the estimator under policy π'' has more information than those under policies π and π'. By following the arguments in Appendix A, one can show that the MMSE estimator under policy π'' is

$$\begin{aligned}\hat{W}_t =&\mathbb{E}[W_t|(S_j, W_{S_j}, D_j)_{j\leq i}, (G_i, W_{G_i}, D_i)]\\ =&\begin{cases} 0, & t \in [0, D_1);\\ W_{G_i}, & t \in [D_i, D_{i+1});\\ W_{S_j}, & t \in [D_j, D_{j+1}),\ j \neq i, j \geq 1.\end{cases}\end{aligned} \tag{66}$$

Notice that, because of the strong Markov property of Wiener process, the estimator under policy π'' uses the fresher sample W_{G_i}, instead of the stale sample W_{S_i}, to construct $\hat{W}_t$ during $[D_i, D_{i+1})$. Because the estimator under policy π'' has more information than that under policy π, one can imagine that policy

π'' has a smaller estimation error than policy π, i.e., for any $T > 0$

$$\begin{aligned}&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t - W_{S_i})^2 dt\right\}\\ \geq&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t - W_{G_i})^2 dt\right\}.\end{aligned} \tag{67}$$

To prove (67), we invoke the orthogonality principle of the MMSE estimator [49, Prop. V.C.2] under policy π'' and obtain

$$\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}2(W_t - W_{G_i})(W_{G_i} - W_{S_i})dt\right\} = 0, \tag{68}$$

where we have used the face that W_{G_i} and W_{S_i} are available by the MMSE estimator under policy π''. Next, from (68), we can get

$$\begin{aligned}&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t - W_{S_i})^2 dt\right\}\\ =&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t - W_{G_i})^2 + (W_{G_i} - W_{S_i})^2 dt\right\}\\ &+\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}2(W_t - W_{G_i})(W_{G_i} - W_{S_i})dt\right\}\\ \\ =&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t - W_{G_i})^2 + (W_{G_i} - W_{S_i})^2 dt\right\}\\ \geq&\mathbb{E}\left\{\int_{D_i\wedge T}^{D_{i+1}\wedge T}(W_t - W_{G_i})^2 dt\right\}.\end{aligned}$$

In other words, the estimation error of policy π'' is no greater than that of policy π. Furthermore, by comparing (65) and (66), we can see that the MMSE estimators under policies π'' and π' are exact the same. Therefore, the estimation error of policy π' is no greater than that of policy π.

By repeating the above arguments for all samples i satisfying $S_i < G_i$, one can show that the sampling policy $\{S_0, G_1, \ldots, G_{i-1}, G_i, G_{i+1}, \ldots\}$ is better than the sampling policy $\pi = \{S_0, S_1, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots\}$. This completes the proof.

APPENDIX G
PROOF OF LEMMA 2

Part (a) is proven in two steps:

Step 1: We will prove that $\mathsf{mse}_{\text{opt}} \leq c$ if and only if $p(c) \leq 0$.

If $\mathsf{mse}_{\text{opt}} \leq c$, then there exists a policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ that is feasible for both (32) and (34), which satisfies

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt\right]}{\sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right]} \leq c. \tag{69}$$

Hence,

$$\lim_{n\to\infty} \frac{\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right]}{\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right]} \leq 0. \tag{70}$$

Because the inter-sampling times $T_i = Y_i + Z_i$ are regenerative, the renewal theory [51] tells us that the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right]$ exists and is positive. By this, we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right] \leq 0. \tag{71}$$

Therefore, $p(c) \leq 0$.

On the reverse direction, if $p(c) \leq 0$, then there exists a policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ that is feasible for both (32) and (34), which satisfies (71). From (71), we can derive (70) and (69). Hence, $\mathsf{mse}_{\text{opt}} \leq c$. By this, we have proven that $\mathsf{mse}_{\text{opt}} \leq c$ if and only if $p(c) \leq 0$.

Step 2: We needs to prove that $\mathsf{mse}_{\text{opt}} < c$ if and only if $p(c) < 0$. This statement can be proven by using the arguments in *Step 1*, in which "$\leq$" should be replaced by "$<$". Finally, from the statement of *Step 1*, it immediately follows that $\mathsf{mse}_{\text{opt}} > c$ if and only if $p(c) > 0$. This completes the proof of part (a).

Part (b): We first show that each optimal solution to (32) is an optimal solution to (34). By the claim of part (a), $p(c) = 0$ is equivalent to $\mathsf{mse}_{\text{opt}} = c$. Suppose that policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ is an optimal solution to (32). Then, $\mathsf{mse}_{\pi} = \mathsf{mse}_{\text{opt}} = c$. Applying this in the arguments of (69)-(71), we can show that policy π satisfies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right] = 0.$$

This and $p(c) = 0$ imply that policy π is an optimal solution to (34).

Similarly, we can prove that each optimal solution to (34) is an optimal solution to (32). By this, part (b) is proven.

Appendix H
Proof of Lemma 3

According to Theorem 2.51 and Exercise 2.15 of [55], $W_t^4 - 6\int_0^t W_s^2 ds$ and $W_t^4 - 6tW_t^2 + 3t^2$ are two martingales of the Wiener process $\{W_t, t \in [0,\infty)\}$. Hence, $\int_0^t W_s^2 ds - tW_t^2 + t^2/2$ is also a martingale of the Wiener process.

Because the minimum of two stopping times is a stopping time and constant times are stopping times [56], it follows that $t \wedge \tau$ is a bounded stopping time for every $t \in [0,\infty)$, where $x \wedge y = \inf[x,y]$. Then, it follows from Theorem 8.5.1 of [56] that for every $t \in [0,\infty)$

$$\mathbb{E}\left[\int_0^{t\wedge\tau} W_s^2 ds\right] = \frac{1}{6}\mathbb{E}\left[W_{t\wedge\tau}^4\right] \tag{72}$$

$$= \mathbb{E}\left[(t\wedge\tau)W_{t\wedge\tau}^2 - \frac{1}{2}(t\wedge\tau)^2\right]. \tag{73}$$

Notice that $\int_0^{t\wedge\tau} W_s^2 ds$ is positive and increasing with respect to t. By applying the monotone convergence theorem [56, Theorem 1.5.5], we can obtain

$$\lim_{t\to\infty} \mathbb{E}\left[\int_0^{t\wedge\tau} W_s^2 ds\right] = \mathbb{E}\left[\int_0^{\tau} W_s^2 ds\right].$$

The remaining task is to show that

$$\lim_{t\to\infty} \mathbb{E}\left[W_{t\wedge\tau}^4\right] = \mathbb{E}\left[W_\tau^4\right]. \tag{74}$$

Towards this goal, we combine (72) and (73), and apply Cauchy-Schwarz inequality to get

$$\begin{aligned}
&\mathbb{E}\left[W_{t\wedge\tau}^4\right] \\
=&\mathbb{E}\left[6(t\wedge\tau)W_{t\wedge\tau}^2 - 3(t\wedge\tau)^2\right] \\
\le&6\sqrt{\mathbb{E}\left[(t\wedge\tau)^2\right]\mathbb{E}\left[W_{t\wedge\tau}^4\right]} - 3\mathbb{E}\left[(t\wedge\tau)^2\right].
\end{aligned}$$

Let $x = \sqrt{\mathbb{E}\left[W_{t\wedge\tau}^4\right]/\mathbb{E}\left[(t\wedge\tau)^2\right]}$, then $x^2 - 6x + 3 \le 0$. By the roots and properties of quadratic functions, we obtain $3-\sqrt{6} \le x \le 3+\sqrt{6}$ and hence

$$\mathbb{E}\left[W_{t\wedge\tau}^4\right] \le (3+\sqrt{6})^2\mathbb{E}\left[(t\wedge\tau)^2\right] \le (3+\sqrt{6})^2\mathbb{E}\left[\tau^2\right] < \infty.$$

Then, we use Fatou's lemma [56, Theorem 1.5.4] to derive

$$\begin{aligned}
&\mathbb{E}\left[W_\tau^4\right]\\
=&\mathbb{E}\left[\lim_{t\to\infty} W_{t\wedge\tau}^4\right]\\
\leq&\liminf_{t\to\infty}\mathbb{E}\left[W_{t\wedge\tau}^4\right]\\
\leq&(3+\sqrt{6})^2\mathbb{E}\left[\tau^2\right]<\infty.
\end{aligned} \tag{75}$$

Further, by using (75), we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,\infty)} W_{t\wedge\tau}^4\right] =&\mathbb{E}\left[\sup_{t\in[0,\tau]} W_t^4\right]\\
\overset{(a)}{=}&\mathbb{E}\left[\mathbb{E}\left[\sup_{t\in[0,\tau]} W_t^4\middle|\tau\right]\right]\\
\overset{(b)}{\leq}&\mathbb{E}\left[\left(\frac{4}{3}\right)^4\mathbb{E}\left[W_\tau^4\middle|\tau\right]\right]\\
\overset{(c)}{=}&\left(\frac{4}{3}\right)^4\mathbb{E}\left[W_\tau^4\right]\\
<&\infty,
\end{aligned}$$

where Step (a) and Step (c) are due to the law of iterated expectations, and Step (b) is due to Doob's inequality [55, Theorem 12.30] and [56, Theorem 5.4.3]. Because $W_{t\wedge\tau}^4 \leq \sup_{t\in[0,\infty)} W_{t\wedge\tau}^4$ and $\sup_{t\in[0,\infty)} W_{t\wedge\tau}^4$ is integrable, (74) follows from dominated convergence theorem [56, Theorem 1.5.6]. This completes the proof.

Appendix I
Proof of (40)

The following lemma is needed in the proof of (40):

Lemma 6. *For any $\lambda \geq 0$, there exists an optimal solution $(Z_0, Z_1, \ldots)$ to (37) in which Z_i is independent of $(W_t, t\in[0,S_i])$ for all $i=1,2,\ldots$*

Proof. Because the Y_i's are *i.i.d.*, Z_i is independent of $Y_{i+1}, Y_{i+2},\ldots$, and the strong Markov property of the Wiener process [55, Theorem 2.16], in the Lagrangian $L(\pi;\lambda)$ the term related to Z_i is

$$\mathbb{E}\left[\int_{S_i+Y_i}^{S_i+Y_i+Z_i+Y_{i+1}}(W_t-W_{S_i})^2dt-(c_{\text{opt}}+\lambda)(Y_i+Z_i)\right], \tag{76}$$

which is determined by the control decision Z_i and the recent information of the system $\mathcal{I}_i = (Y_i, (W_{S_i+t} - W_{S_i},\, t \geq 0))$. According to [58, p. 252] and [59, Chapter 6], $\mathcal{I}_i$ is a *sufficient statistics* for determining Z_i in (37). Therefore, there exists an optimal policy $(Z_0, Z_1, \ldots)$ in which Z_i is determined based on only $\mathcal{I}_i$, which is independent of $(W_t : t \in [0, S_i])$. This completes the proof. □

Proof of (40). By using (31) and Lemma 6, we obtain that for given Y_i and Y_{i+1}, Y_i and $Y_i + Z_i + Y_{i+1}$ are stopping times of the time-shifted Wiener process $\{W_{S_i+t} - W_{S_i}, t \geq 0\}$. Hence,

$$
\begin{aligned}
&= \mathbb{E}\left\{\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt\right\} \\
&= \mathbb{E}\left\{\int_{Y_i}^{Y_i+Z_i+Y_{i+1}} (W_{S_i+t} - W_{S_i})^2 dt\right\} \\
&\overset{(a)}{=} \mathbb{E}\left\{\mathbb{E}\left\{\int_{Y_i}^{Y_i+Z_i+Y_{i+1}} (W_{S_i+t} - W_{S_i})^2 dt \,\middle|\, Y_i, Y_{i+1}\right\}\right\} \\
&\overset{(b)}{=} \frac{1}{6}\mathbb{E}\left\{\mathbb{E}\left\{(W_{S_i+Y_i+Z_i+Y_{i+1}} - W_{S_i})^4 \,\middle|\, Y_i, Y_{i+1}\right\}\right\} \\
&\quad - \frac{1}{6}\mathbb{E}\left\{\mathbb{E}\left\{(W_{S_i+Y_i} - W_{S_i})^4 \,\middle|\, Y_i, Y_{i+1}\right\}\right\} \\
&\overset{(c)}{=} \frac{1}{6}\mathbb{E}\left[(W_{S_i+Y_i+Z_i+Y_{i+1}} - W_{S_i})^4\right] - \frac{1}{6}\mathbb{E}\left[(W_{S_i+Y_i} - W_{S_i})^4\right],
\end{aligned} \tag{77}
$$

where Step (a) and Step (c) are due to the law of iterated expectations, and Step (b) is due to Lemma 3.

Because $S_{i+1}=S_i+Y_i+Z_i$, we have

$$\begin{aligned}
&\mathbb{E}\left[(W_{S_i+Y_i+Z_i+Y_{i+1}}-W_{S_i})^4\right]\\
=&\mathbb{E}\left\{[(W_{S_i+Y_i+Z_i}-W_{S_i})+(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})]^4\right\}\\
=&\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^4\right]\\
&+4\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^3(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})\right]\\
&+6\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^2(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^2\right]\\
&+4\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^3\right]\\
&+\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^4\right]\\
=&\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^4\right]\\
&+4\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^3\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})\right]\\
&+6\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^2\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^2\right]\\
&+4\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^3\right]\\
&+\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^4\right],
\end{aligned}$$

where in the last equation we have used the fact that Y_{i+1} is independent of Y_i and Z_i, and the strong Markov property of the Wiener process [55, Theorem 2.16]. Because

$$\begin{aligned}
&\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^3|Y_{i+1}\right]\\
=&\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})|Y_{i+1}\right]=0,
\end{aligned}$$

by the law of iterated expectations, we have

$$\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^3\right]=\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})\right]=0.$$

In addition, Wald's identity tells us that $\mathbb{E}\left[W_\tau^2\right]=\mathbb{E}\left[\tau\right]$ for any stopping time τ with $\mathbb{E}\left[\tau\right]<\infty$. Hence,

$$\begin{aligned}
&\mathbb{E}\left[(W_{S_i+Y_i+Z_i+Y_{i+1}}-W_{S_i})^4\right]\\
=&\mathbb{E}\left[(W_{S_i+Y_i+Z_i}-W_{S_i})^4\right]+6\mathbb{E}\left[Y_i+Z_i\right]\mathbb{E}\left[Y_{i+1}\right]\\
&+\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^4\right]. \qquad (78)
\end{aligned}$$

Finally, because $(W_{S_i+t}-W_{S_i})$ and $(W_{S_{i+1}+t}-W_{S_{i+1}})$ are both Wiener processes, and the Y_i's are *i.i.d.*,

$$\mathbb{E}\left[(W_{S_i+Y_i}-W_{S_i})^4\right]=\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}}-W_{S_{i+1}})^4\right]. \tag{79}$$

Combining (77)-(79), yields (40). □

APPENDIX J
PROOF OF THEOREM 5

By (40), (76) can be rewritten as

$$\begin{aligned}
&\mathbb{E}\left[\int_{S_i+Y_i}^{S_i+Y_i+Z_i+Y_{i+1}}(W_t-W_{S_i})^2dt-(c_{\text{opt}}+\lambda)(Y_i+Z_i)\right]\\
=&\mathbb{E}\left[\frac{1}{6}(W_{S_i+Y_i+Z_i}-W_{S_i})^4-(c_{\text{opt}}+\lambda-\mathbb{E}[Y])(Y_i+Z_i)\right]\\
=&\mathbb{E}\left[\frac{1}{6}(W_{S_i+Y_i+Z_i}-W_{S_i})^4-\frac{\beta}{3}(Y_i+Z_i)\right]\\
=&\mathbb{E}\left[\frac{1}{6}[(W_{S_i+Y_i}-W_{S_i})+(W_{S_i+Y_i+Z_i}-W_{S_i+Y_i})]^4\right.\\
&\left.-\frac{\beta}{3}(Y_i+Z_i)\right].
\end{aligned} \tag{80}$$

Because the Y_i's are *i.i.d.* and the strong Markov property of the Wiener process [55, Theorem 2.16], the expectation in (80) is determined by the control decision Z_i and the information $\mathcal{I}_i'=(W_{S_i+Y_i}-W_{S_i},Y_i,(W_{S_i+Y_i+t}-W_{S_i+Y_i},\ t\geq 0))$. According to [58, p. 252] and [59, Chapter 6], $\mathcal{I}_i'$ is a ***sufficient statistics*** for determining the waiting time Z_i in (37). Therefore, there exists an optimal policy $(Z_0,Z_1,\ldots)$ in which Z_i is determined based on only $\mathcal{I}_i'$. By this, (37) is decomposed into a sequence of per-sample control problems (41). Combining (32), (40), and Lemma 2, yields $c_{\text{opt}}\geq\mathbb{E}[Y]$. Hence, $\beta\geq 0$.

We note that, because the Y_i's are *i.i.d.* and the strong Markov property of the Wiener process, the Z_i's in this optimal policy are *i.i.d.* Similarly, the $(W_{S_i+Y_i+Z_i}-W_{S_i})$'s in this optimal policy are *i.i.d.*

APPENDIX K
PROOF OF LEMMA 4

Case 1: If $b^2\geq\beta$, then (46) tells us that

$$\tau^*=0 \tag{81}$$

and

$$u(x) = \mathbb{E}[g(X_0)|X_0 = x] = g(x) = \beta s - \frac{1}{2}b^4.$$

Case 2: If $b^2 < \beta$, then $\tau^* > 0$ and $(b + W_{\tau^*})^2 = \beta$. Invoking Theorem 8.5.5 in [56], yields

$$\mathbb{E}_x\tau^* = -(\sqrt{\beta} - b)(-\sqrt{\beta} - b) = \beta - b^2. \tag{82}$$

Using this, we can obtain

$$\begin{aligned} u(x) &= \mathbb{E}_x g(X(\tau^*)) \\ &= \beta(s + \mathbb{E}_x\tau^*) - \frac{1}{2}\mathbb{E}_x\left[(b + W_{\tau^*})^4\right] \\ &= \beta(s + \beta - b^2) - \frac{1}{2}\beta^2 \\ &= \beta s + \frac{1}{2}\beta^2 - b^2\beta. \end{aligned}$$

Hence, in Case 2,

$$u(x) - g(x) = \frac{1}{2}\beta^2 - b^2\beta + \frac{1}{2}b^4 = \frac{1}{2}(b^2 - \beta)^2 \geq 0.$$

By combining these two cases, Lemma 4 is proven.

Appendix L

Proof of Lemma 5

The function $u(s, b)$ is continuous differentiable in (s, b). In addition, $\frac{\partial^2}{\partial^2 b}u(s, b)$ is continuous everywhere but at $b = \pm\sqrt{\beta}$. By the Itô-Tanaka-Meyer formula [55, Theorem 7.14 and Corollary 7.35], we obtain that almost surely

$$\begin{aligned} & u(s + t, b + W_t) - u(s, b) \\ =& \int_0^t \frac{\partial}{\partial b}u(s + r, b + W_r)dW_r \\ & + \int_0^t \frac{\partial}{\partial s}u(s + r, b + W_r)dr \\ & + \frac{1}{2}\int_{-\infty}^{\infty} L^a(t)\frac{\partial^2}{\partial b^2}u(s + r, b + a)da, \end{aligned} \tag{83}$$

where $L^a(t)$ is the local time that the Wiener process spends at the level a, i.e.,

$$L^a(t) = \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t 1_{\{|W_s - a| \leq \epsilon\}}ds, \tag{84}$$

and 1_A is the indicator function of event A. By the property of local times of the Wiener process [55, Theorem 6.18], we obtain that almost surely

$$\begin{aligned} & u(s+t, b+W_t) - u(s,b) \\ &= \int_0^t \frac{\partial}{\partial b} u(s+r, b+W_r) dW_r \\ &\quad + \int_0^t \frac{\partial}{\partial s} u(s+r, b+W_r) dr \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial b^2} u(s+r, b+W_r) dr. \end{aligned} \tag{85}$$

Because

$$\frac{\partial}{\partial b} u(s,b) = \begin{cases} -2b^3, & \text{if } b^2 \geq \beta; \\ -2\beta b, & \text{if } b^2 < \beta, \end{cases}$$

we can obtain that for all $t \geq 0$ and all $x = (s,b) \in \mathbb{R}^2$

$$\mathbb{E}_x \left\{ \int_0^t \left[\frac{\partial}{\partial b} u(s+r, b+W_r) \right]^2 dr \right\} < \infty.$$

This and Thoerem 7.11 of [55] imply that $\int_0^t \frac{\partial}{\partial b} u(s+r, b+W_r) dW_r$ is a martingale and

$$\mathbb{E}_x \left[\int_0^t \frac{\partial}{\partial b} u(s+r, b+W_r) dW_r \right] = 0, \ \forall \ t \geq 0. \tag{86}$$

By combining (43), (85), and (86), we get

$$\mathbb{E}_x \left[u(X_t) \right] - u(x) = \mathbb{E}_x \left\{ \int_0^t \left[\frac{\partial}{\partial s} u(X_r) + \frac{1}{2} \frac{\partial^2}{\partial b^2} u(X_r) \right] dr \right\}. \tag{87}$$

It is easy to compute that if $b^2 > \beta$,

$$\frac{\partial}{\partial s} u(s,b) + \frac{1}{2} \frac{\partial^2}{\partial b^2} u(s,b) = \beta - 3b^2 \leq 0;$$

and if $b^2 < \beta$,

$$\frac{\partial}{\partial s} u(s,b) + \frac{1}{2} \frac{\partial^2}{\partial b^2} u(s,b) = \beta - \beta = 0.$$

Hence,

$$\frac{\partial}{\partial s} u(s,b) + \frac{1}{2} \frac{\partial^2}{\partial b^2} u(s,b) \leq 0 \tag{88}$$

for all $(s,b) \in \mathbb{R}^2$ except for $b = \pm\sqrt{\beta}$. Since the Lebesgue measure of those r for which $b + W_r = \pm\sqrt{\beta}$

is zero, we get from (87) and (88) that $\mathbb{E}_x\left[u(X_t)\right] \leq u(x)$ for all $x \in \mathbb{R}^2$ and $t \geq 0$. This completes the proof.

Appendix M
Proof of Corollary 1

Because (46) is the optimal solution to (44), by choosing $s = Y_i$, $b = W_{S_i+Y_i} - W_{S_i}$, and using $W_{S_i+Y_i+t} - W_{S_i}$ to replace W_t, it is immediate that (50) is the optimal solution to (41).

The remaining task is to prove (51) and (52). According to (50) with $\beta \geq 0$, we have

$$\begin{aligned} & W_{S_i+Y_i+Z_i} - W_{S_i} \\ =& \begin{cases} W_{S_i+Y_i} - W_{S_i}, & \text{if } |W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}; \\ \sqrt{\beta}, & \text{if } |W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}. \end{cases} \end{aligned}$$

Hence,

$$\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4] = \mathbb{E}[\max(\beta^2, (W_{S_i+Y_i} - W_{S_i})^4)]. \tag{89}$$

In addition, from (81) and (82) we know that if $|W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}$, then

$$\mathbb{E}[Z_i|Y_i] = 0;$$

otherwise, if $|W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}$, then

$$\mathbb{E}[Z_i|Y_i] = \beta - (W_{S_i+Y_i} - W_{S_i})^2.$$

Hence,

$$\mathbb{E}[Z_i|Y_i] = \max[\beta - (W_{S_i+Y_i} - W_{S_i})^2, 0].$$

Using the law of iterated expectations, the strong Markov property of the Wiener process, and Wald's identity $\mathbb{E}[(W_{S_i+Y_i} - W_{S_i})^2] = \mathbb{E}[Y_i]$, yields

$$\begin{aligned} & \mathbb{E}[Z_i + Y_i] \\ =& \mathbb{E}[\mathbb{E}[Z_i|Y_i] + Y_i] \\ =& \mathbb{E}[\max(\beta - (W_{S_i+Y_i} - W_{S_i})^2, 0) + Y_i] \\ =& \mathbb{E}[\max(\beta - (W_{S_i+Y_i} - W_{S_i})^2, 0) + (W_{S_i+Y_i} - W_{S_i})^2] \\ =& \mathbb{E}[\max(\beta, (W_{S_i+Y_i} - W_{S_i})^2)]. \end{aligned} \tag{90}$$

Finally, because W_t and $W_{S_i+t} - W_{S_i}$ are of the same distribution, (51) and (52) follow from (90) and (89), respectively. This completes the proof.

Appendix N
Proof of Theorem 7

According to [53, Prop. 6.2.5], if we can find $\pi^\star = (Z_0, Z_1, \ldots)$ and $\lambda^\star$ satisfying the following conditions:

$$\pi^\star \in \Pi, \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right] - \frac{1}{f_{\max}} \geq 0, \tag{91}$$

$$\lambda^\star \geq 0, \tag{92}$$

$$L(\pi^\star; \lambda^\star) = \inf_{\pi\in\Pi_1} L(\pi; \lambda^\star), \tag{93}$$

$$\lambda^\star \left\{ \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right] - \frac{1}{f_{\max}} \right\} = 0, \tag{94}$$

then $\pi^\star$ is an optimal solution to the primal problem (34) and $\lambda^\star$ is a geometric multiplier [53] for the primal problem (34). Further, if we can find such $\pi^\star$ and $\lambda^\star$, then the duality gap between (34) and (38) must be zero, because otherwise there is no geometric multiplier [53, Prop. 6.2.3(b)]. We note that (91)-(94) are different from the Karush-Kuhn-Tucker (KKT) conditions because of (93).

The remaining task is to find $\pi^\star$ and $\lambda^\star$ that satisfies (91)-(94). According to Theorem 5 and Corollary 1, the solution $\pi^\star$ to (93) is given by (50) where $\beta = 3(c_{\text{opt}} + \lambda^\star - \mathbb{E}\left[Y\right])$. In addition, as shown in the proof of Theorem 5, the Z_i's in policy $\pi^\star$ are *i.i.d.* Using (91), (92), and (94), the value of $\lambda^\star$ can be obtained by considering two cases: If $\lambda^\star > 0$, because the Z_i's are *i.i.d.*, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right] = \mathbb{E}\left[Y_i + Z_i\right] = \frac{1}{f_{\max}}. \tag{95}$$

If $\lambda^\star = 0$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[Y_i + Z_i\right] = \mathbb{E}\left[Y_i + Z_i\right] \geq \frac{1}{f_{\max}}. \tag{96}$$

Next, we use (95), (96), and $\beta = 3(c_{\text{opt}} + \lambda^\star - \mathbb{E}\left[Y\right])$ to determine $\lambda^\star$. By $p(c_{\text{opt}}) = 0$ and Lemma

2(a), we have $\mathsf{mse}_{\text{opt}} = c_{\text{opt}}$. To compute c_{opt}, we substitute policy $\pi^\star$ and (40) into (32), which yields

$$\begin{aligned} c_{\text{opt}} &= \mathsf{mse}_{\text{opt}} \\ &= \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4 + (Y_i + Z_i)\mathbb{E}[Y]\right]}{6\sum_{i=0}^{n-1}\mathbb{E}\left[Y_i + Z_i\right]} \\ &= \frac{\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right]}{6\mathbb{E}\left[Y_i + Z_i\right]} + \mathbb{E}[Y], \end{aligned} \tag{97}$$

where in the last equation we have used that the Z_i's are *i.i.d.* and the $(W_{S_i+Y_i+Z_i} - W_{S_i})$'s are *i.i.d.*, which were shown in the proof of Theorem 5. Hence, the value of $\beta = 3(c_{\text{opt}} + \lambda^\star - \mathbb{E}[Y])$ can be obtained by considering the following two cases:

Case 1: If $\lambda^\star > 0$, then (97) and (95) imply that

$$\mathbb{E}[Y_i + Z_i] = \frac{1}{f_{\max}}, \tag{98}$$

$$\beta > 3(c_{\text{opt}} - \mathbb{E}[Y]) = \frac{\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right]}{2\mathbb{E}\left[Y_i + Z_i\right]}. \tag{99}$$

Case 2: If $\lambda^\star = 0$, then (97) and (96) imply that

$$\mathbb{E}[Y_i + Z_i] \geq \frac{1}{f_{\max}}, \tag{100}$$

$$\beta = 3(c_{\text{opt}} - \mathbb{E}[Y]) = \frac{\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right]}{2\mathbb{E}\left[Y_i + Z_i\right]}. \tag{101}$$

Combining (98)-(101), yields that β is the root of

$$\mathbb{E}[Y_i + Z_i] = \max\left(\frac{1}{f_{\max}}, \frac{\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4]}{2\beta}\right). \tag{102}$$

Substituting (51) and (52) into (102), we obtain that β is the root of (13). Further, (50) can be rewritten as (9). Hence, if we choose $\pi^\star$ as the sampling policy in (9) and choose $\lambda^\star = \beta/3 - c_{\text{opt}} + \mathbb{E}[Y]$ where β is the root of (13), then $\pi^\star$ and $\lambda^\star$ satisfies (91)-(94). By using the properties of geometric multiplier mentioned above, (9) and (13) is an optimal solution to the primal problem (34).

Because the problems (11), (32), and (34) are equivalent, (9) and (13) is also an optimal solution to (11) and (32).

The optimal objective value $\mathsf{mse}_{\text{opt}}$ is given by (97). Substituting (51) and (52) into (97), (14) follows. This completes the proof.

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