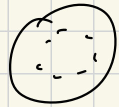
$\boxed{403}$

1 <u>VQ:</u> IT upper bound

2 Gaussian case 

Universal Quant.

3 Non-white source, non-MSE distortion.

(1) <u>Recall</u>: $D_{vec}(R) = \inf\{\mathbb{E}\|X-Y\|^2 : I(X;Y) \le R\}$

<u>Claim</u>: if $\exists$ cpl. between vectors $\underline{X}, \underline{Y}$
$\Rightarrow \exists$ good quantizer.

$i(\underline{X};\underline{Y}) \triangleq \log \frac{P_{\underline{X},\underline{Y}}}{P_{\underline{X}} Q_{\underline{Y}}}$

$\boxed{\text{Th}}$ Consider any $(\underline{X},\underline{Y})$. $\forall M, \gamma$ there exist an $(M+1)$-quantiser s.t.

$$\mathbb{E}\|X - q(X)\|^2 \le \mathbb{E}\|\underline{X}-\underline{Y}\|^2 + e^{-M/\gamma}\mathbb{E}\|\underline{X}\|^2 +$$
$$+ \mathbb{E}\left[\|\underline{X}\|^2 1\{i(\underline{X},\underline{Y}) > \log\gamma\}\right]$$

<u>Pf</u>: $c_1, \dots, c_M \overset{iid}{\sim} P_{\underline{Y}}$, $c_{M+1} = 0$.

$$\int_0^\infty \mathbb{P}[d(x,\hat{x}) > t]\, dt = \mathbb{E}_X\left\{\int_0^{d(x,y_0)} \mathbb{P}[\min_j d(x,c_j) > t]\, dt\right\}.$$

$$\mathbb{P}[\min_j d(x,c_j) > t] = \left(1 - \mathbb{P}[d(x,c_j) \le t]\right)^M.$$

<u>Step 1</u> $\left(1 - \mathbb{P}[d(x_0,\overline{Y}) \le t]\right)^M \le e^{-M\mathbb{P}[d(x_0,\overline{Y}) \le t \mid x = x_0]}$

<u>Steps 2...4</u>:

$\mathbb{P}[d(x_0,\bar{Y}) \le t \mid x = x_0] = \mathbb{E}\left[e^{-i(x_0,Y)} 1\{d \le t\} \mid x = x_0\right]$

by Jensen.

$\le \mathbb{E}\left[e^{-Me^{-i}1\{d\le t\}} \mid x = x_0\right]$

Now consider cases:

$i > \log \gamma$ or $d > t$ $\Rightarrow$ ≤ 1.

$i \leq \log \gamma$ and. $d \leq t$ $\Rightarrow$ $\leq e^{-M/\gamma} 1\{d \leq t\}$

$$(\leq) \ \mathbb{P}[i > \log \gamma \cup d > t \mid x = x_0] + e^{-M/\gamma}.$$

Step 5: Putting it back together:

$$\mathbb{E} d \leq \mathbb{E}_X \int_0^{d(x,y_0)} e^{-M/\gamma} + \mathbb{P}[i(x,Y) > \log \gamma \cup d > t \mid x] \, dt.$$

$$\leq e^{-M/\gamma} \mathbb{E} d(X, y_0) + \mathbb{E}\left[\int_0^{d(x,y_0)} dt \cdot 1\{i > \log \gamma\} + 1\{d > t\}\right]$$

$$= e^{-M/\gamma} \mathbb{E} d(X, y_0) + \mathbb{E} d(X, y_0) 1\{i > \log \gamma\} +$$

$$+ \underbrace{\mathbb{E}[d(X,Y) \wedge d(\cancel{X}, y_0)]}_{\leq \mathbb{E} d(X,Y)}$$

QED.

(16) Asymptotics

$\boxed{\text{Th}}$ Let $\underline{X} = (X_1, \dots, X_n) \overset{iid}{\sim} P_X$ with $\mathbb{E}|X_1|^2 < \infty$.
then $\forall R \; \exists$ VQ of rate $\leq R$ and.

$$\mathbb{E}\|\underline{X} - q(\underline{X})\|^2 \leq n D_{vec}(R) + o(n).$$

Pf: Take coupling $P_{X,Y}$ and create $\hat{X}^n = Y^n$ iid.
take $R > I(X;Y)$

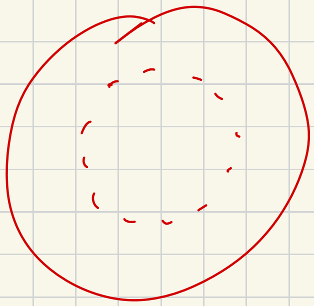
$$\Downarrow$$

$$\mathbb{E}\, d_n(X^n, \hat{X}^n) \leq e^{-M/\gamma} n\mathbb{E}|X_1|^2 + n\mathbb{E}(X - Y)^2.$$
$$+ \mathbb{E}\left[\|X^n\|^2 \; 1\{ i(X^n;Y^n) > \log\gamma\}\right]$$

- $\log \gamma = n(I(X;Y) + \varepsilon)$, $M = 2^{nR}$.
- $e^{-2^{n(R - I - \varepsilon)}} \to 0.$
- $\mathbb{E}d(X,Y) < D$
- $E_n = \{ i(X^n;Y^n) > \log\gamma\}$
- $g_{n,i} = \mathbb{E}\, |X_i|^2 \, 1_{E_n} \to 0$ since $\mathbb{P}[E_n] \to 0$ (uniform integrability)
- Note: from symmetry all $g_{n,i}$ $i \in [n]$ are equal

$\Rightarrow \mathbb{E}\left[\|X^n\|^2 \, 1\{ \;\}\right] = o(n)$

(2) So for iid Gsn can ach 2^{-2R} MSE.
Need to place rep. pts at smaller radius



Why Gsn special?

Th For any P_X with $\mathrm{Var}\, X = 1$

$$2^{-2(R + D(P_X \| \mathcal{N}))} \leq D_{rec}(R) \leq 2^{-2R}$$

Q: How to convert Gsn quant to quant for any P_X?

• Random rotation: Any $X \Rightarrow$ gsn $+ \varepsilon)$

$$X_G^n \longrightarrow \hat{X}_G^n$$
$$|$$
$$Y^n$$

$$\mathbb{E}\|Y^n - X_G^n\|_2^2 \le n\varepsilon \Rightarrow \mathbb{E}\|\hat{X}^n_{G} - Y^n\|_2^2 \le n\left(D + \varepsilon + 2\sqrt{D\varepsilon}\right)$$

$$\mathbb{E}\|a+b\|^2 = \mathbb{E}\|a\|^2 + \mathbb{E}\|b\|^2 + 2\,\mathbb{E}\langle a, b\rangle$$
$$\le nD + n\varepsilon + 2n\sqrt{D\varepsilon}$$

take: $Y^N = UX^N$

Given X^N: $W_2\left(Y^n, X_G^n\right) \le f(N, n)$

$$\left(Y_1^n, Y_{n+1}^{2n}, \dots, Y_{N-n+1}^N\right)$$
$$\left(\hat{Y}_1^n, \dots, \hat{Y}_{N-n+1}^N\right)$$

we know $\mathbb{E}_U \|\hat{Y}_1^n - Y_1^n\|_2^2 \le n(D+\delta)$

$$\Downarrow$$
$$\mathbb{E}_U \|Y^N - \hat{Y}^N\|_2^2 \le N(D+\delta)$$
$$\Updownarrow$$
$$\mathbb{E}_U \|U^{-1}Y^N - U^{-1}\hat{Y}^N\|_2^2 \le N(D+\delta)$$
$$\Updownarrow$$
$$\mathbb{E}_U \|X^N - U^{-1}\hat{Y}^N\|_2^2 \le N(D+\delta)$$
$$\Downarrow$$
$$\exists U \text{ s.t. } \mathbb{E}_{X^N}\|X^N - U^{-1}\hat{Y}^N\|_2^2 \le N(D+\delta)$$

Final Quantizer:
- Send $\|X^N\|$
- UX^N
- Compress n-blocks via Gsn Quant.

Note: this requires sending $\|X^N\|$ separately and then redefining $X^N \leftarrow \frac{X^N}{\|X^N\|}$

+ explain how outliers are removed:

$X \sim (1-\delta)\, U_{[-1,1]} + \delta/2\, \delta_{1/\sqrt{\delta}} + \delta/2\, \delta_{-1/\sqrt{\delta}} \Rightarrow \mathcal{N}(0, 1/3)$

(3) Water filling

Premise: dumbest part of modern quant. algos is uniform rate

- $\underline{X} \sim \mathcal{N}(0, \Sigma)$ metric $\mathbb{E}\, \Delta^T M \Delta$

$$\min \left\{ I(\underline{X}; \underline{Y}) : \mathbb{E}\, (\underline{x}-\underline{y})^T M (\underline{x}-\underline{y}) \le nD \right\}$$

$\underline{X} = QU \Rightarrow$ metric on U.

$$\mathbb{E}\, (u-\hat{u})^T \underbrace{Q^T M Q}_{\text{new } M} (u-\hat{u}).$$

- $\underline{X} \sim \mathcal{N}(0, I_n)$ metric $\mathbb{E}\, \Delta^T M \Delta$

- WLOG: M - diagonal: (rotation)

$$\min \left\{ I(X^n; Y^n) : \underbrace{\mathbb{E} \sum_i \sigma_i^2 (X_i - Y_i)^2}_{\text{dist}} \le nD \right\}$$

$$\ge \sum_i I(X_i; Y_i)$$

tight if $(X_1, Y_1) \perp\!\!\!\perp \ldots \perp\!\!\!\perp (X_n, Y_n)$ and „dist" is unaffected by replacing with $\prod P_{X_i, Y_i}$!

Again, WLOG.

$X_i = Y_i + \tau_i Z_i \qquad Y_i \sim \mathcal{N}(0, 1-\tau_i^2)$

$$\min_{0 \le \tau_i \le 1} \left\{ \sum_i \frac{1}{2} \log 1/\tau_i^2 : \sum_i \sigma_i^2 \tau_i^2 \le nD \right\}$$

Lagrange dual:

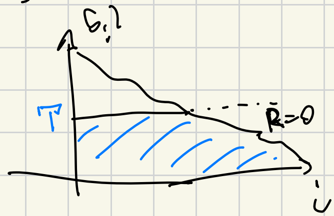
$$\min_{x_i} \sum_i -\ln x_i + \lambda \sigma_i^2 x_i$$

$$-\frac{1}{x_i} + \lambda \sigma_i^2 = 0.$$

$$x_i = \frac{1}{\lambda \sigma_i^2} \wedge 1.$$

$$\Rightarrow \tau_i^2 = \begin{cases} \frac{T}{\sigma_i^2}, & \sigma_i^2 > T = 1/\lambda \\ 1, & \sigma_i^2 < T \end{cases}$$

So i-th direction gets $R_i = \begin{cases} \frac{1}{2} \log \sigma_i^2/T, \\ 0, \end{cases}$



distortion contrib: $\sigma_i^2 \tau_i^2 = \begin{cases} T, & \sigma_i^2 > T \\ \sigma_i^2, & \sigma_i^2 < T \end{cases}$

$\Rightarrow$ to select T fill water until dist. target is met. $\Rightarrow$ get rate allocation.

Note: Generally cannot use this for better GPTQ: need eigen vectors!