Efficient Representation of Large-Alphabet Probability Distributions

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Abstract

A number of engineering and scientific problems require representing and manipulating probability distributions over large alphabets, which we may think of as long vectors of reals summing to 1. In some cases it is required to represent such a vector with only \( b \) bits per entry. A natural choice is to partition the interval \([0, 1]\) into \( 2^b \) uniform bins and quantize entries to each bin independently. We show that a minor modification of this procedure – applying an entrywise non-linear function (compander) \( f(x) \) prior to quantization – yields an extremely effective quantization method. For example, for \( b = 8(16) \) and \( 10^5 \)-sized alphabets, the quality of representation improves from a loss (under KL divergence) of \( 0.5(0.1) \) bits/entry to \( 10^{-4}(10^{-9}) \) bits/entry. Compared to floating point representations, our compander method improves the loss from \( 10^{-1}(10^{-6}) \) to \( 10^{-4}(10^{-9}) \) bits/entry. These numbers hold for both real-world data (word frequencies in books and DNA \( k \)-mer counts) and for synthetic randomly generated distributions. Theoretically, we set up a minimax optimality criterion and show that the compander \( f(x) \propto \text{ArcSinh}(\sqrt{(1/2)(K \log K)}x) \) achieves near-optimal performance, attaining a KL-quantization loss of \( \approx 2^{-2b} \log^2 K \) for a \( K \)-letter alphabet and \( b \to \infty \). Interestingly, a similar minimax criterion for the quadratic loss on the hypercube shows optimality of the standard uniform quantizer. This suggests that the \( \text{ArcSinh} \) quantizer is as fundamental for KL-distortion as the uniform quantizer for quadratic distortion.

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I. COMPANDER BASICS AND DEFINITIONS

Consider the problem of quantizing the probability simplex \( \Delta_{K-1} = \{ x \in \mathbb{R}^K : x \geq 0, \sum_i x_i = 1 \} \) of alphabet size \( K \), i.e. of finding a finite subset \( \mathcal{Y} \subseteq \Delta_{K-1} \) to represent the entire simplex. Each \( x \in \Delta_{K-1} \) is associated with some \( y = y(x) \in \mathcal{Y} \), and the objective is to find a set \( \mathcal{Y} \) and an assignment such that the difference between the values \( x \in \Delta_{K-1} \) and their representations \( y \in \mathcal{Y} \) are minimized; while this can be made arbitrarily small by making \( \mathcal{Y} \) arbitrarily large, the objective is to do this efficiently for any given fixed size \( M \). Since \( x, y \in \Delta_{K-1} \), they both represent probability distributions over a size-\( K \) alphabet. Hence, a natural way to measure the quality of the quantization is to use the KL (Kullback-Leibler) divergence \( D_{\text{KL}}(x \| y) \), which corresponds to the excess code length for lossless compression and is commonly used as a way to compare probability distributions. (Note that we want to minimize the KL divergence.)

While one can consider how to best represent the vector \( x \) as a whole, in this paper we consider only scalar quantization methods in which each element \( x_j \) of \( x \) is handled separately, since we showed in [1] that for Dirichlet priors on the simplex, methods using scalar quantization perform nearly as well as optimal vector quantization. Scalar quantization is also typically simpler and faster to use, and can be parallelized easily. Our scalar quantizer is based on companders (portmanteau of ‘compressor’ and ‘expander’), a simple, powerful and flexible technique first explored by Bennett in 1948 [2] in which the value \( x_j \) is passed through a nonlinear function \( f \) before being quantized. We discuss the background in greater depth in Section III, in particular to compare certain previously-known results to Theorem 1.

In what follows, \( \log \) is always base-\( e \) unless otherwise specified.

1) Encoding: Companders require two things: a monotonically increasing function \( f : [0, 1] \rightarrow [0, 1] \) (we denote the set of such functions as \( \mathcal{F} \)) and an integer \( N \) representing the number of quantization levels, or granularity. To simplify the problem and algorithm, we use the same \( f \) for each element of the vector \( x = (x_1, \ldots, x_K) \in \Delta_{K-1} \) (see Remark 1). To quantize \( x \in [0, 1] \), the compander computes \( f(x) \) and applies a uniform quantizer with \( N \) levels, i.e. encoding \( x \) to \( n_N(x) \) \( \in [N] \) if \( f(x) \in \left( \frac{n-1}{N}, \frac{n}{N} \right] \); this is equivalent to \( n_N(x) = \lfloor f(x)N \rfloor \).

\( ^1 \)While the alphabet has \( K \) letters, \( \Delta_{K-1} \) is \( K - 1 \) dimensional due to the constraint that the entries sum to 1.
This encoding system partitions \([0, 1]\) into bins \(I^{(n)}\):

\[
x \in I^{(n)} = f^{-1}\left(\left(\frac{n-1}{N}, \frac{n}{N}\right)\right) \iff n_N(x) = n
\]

where \(f^{-1}\) denotes the preimage under \(f\).

2) Decoding: To decode \(n \in [N]\), we pick some \(\hat{y}(n) \in I^{(n)}\) to represent all \(x \in I^{(n)}\); for a given \(x\) (at granularity \(N\)), its representation is denoted \(\hat{y}(x) = \hat{y}(n_N(x))\). This is usually the midpoint of the bin or, if \(x\) is drawn randomly from a prior\(^2\) \(p\), the centroid (the mean within bin \(I^{(n)}\)). The midpoint of \(I^{(n)}\) can be computed as \((f^{-1}(\frac{n-1}{N}) + f^{-1}(\frac{n}{N}))/2\) and we denote it as \(\tilde{y}(n)\). The centroid of \(I^{(n)}\) is defined as

\[
\tilde{y}(n) = \mathbb{E}_{X \sim p}[X \mid X \in I^{(n)}].
\]

We will discuss this in greater detail in Section I-4.

Handling each element of \(x\) separately means the decoded values may not sum to 1, so we normalize the vector after decoding. Thus, if \(x\) is the input,

\[
y_i(x) = \frac{\hat{y}(x_i)}{\sum_{j=1}^{K} \hat{y}(x_j)}
\]

the vector \(y = y(x) = (y_1(x), \ldots, y_K(x)) \in \Delta_{K-1}\) is the output of the compander.\(^3\) We refer to \(\hat{y} = \hat{y}(x) = (\hat{y}(x_1), \ldots, \hat{y}(x_K))\) as the raw reconstruction of \(x\), and \(y\) as the normalized reconstruction. (We generally use the \(\hat{\ }\) accent to mark values dependent on the raw reconstruction.) If the raw reconstruction uses centroid decoding, we likewise denote it using \(\tilde{y} = \tilde{y}(x) = (\tilde{y}(x_1), \ldots, \tilde{y}(x_K))\).

Thus, any \(x \in \Delta_{K-1}\) requires \(K\lceil \log_2 N \rceil\) bits to store; to encode and decode, only \(f\) and \(N\) need to be stored (as well as the prior if using centroid decoding). Another major advantage is that a single \(f\) can work well over many or all choices of \(N\), making the design more flexible.

3) KL divergence loss: The loss incurred by representing \(x\) as \(y(x)\) is the KL divergence

\[
D_{KL}(x \parallel y(x)) = \sum_{i=1}^{K} x_i \log \frac{x_i}{y_i(x)}.
\]

Although this loss function has some strange properties (for instance \(D_{KL}(x \parallel y) \neq D_{KL}(y \parallel x)\) and it doesn’t obey the triangle inequality) it measures the amount of ‘mis-representation’ created

\(^2\)Priors on \(\Delta_{K-1}\) induce priors over \([0, 1]\) for each entry.

\(^3\)This notation reflects the fact that each entry of the normalized reconstruction depends on the entire vector \(x\) due to the normalization step.
by representing the probability vector $x$ by another probability vector $y(x)$, and is hence a natural thing to minimize. In particular, it represents the excess code length created by trying to encode the output of $x$ using a code built for $y(x)$, as well as having connections to hypothesis testing (a natural setting in which the ‘difference’ between probability distributions is studied).

4) **Distributions from a prior:** Much of our work concerns the case where $x \in \Delta_{K-1}$ is drawn from some prior $P_x$ (to be commonly denoted as simply $P$). Using a single $f$ for each entry means we can WLOG assume that $P$ is symmetric over the alphabet, as permuting the letter indices does not affect the KL divergence. We denote the set of such priors as $\mathcal{P}_K^\Delta$.

**Remark 1.** In principle, given a nonsymmetric prior $P_x$ over $\Delta_{K-1}$ with marginals $p_1, \ldots, p_K$, we could quantize each letter’s value with a different compander $f_1, \ldots, f_K$, giving more accuracy than using a single $f$ (at the cost of higher complexity). However, the symmetrization of $P_x$ over the letters (by permuting the indices randomly after generating $X \sim P_x$) yields a prior in $\mathcal{P}_K^\Delta$ on which any single $f$ will have the same (overall) performance and cannot be improved on by using varying $f_i$. Thus, considering symmetric $P_x$ suffices to derive our minimax compander (which performs well across all $P_x$ whose marginals are continuous probability distributions).

While the random probability vector comes from a prior $P \in \mathcal{P}_K^\Delta$, our analysis will rely on decomposing the loss so we can deal with one letter at a time. Hence, we work with the marginals $p$ of $P$ (which are identical since $P$ is symmetric), which we refer to as *single-letter distributions* and are probability distributions over $[0, 1]$.

We let $\mathcal{P}$ denote the class of probability distributions over $[0, 1]$ that are absolutely continuous with respect to the Lebesgue measure. We denote elements of $\mathcal{P}$ by their probability density functions (PDF), e.g. $p \in \mathcal{P}$; the cumulative distribution function (CDF) associated with $p$ is denoted $F_p$ and satisfies $F_p'(x) = p(x)$ and $F_p(x) = \int_0^x p(t) dt$ (since $F_p$ is monotonic, its derivative exists almost everywhere). Note that while $p \in \mathcal{P}$ does not have to be continuous, its CDF $F_p$ must be absolutely continuous. Following common terminology [3], we refer to such probability distributions as *continuous*.

Let $\mathcal{P}_{1/K} = \{p \in \mathcal{P} : \mathbb{E}_{X \sim p}[X] = 1/K\}$. Note that $P \in \mathcal{P}_K^\Delta$ implies its marginals are in $\mathcal{P}_{1/K}$.

5) **Expected loss and preliminary results:** For $P \in \mathcal{P}_K^\Delta$, $f \in \mathcal{F}$ and granularity $N$, we define the expected loss:

$$
\mathcal{L}_K(P, f, N) = \mathbb{E}_{X \sim p}[D_{KL}(X\|y(X))].
$$

(5)
This is the value we want to minimize.

**Remark 2.** While $X$ and $y(X)$ are random, they are also probability vectors. The KL divergence $D_{\text{KL}}(X \| y(X))$ is the divergence between $X$ and $y(X)$ themselves, not the prior distributions over $\triangle_{K-1}$ they are drawn from.

Note that $\mathcal{L}_K(P, f, N)$ can almost be decomposed into a sum of $K$ separate expected values (one per entry), except the normalization step (3) depends on the vector as a whole. Hence, we define the raw loss:

$$\tilde{\mathcal{L}}_K(P, f, N) = \mathbb{E}_{X \sim P} \left[ \sum_{i=1}^{K} X_i \log \left( \frac{X_i}{\tilde{y}(X_i)} \right) \right]. \quad (6)$$

We also define for $p \in \mathcal{P}$, the single-letter loss as

$$\tilde{\mathcal{L}}(p, f, N) = \mathbb{E}_{X \sim p} \left[ X \log \left( \frac{X}{\tilde{y}(X)} \right) \right] \quad (7)$$

The raw loss is useful because it bounds the (normalized) expected loss and is decomposable into single-letter losses. Note that both raw and single-letter loss are defined with centroid decoding.

**Proposition 1.** For $P \in \mathcal{P}_K$ with marginals $p$,

$$\mathcal{L}_K(P, f, N) \leq \tilde{\mathcal{L}}_K(P, f, N) = K \tilde{\mathcal{L}}(p, f, N). \quad (8)$$

**Proof.** Since

$$\mathcal{L}(P, f, N) = \mathbb{E}_{X \sim P} D_{\text{KL}}(X \| Y) \quad (9)$$

$$= \tilde{\mathcal{L}}_K(P, f, N) + \mathbb{E}_{X \sim P} \left[ \log \left( \sum_{k=1}^{K} \tilde{Y}_k \right) \right]. \quad (10)$$

Since $\mathbb{E}[\tilde{Y}_k] = \mathbb{E}[X_k]$ for all $k$, $\sum_{k=1}^{K} \mathbb{E}[\tilde{Y}_k] = \sum_{k=1}^{K} \mathbb{E}[X_k] = 1$. Because $\log$ is concave, by Jensen’s Inequality

$$\mathbb{E}_{X \sim P} \left[ \log \left( \sum_{k=1}^{K} \tilde{Y}_k \right) \right] \leq \log \left( \mathbb{E} \left[ \sum_{k=1}^{K} \tilde{Y}_k \right] \right) = \log(1) = 0 \quad (11)$$

and we are done. \qed

To derive our results about worst-case priors (for instance, Theorem 3), we will also be interested in $\tilde{\mathcal{L}}(p, f, N)$ even when $p$ is not known to be a marginal of some $P \in \mathcal{P}_K$.

**Remark 3.** Though one can define raw loss and single-letter loss without centroid decoding (replacing $\tilde{y}$ in (6) or (7) with another decoding method $\hat{y}$), doing so removes much of their
This is because the resulting expected loss can be dominated by the difference between \( E[X] \) and \( E[\hat{y}(X)] \), potentially even making it negative; specifically, the Taylor expansion of \( X \log(X/\hat{y}(X)) \) has \( X - \hat{y}(X) \) in its first term, which can have negative expectation.

While this can make the expected ‘raw loss’ general decoding negative, it cannot be exploited to make the (normalized) expected loss negative\(^4\) because the normalization step \( y_i(X) = \hat{y}(X_i)/\sum_j \hat{y}(X_j) \) cancels out the problematic term. Centroid decoding avoids this problem by ensuring \( E[X] = E[\hat{y}(X)] \), removing the issue.

As we will show, when \( N \) is large these values are roughly proportional to \( N^{-2} \) (for well-chosen \( f \)) and hence we define the asymptotic single-letter loss:

\[
\tilde{L}(p, f) = \lim_{N \to \infty} N^2 \tilde{L}(p, f, N).
\]

We similarly define \( \tilde{L}_K(P, f) \) and \( L_K(P, f) \). While the limit in (12) does not exist for every \( p, f \), we will show that one can ensure it exists by choosing an appropriate \( f \) (which works against any \( p \in \mathcal{P} \)), and cannot gain much by not doing so.

II. MAIN RESULTS

We demonstrate, theoretically and experimentally, the efficacy of companding for quantizing probability distributions with KL divergence loss. Though our theoretical results are asymptotic as \( N \to \infty \) and focus on raw loss (which uses centroid decoding), the experimental loss of the various companders with midpoint decoding (normalized) closely tracks the raw loss predicted theoretically, even for quantization levels as low as \( N = 256 \) (8 bits per value).

A. Theoretical Results

We first define a set of ‘well-behaved’ companders:

**Definition 1.** Define \( \mathcal{F}^+ \subseteq \mathcal{F} \) to be the set of \( f \) such that for each \( f \) there exist constants \( c > 0 \) and \( \alpha \in (0, 1/2] \) for which \( f(x) - cx^\alpha \) is still monotonically increasing.

This is equivalent to \( f'(x) \geq c \alpha x^{\alpha - 1} \) for all \( x \) where \( f' \) is defined (which is almost everywhere since \( f \) is monotonic). We also define the following function on \( p \) and \( f \):

\(^4\) As expected, since negative KL loss between probability distributions is not possible.
**Definition 2.** For $p \in \mathcal{P}$ and $f \in \mathcal{F}$, let

$$L^t(p, f) = \frac{1}{24} \int_0^1 p(x)f'(x)^{-2}x^{-1} \, dx$$

$$= \int_{[0,1]} \frac{1}{24} f'(x)^{-2}x^{-1} \, dp .$$

Then the asymptotic loss of $f$ against $p$ satisfies:

**Theorem 1.** For any $p \in \mathcal{P}$ and $f \in \mathcal{F}$,

$$\lim \inf_{N \to \infty} N^2 \tilde{L}(p, f, N) \geq L^t(p, f).$$

Furthermore, if $f \in \mathcal{F}^*$ then an exact result holds:

$$\tilde{L}(p, f) = L^t(p, f) < \infty .$$

Essentially, as long as you select a compander $f$ from the ‘well-behaved’ set $\mathcal{F}^*$, for large granularities $N$ the single-letter loss will be approximated by

$$\tilde{L}(p, f, N) \approx N^{-2} L^t(p, f).$$

The lower bound (15) shows that even for $f \notin \mathcal{F}^*$,

$$\tilde{L}(p, f, N) \approx N^{-2} L^t(p, f)$$

i.e. the quantizer cannot do better than $N^{-2} L^t(p, f)$ loss (as $N \to \infty$) by choosing $f \notin \mathcal{F}^*$.

**Theorem 2.** The best loss against source $p \in \mathcal{P}$ is

$$\inf_{f \in \mathcal{F}} \tilde{L}(p, f) = \min_{f \in \mathcal{F}} L^t(p, f) = \frac{1}{24} \left( \int_0^1 (p(x)x^{-1})^{1/3} \, dx \right)^3$$

where the optimal compander against $p$ is

$$f_p(x) = \arg \min_{f \in \mathcal{F}} L^t(p, f) = \frac{\int_0^1 (p(t)t^{-1})^{1/3} \, dt}{\int_0^1 (p(t)t^{-1})^{1/3} \, dt}$$

(satisfying $f'_p(x) \propto (p(x)x^{-1})^{1/3}$).

If $f_p \in \mathcal{F}^*$, it achieves the value from (19) and (as minimizer of $L^t(p, f)$) it has the smallest asymptotic loss against $p$. If $f_p \notin \mathcal{F}^*$, we use the following:

**Proposition 2.** For any $f \in \mathcal{F}$ and $\delta \in (0, 1]$, the functions

$$f_{p, \delta}(x) = (1 - \delta)f_p(x) + \delta x^{1/2}$$

(21)
satisfy \( f_{p,\delta} \in \mathcal{F}^\dagger \) and

\[
\lim_{\delta \to 0} \tilde{L}(p, f_{p,\delta}) = \lim_{\delta \to 0} L^\dagger(p, f_{p,\delta}) = L(p, f_p). \tag{22}
\]

Thus, you can imitate \( f_p \) arbitrarily closely by mixing it with \( x^{1/2} \) (or any \( x^\alpha \) for \( \alpha \in (0, 1/2] \) will also work); the mixture is by definition in \( \mathcal{F}^\dagger \). This (with Theorem 1) shows there is no real advantage to using \( f \notin \mathcal{F}^\dagger \), so we restrict our analysis to \( f \in \mathcal{F}^\dagger \), for which (13) holds.

Since the prior \( P \) generating \( x \) is usually unknown, we give a compander which performs well against any prior. This is closely linked to the following probability density on \( r_0, 1 \):

**Proposition 3.** For alphabet size \( K > 4 \), there is a unique \( c_K \in [\frac{1}{4}, \frac{3}{4}] \) such that if \( a_K = (4/(c_K K \log K + 1))^{1/3} \) and \( b_K = 4/a^2_K - a_K \), then the following density is in \( \mathcal{P}_{1/K} \):

\[
p^*_K(x) = (a_K x^{1/3} + b_K x^{4/3})^{-3/2} \tag{23}
\]

Furthermore, \( \lim_{K \to \infty} c_K = 1/2 \).

We call \( p^*_K \) the maximin single-letter density.

The optimal compander against \( p^*_K \) is the minimax compander:

\[
f^*_K(x) = \frac{\text{ArcSinh}(\sqrt{c_K(K \log K)} x)}{\text{ArcSinh}(\sqrt{c_K K \log K})}. \tag{24}
\]

Note that \( f^*_K \in \mathcal{F}^\dagger \) (see Remark 4). The source \( p^*_K \) and compander \( f^*_K \) then form an ‘equilibrium’:

**Theorem 3.** The minimax compander \( f^*_K \) and maximin single-letter density \( p^*_K \) satisfy

\[
\sup_{p \in \mathcal{P}_{1/K}} \tilde{L}(p, f^*_K) = \inf_{f \in \mathcal{F}^\dagger} \sup_{p \in \mathcal{P}_{1/K}} \tilde{L}(p, f) \tag{25}
\]

\[
= \sup_{p \in \mathcal{P}_{1/K}} \inf_{f \in \mathcal{F}^\dagger} \tilde{L}(p, f) = \inf_{f \in \mathcal{F}^\dagger} \tilde{L}(p^*_K, f) \tag{26}
\]

which is equal to \( \tilde{L}(p^*_K, f^*_K) \) and satisfies

\[
\tilde{L}(p^*_K, f^*_K) = \Theta(K^{-1} \log^2 K). \tag{27}
\]

This theorem importantly implies the following:

**Corollary 1.** For any prior \( P \in \mathcal{P}_K^\triangle \),

\[
\mathcal{L}_K(P, f^*_K) \leq \tilde{\mathcal{L}}_K(P, f^*_K) = \Theta(\log^2 K). \tag{28}
\]
There also exists $P^* \in \mathcal{P}_K^\Delta$ such that for any $P \in \mathcal{P}_K^\Delta$

$$\inf_{f \in \mathcal{F}} \tilde{L}_K(P^*, f) \geq \frac{K - 1}{2K} \tilde{L}_K(P, f^*_K) = \Theta(\log^2 K).$$  

(29)

The constant-factor gap exists because $P^* \in \mathcal{P}_K^\Delta$ is a stronger constraint than $p^*_K \in \mathcal{P}_{1/K}$.

For any $K$, $c_K$ can be approximated numerically. To simplify the quantizer, we can use $c_K \approx \frac{1}{2}$ for large $K$ to get the approximate minimax compander:

$$f^*_K(x) = \frac{\text{ArcSinh} \left( \sqrt{(1/2)(K \log K)} \frac{1}{x} \right)}{\text{ArcSinh} \left( \sqrt{(1/2)K \log K} \right)}.$$  

(30)

This is close to optimal without needing to compute $c_K$:

**Theorem 4.** Suppose that $K$ is sufficiently large so that $c_K \in \left[ \frac{1}{2(1 + \varepsilon)}, \frac{1 + \varepsilon}{2} \right]$. Then for any $p \in \mathcal{P}$, $\tilde{L}(p, f^*_K) \leq (1 + \varepsilon) \tilde{L}(p, f^*_K)$.  

(31)

**Remark 4.** While $f^*_K$ and $f^**_K$ might appear complicated, $\text{ArcSinh}(\sqrt{z}) = \log(\sqrt{z} + \sqrt{z + 1})$ is fairly simple. Taking the Taylor expansion also confirms that they are in $\mathcal{F}^\dagger$.

Note that (20) (Theorem 2) suggests that the natural form of an optimal compander against $p$ is a normalized incomplete integral, which is hard to use. Thus, the closed-form expressions of $f^*_K$ and $f^**_K$ is a welcome surprise.

Using the minimax compander $f^*_K$ or approximate minimax compander $f^**_K$ on $P \in \mathcal{P}_K^\Delta$ with granularity $N$, we have a bound on the average KL divergence:

$$\mathbb{E}_{X \sim P} [D_{KL}(X \| Y)] = O \left( N^{-2} \log^2 K \right).$$  

(32)

**Remark 5.** If we use the uniform quantizer, by contrast, there exists a $P \in \mathcal{P}_K^\Delta$ where we have

$$\mathbb{E}_{X \sim P} [D_{KL}(X \| Y)] = \Theta \left( K^2 N^{-2} \log N \right).$$  

(33)

The dependence on $N$ is greater than $N^{-2}$ (thus yielding $\tilde{L}(p, f) = \infty$) and the dependence on $K$ is quadratic. This shows that, generally speaking, the uniform quantizer is very bad since it risks increasingly high loss relative to more suitable companders; specifically, by Theorem 1, any $f \in \mathcal{F}^\dagger$ is guaranteed to have loss $\propto N^{-2}$ for all priors.

Additionally, (33) is by no means the worst possible for the uniform quantizer; it is just sufficient to show how it may perform much, much worse than well-chosen companders.

**Remark 6.** Instead of the KL divergence loss on the simplex, we can do a similar analysis to find the minimax compander for $L_2^2$ loss on the unit hypercube. It turns out, the solution is
given by the identity function \( f(x) = x \) corresponding to the standard (non-companded) uniform quantization. (See Section VI.)

The above are all ‘average case’ results, where \( X \) is drawn from a prior \( P \) (which is fixed as \( N \to \infty \)). In the worst-case problem, \( x \) is chosen to maximize loss and can depend on \( N \) (decoding is midpoint by default since there is no prior and hence no well-defined centroid):

**Theorem 5.** The minimax compander with midpoint decoding achieves worst-case loss

\[
\max_{\Delta_{K-1}} D_{\text{KL}}(x\|y) = O \left( N^{-2} \log^2 K \right).
\]

**Remark 7.** When \( b \) is the number of bits used to quantize each value in the probability vector, using the approximate minimax compander yields a worst-case loss on the order of \( 2^{-2b} \log^2 K \). Using optimal vector quantization (as explored in [4]), the loss is an order between \( 2^{-2b} K^{K-1} \) and \( 2^{-2b} K^{K-1} \log K \). Thus, our result using companders is within a factor \( 2^{-2b} K^{K-1} \log K \) of the optimal loss. (The bound \( 2^{-2b} K^{K-1} \log K \) is not associated with an explicit quantization scheme. One is only shown to exist.)

### B. Experimental Results

We compare the performance of five quantizers, with granularities \( N = 2^8 \) and \( N = 2^{16} \), on three types of datasets: (i) random synthetic distributions drawn from the uniform prior over the simplex;\(^5\) (ii) frequency of words in books;\(^6\) and (iii) frequency of \( k \)-mers in DNA.\(^7\) Our quantizers are:

- **Approximate Minimax Compander:** As given by (30). Using the approximate minimax compander is much simpler than the minimax compander since the constant \( c_K \) does not need to be computed, and by Theorem 4 it has almost identical performance for large \( K \).
- **Truncation:** Uniform quantization (equivalent to \( f(x) = x \)), which truncates the least significant bits. This is the natural way of quantizing values in \([0, 1]\).

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\(^5\)We draw 1000 random samples and average over these for our results.

\(^6\)These frequencies are computed from text available on the Natural Language Toolkit (NLTK) libraries for Python. For each text, we get tokens (single words or punctuation) from each text and simply count the occurrence of each token.

\(^7\)For a given sequence of DNA, the set of \( k \)-mers are the set of length \( k \) substrings which appear in the sequence. We use the human genome as the source for our DNA sequences. Parts of the sequence marked as repeats are removed.
Float and bfloat16: For 8-bit encodings \((N = 2^8)\), we use a floating point implementation which allocates 4 bits to the exponent and 4 bits to the mantissa. For 16-bit encodings \((N = 2^{16})\), we use bfloat16, a standard which is commonly used in machine learning [5].

Exponential Density Interval (EDI): This is the quantization method we used in an achievability proof in [1]. It is designed for the uniform prior over the simplex.

Power Compander: The compander is \(f(x) = x^s\), a natural class of functions from \([0, 1]\) to \([0, 1]\). We optimize \(s\) and find that \(s = \frac{1}{\log K}\) asymptotically minimizes KL divergence, and also gives close to the best performance empirically. To see the effects of different powers \(s\) on the performance of the power compander, see Figure 1.

Because a well-defined prior doesn’t always exist for these datasets (and for simplicity) we use midpoint decoding for all the companders. When a probability value of exactly 0 appears, we do not use companding and instead quantize the value to 0, i.e. the value 0 has its own bin.

Our main experimental results are given in Figure 2, showing the KL divergence between the original distribution \(x\) and its quantized version \(y\) versus alphabet size \(K\). The approximate minimax compander performs well against all sources. For truncation, the KL divergence increases with \(K\) and is generally fairly large. The EDI quantizer works well for the synthetic uniform prior (as it should), but for real-world datasets like word frequency in books, it performs
Fig. 2. Plot comparing the performance of the truncation compander, the EDI compander, floating points, the power compander, and the approximate minimax compander (30) on probability distributions of various sizes.

badly (sometimes even worse than truncation). The power compander performs similarly to the minimax compander and is worse only by a constant.\(^8\)

The experiments demonstrate that the approximate minimax compander achieves low loss on the entire ensemble of data (even for relatively small granularity, such as \(N = 256\)) and outperforms both truncation and floating-point implementations on the same number of bits. Additionally, its closed-form expression (and entrywise application) makes it simple to implement

\(^8\)Theorem 5 also holds for the power compander with different constants.
and computationally inexpensive. Thus it can be easily added to existing systems to lower storage requirements at little or no cost to fidelity.

C. Paper Organization

We provide background and discuss previous work on companders in Section III. We prove Theorem 1 in Section IV (though proofs of some lemmas and propositions leading up to it are given in Section A). In Section V, we optimize over (13) to get the maximin single-letter distribution and minimax compander, thus showing Theorems 2 and 3 and Corollary 1 (leaving Theorem 4 for Section C-C). We prove Theorem 5 in Section D. In Section VI we discuss companders for losses other than KL divergence.

III. BACKGROUND

Companders (also spelled “compandors”) were introduced by Bennett in 1948 [2] as a way to quantize speech signals, where it is advantageous to give finer quantization levels to weaker signals and coarser levels to larger signals. Bennett gives a first order approximation that the mean-square error in this system is given by

\[ \frac{1}{12N^2} \int_{a}^{b} \frac{p(x)}{(f'(x))^2} dx \]  

(35)

where \( N \) is the number quantization levels, \( a \) and \( b \) are the minimum and maximum values of the input signal, \( p \) is the probability density of the input signal, and \( f' \) is the slope of the compressor function placed before the uniform quantization. This formula is similar to our (13) except that we have an extra \( x^{-1} \) since we are working with KL divergence. Others have expanded on this line of work. In [6], the authors studied the same problem and determined the optimal compressor under mean-square error, a result which parallels our result (19). However, results like those in [2], [6] are stated either as first order approximations or make simplifying assumptions. For example, in [6], the authors state that they assume the values \( \hat{y}_{(n)} \) are close together enough that probabilities over each bin are approximately uniform. Their results proceed from this assumption. Our work rigorously addresses these assumptions, showing they hold under very general conditions.

Generalizations of Bennett’s formula are also studied when instead of mean-square error, the loss is the expected \( r \)th moment loss \( \mathbb{E} \cdot \| \cdot \|^r \). This is computed for vectors of length \( K \) in [7] and [8]. The typical examples of companders used in engineering are \( \mu \)-law and \( A \)-law companders
[9]. For the $\mu$-law compander, [6] and [10] argue that for a large enough constant $\mu$, in the case of mean-squared error, the distortion becomes independent of the signal.

Quantizing probability distributions is a common topic, though typically the loss function is a norm and not KL divergence [11]. Quantizing for KL divergence is considered in our earlier work [1], where we focus on average KL loss for Dirichlet priors.

A similar problem to quantizing under KL divergence is information $k$-means. This is the problem of clustering $n$ points $a_i$ to $k$ centers $\hat{a}_j$ to minimize the KL divergences between the points and their associated centers. Theoretical aspects of this are explored in [12] and [13]. Information $k$-means has been implemented for several different applications [14], [15], [16]. There are also other works that study clustering with a slightly different but related metric [17], [18], [19]; however, the focus of these works is to analyze data rather than reduce storage.

IV. ASYMPTOTIC SINGLE-LETTER LOSS

In this section we give the proof of Theorem 1 (though the proofs of some lemmas must be sketched). We use the following notation:

- Given an interval $I$ we define $\bar{y}_I$ to be its midpoint and $r_I$ to be its width, so that by definition
  \[
  I = [\bar{y}_I - r_I/2, \bar{y}_I + r_I/2].
  \] (36)
  Note that if $I \subseteq [0,1]$ then $r_I \leq 2\bar{y}_I$.

- Given probability distribution $p$ and interval $I$, $p|_I$ denotes $p$ restricted to $I$, i.e. $X \sim p|_I$ is the same as $X \sim p$ conditioned on $X \in I$. We also define the probability mass of $I$ under $p$ as $\pi_{p,I} = \mathbb{P}_{X \sim p}[X \in I]$. If $\pi_{p,I} = 0$, we let $p|_I$ be uniform on $I$ by default.

- Given probability distribution $p$ and interval $I$, we denote the centroid of $I$ under $p$ as
  \[
  \bar{y}_{p,I} := \mathbb{E}_{X \sim p|_I}[X] = \mathbb{E}_{X \sim p}[X \mid X \in I].
  \] (37)
  If this is undefined because $\mathbb{P}_{X \sim p}[X \in I] = 0$ then by the convention on $p|_I$, we have $\bar{y}_{p,I} = \bar{y}_I$ (the centroid under a uniform distribution is the midpoint).

- Given two probability distributions $p, q$ (over the same domain), we define their Kolmogorov-Smirnov distance (KS distance) to be
  \[
  d_{KS}(p, q) = \|F_p - F_q\|_\infty = \sup_x |F_p(x) - F_q(x)|
  \] (38)
  (recall that $F_p, F_q$ are the CDFs of $p, q$).
We use standard order-of-growth notation (which are also used in Section II). We review these definitions here for clarity, especially as we will use some of the rarer concepts (in particular, small-\( \omega \)). For a parameter \( t \) and functions \( a(t), b(t) \), we say:

\[
a(t) = O(b(t)) \iff \limsup_{t \to \infty} |a(t)/b(t)| < \infty
\]

\[
a(t) = \Omega(b(t)) \iff \liminf_{t \to \infty} |a(t)/b(t)| > 0
\]

\[
a(t) = \Theta(b(t)) \iff a(t) = O(b(t)), a(t) = \Omega(b(t)).
\]

We use small-\( o \) notation to denote the strict versions of these:

\[
a(t) = o(b(t)) \iff \lim_{t \to \infty} |a(t)/b(t)| = 0
\]

\[
a(t) = \omega(b(t)) \iff \lim_{t \to \infty} |a(t)/b(t)| = \infty.
\]

Sometimes we will want to indicate order-of-growth as \( t \to 0 \) instead of \( t \to \infty \); this will be explicitly mentioned in that case.

When \( I = I(n) \) is a bin of the compander, we can replace it with \( (n) \) in the notation, i.e. \( \bar{y}_{(n)} = \bar{y}_{I(n)} \) (so the midpoint of the bin containing \( x \) at granularity \( N \) is denoted \( \bar{y}_{(n,N(x))} \) and the width of the bin is \( r_{(n,N(x))} \)). When \( I \) and/or \( p \) are fixed, we will sometimes drop them from the notation, i.e. \( \bar{y}_I \) or even just \( \bar{y} \) to denote the centroid of \( I \) under \( p \).

A. The Local Loss Function

One key to the proof is the following perspective: instead of considering \( X \sim p \) directly, we (equivalently) first select bin \( I(n) \) with probability \( \pi_{p,(n)} \), and then select \( X \sim p|_{(n)} \). The expected loss can then be considered within bin \( I(n) \). This makes it useful to define:

**Definition 3.** Given probability measure \( p \) and interval \( I \subseteq [0,1] \), the single-interval loss of \( I \) under \( p \) is

\[
\ell_{p,I} = \mathbb{E}_{X \sim p|_I} [X \log(X/\bar{y}_{p,I})].
\]

As before, if \( p \) and/or \( I \) is fixed and clear, we can drop it from the notation (and if \( I = I(n) \) is a bin, we can denote the local loss as \( \ell_{p,(n)} \)). This can be interpreted as follows: if we quantize
all \( x \in I \) to the centroid \( \bar{y} \), then \( \ell_{p,I} \) is the expected loss of \( X \sim p \) conditioned on \( X \in I \). Thus the values of \( \ell_{p,(n)} \) can be used as an alternate means of computing the single-letter loss:

\[
\tilde{L}(p, f, N) = \mathbb{E}_{X \sim p}[X \log(X/\bar{y}(X))] = \sum_{n=1}^{N} \pi_{p,(n)} \mathbb{E}_{X \sim p,(n)}[X \log(X/\bar{y}_{p,(n)})] = \sum_{n=1}^{N} \pi_{p,(n)} \ell_{p,(n)} = \int_{[0,1]} \ell_{p,(nN(x))} \, dp.
\]

(45)

(46)

(47)

(48)

Thus the normalized single-letter loss (whose limit is the asymptotic single-letter loss (12)) is

\[
N^2 \tilde{L}(p, f, N) = \int_{[0,1]} N^2 \ell_{p,(nN(x))} \, dp.
\]

(49)

For single-letter density \( p \) and compander \( f \), we define the local loss function at granularity \( N \):

\[
g_N(x) = N^2 \ell_{p,(nN(x))}.
\]

(50)

We also define the asymptotic local loss function:

\[
g(x) = \frac{1}{24} f'(x)^{-2} x^{-1}.
\]

(51)

Theorem 1 is therefore equivalent to:

\[
\lim \inf_{N \to \infty} \int g_N \, dp \geq \int g \, dp \quad \text{for all } p \in \mathcal{P}, f \in \mathcal{F}
\]

(52)

and

\[
\lim_{N \to \infty} \int g_N \, dp = \int g \, dp \quad \text{for all } p \in \mathcal{P}, f \in \mathcal{F}^\dagger.
\]

(53)

To prove (52) and (53), we show:

**Proposition 4.** For all \( p \in \mathcal{P}, f \in \mathcal{F} \), if \( X \sim p \) then

\[
\lim_{N \to \infty} g_N(X) = g(X) \quad \text{almost surely.}
\]

(54)

**Proposition 5.** Let \( f \) be a compander and \( c > 0 \) and \( \alpha \in (0,1] \) such that \( f(x) - cx^\alpha \) is monotonically increasing. Letting \( g_N \) be the local loss functions as in (50) and

\[
h(x) = (2^{2/\alpha} + \alpha^2 2^{1/\alpha - 2}) (c\alpha)^{-2} x^{1-2\alpha} + c^{-1/\alpha} 2^{1/\alpha - 2}
\]

(55)

then \( g_N(x) \leq h(x) \) for all \( x, N \). Additionally, if \( \alpha \leq 1/2 \) then \( \int_{[0,1]} h \, dp < \infty \).
The lower bound (52) then follows immediately from Proposition 4 and Fatou’s Lemma; and when \( f \in \mathcal{F}^t \), by Proposition 5 there is some \( h \) which is integrable over \( p \) and dominates all \( g_N \), thus showing (53) by the Dominated Convergence Theorem.

To prove Proposition 4, we use the following facts:

- For any \( x \) at which \( f \) is differentiable, when \( N \) is large
  \[
  r_{(n_N(x))} \approx N^{-1} f'(x)^{-1}. \tag{56}
  \]

- For any \( x \) at which \( F_p \) is differentiable, \( p|_I \) will be approximately uniform over any sufficiently small \( I \) containing \( x \).

- For a sufficiently small interval \( I \) containing \( x \) and such that \( p|_I \) approximately uniform,
  \[
  \ell_{p,I} \approx \frac{1}{24} r^2_{p,I}^{-1}. \tag{57}
  \]

Putting these together, we get that if \( F_p \) and \( f \) are both differentiable at \( x \) then when \( N \) is large,
\[
 g_N(x) = N^2 \ell_{p,(n_N(x))} \approx N^2 \frac{1}{24} r^2_{(n_N(x))}^{-1} \approx \frac{1}{24} f'(x)^{-2} x^{-1} = g(x) \tag{58}
\]
as we wanted. We formally state each of these steps in Section IV-B (due to space constraints, most proofs are in the appendix), and combine them to prove Proposition 4 in Section IV-C.

The proof of Proposition 5 is given in Section IV-D, along with its own set of definitions and lemmas needed to show it.

**B. Preliminaries for Proposition 4**

We first generalize the idea of *bins* (1). The bin around \( x \in [0,1] \) at granularity \( N \) is the interval \( I = I^{(n)} \) containing \( x \) such that \( f(I) = [(n - 1)/N, n/N] \) for some \( n \in [N] \). This notion relies on integers because \( f(I) = [(n - 1)/N, n/N] \) for integers \( n, N \). We remove the dependence on integers while keeping the basic structure (an interval \( I \) about \( x \) whose image \( f(I) \) is a given size):

**Definition 4.** For any \( x \in [0,1] \), \( \theta \in [0,1] \), and \( \varepsilon > 0 \), we define the pseudo-bin \( I^{(x,\theta,\varepsilon)} \) as the interval satisfying:
\[
 I^{(x,\theta,\varepsilon)} = [x - \theta r^{(x,\theta,\varepsilon)}, x + (1 - \theta) r^{(x,\theta,\varepsilon)}] \text{ where } \tag{60}
\]
\[
 r^{(x,\theta,\varepsilon)} = \inf \{ r : f(x + (1 - \theta) r) - f(x - \theta r) \geq \varepsilon \} \tag{61}
\]
The interpretation of this is that \( I^{(x,\theta,\varepsilon)} \) is the minimal interval \( x \) such that \(|f(I^{(x,\theta,\varepsilon)})| \geq \varepsilon \) and such that \( x \) occurs at \( \theta \) within \( I^{(x,\theta,\varepsilon)} \), i.e. a \( \theta \) fraction of \( I^{(x,\theta,\varepsilon)} \) falls below \( x \) and \( 1-\theta \) falls above. Its width is \( r^{(x,\theta,\varepsilon)} \). This implies that bins are a special type of pseudo-bins. Specifically, for any \( x \) and \( N \) (and any compander \( f \)),

\[
I^{(n_N(x))} = I^{(x,\theta,1/N)} \text{ for some } \theta \in [0,1].
\] (62)

We now consider the size of pseudo-bins as \( \varepsilon \to 0 \):

**Lemma 1.** If \( f \) is differentiable at \( x \), then

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} r^{(x,\theta,\varepsilon)} = f'(x)^{-1}
\] (including going to \( \infty \) when \( f'(x) = 0 \)). The limit converges uniformly over \( \theta \in [0,1] \).

The proof is given in Section A-A. Note that applying this to bins means \( \lim_{N \to \infty} N r^{(n_N(x))} = f'(x)^{-1} \), and hence when \( f'(x) > 0 \) we have \( r^{(n_N(x))} = N^{-1} f'(x)^{-1} + o(N^{-1}) \).

For any interval \( I \), we want to measure how close \( p \) is to uniform over \( I \) using the distance measure \( d_{KS}(p,q) \) from (38). We will show that when \( F'_p(x) = p(x) \) is well-defined and positive at \( x \), \( p \) is approximately uniform on any sufficiently small interval \( I \) around \( x \). Formally:

**Proposition 6.** If \( p(x) = F'_p(x) > 0 \) is well-defined, then for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for all intervals \( I \) such that \( x \in I \) and \( r_I \leq \delta \),

\[
d_{KS}(p|_I, \text{unif}_I) \leq \varepsilon.
\] (64)

We give the proof in Section A-B. This allows us to use the following:

**Proposition 7.** Let \( p \) be a probability measure and \( I \) be an interval containing \( x \) such that \( r_I \leq x/4 \) and \( d_{KS}(p|_I, \text{unif}_I) \leq \varepsilon \) where \( \varepsilon \leq 1/2 \). Then

\[
|\ell_{p,I} - \ell_{\text{unif}_I}| \leq 2 \varepsilon r_I^2 x^{-1} + O(r_I^3 x^{-2})
\] (65)

Recall that \( \ell_{p,I} \) is the interval loss of \( I \) under distribution \( p \) when all points in \( I \) are quantized to \( \tilde{y}_{p,I} \), the centroid of the interval. We give the proof of Proposition 7 in Section A-C.

**Proposition 8.** For any \( x > 0 \) and any sequence of intervals \( I_1, I_2, \cdots \subseteq [0,1] \) all containing \( x \) such that \( r_{I_i} \to 0 \) as \( i \to \infty \),

\[
\ell_{\text{unif}_{I_i}} = \frac{1}{24} r_{I_i}^2 x^{-1} + O(r_{I_i}^3 x^{-2}).
\] (66)
The proof is in Section A-D.

Note that the above lemmas are all about asymptotic behavior as intervals shrink to 0 in width; to deal with the (edge) case where they don’t, we need the following lemma:

**Lemma 2.** For any \( I \) such that \( \mathbb{P}_{X \sim p}[X \in I] > 0 \), there is some \( a_I > 0 \) such that

\[
\ell_{p,J} \geq a_I \text{ for any } J \supseteq I.
\]  

(67)

We give the proof in Section A-E.

**C. Proof of Proposition 4**

We now combine the above results to prove Proposition 4, i.e. that \( \lim_{N \to \infty} g_N(X) = g(X) \) almost surely when \( X \sim p \). Because \( p \in \mathcal{P} \) (i.e. it is a continuous probability distribution) we will treat the bins as closed sets, i.e. \( I^{(n)} = [f\left(\frac{n-1}{N}\right)^{-1}, f\left(\frac{n}{N}\right)^{-1}] \); this does not affect anything since the resulting overlap is only a finite set of points.

**Proof.** Since \( p \in \mathcal{P} \) then when \( X \sim p \) the following hold with probability 1:

1) \( 0 < X < 1 \);
2) \( f'(X) \) is well-defined;
3) \( p(X) = F_p'(X) \) is well defined;
4) \( p(X) > 0 \).

This is because if \( p \in \mathcal{P} \), and \( |S| \) denotes the Lebesgue measure of set \( S \), then

\[
|S| = 0 \implies \mathbb{P}_{X \sim p}[X \in S] = 0
\]  

(68)

This immediately implies 1) since \( \{0, 1\} \) is measure-0.

Additionally, by Lebesgue’s differentiation theorem for monotone functions, any monotonic function on \([0, 1]\) is differentiable almost everywhere on \([0, 1]\) (i.e. excluding at most a measure-0 set), and compander \( f \) and CDF \( F_p \) are monotonic. This implies 2) and 3). Finally, 4) follows because the set of \( X \) such that \( p(X) = 0 \) has probability 0 under \( p \) by definition.

Therefore, we can fix \( X \sim p \) and assume it satisfies the above properties.

We now consider the bin size \( r_{(n_N(X))} \) as \( N \to \infty \); there are two cases, (a) \( \lim_{N \to \infty} r_{(n_N(X))} = 0 \) and (b) \( \lim \sup_{N \to \infty} r_{(n_N(X))} > 0 \). For case (b), since the length of the interval does not go to zero, \( g_N(X) = N^2 \ell_{p,(n_N(X))} \to \infty \); additionally, \( g(X) = \infty \) by default since case (b) requires that \( f'(X) = 0 \), and so \( g_N(X) \to g(X) \) as we want.
Case (a): In this case (which holds for all $X$ if $f \in \mathcal{F}'$), any $\delta > 0$ there is some sufficiently large $N^*$ (which can depend on $X$) such that

$$N \geq N^* \implies r_{(n_N(X))} \leq \delta.$$  \hfill (69)

By Proposition 6, for any $\varepsilon > 0$ there is some $\delta > 0$ such that for all intervals $I$ where $X \in I$ and $r_I \leq \delta$, we have $d_{KS}(p|_I, \text{unif}_I) \leq \varepsilon$. Putting this together implies that for any $\varepsilon > 0$, there is some sufficiently large $N^*_\varepsilon$ such that for all $N \geq N^*_\varepsilon$,

$$d_{KS}(p|_{(n_N(X))}, \text{unif}_{(n_N(X))}) \leq \varepsilon.$$  \hfill (70)

i.e. $p$ is $\varepsilon$ close to uniform on $I^{(n_N(X))}$. Furthermore, we can always choose $\varepsilon \leq 1/2$ and $N^*_\varepsilon$ sufficiently large that $r_{(n_N(X))} \leq X/4$ (since $\lim_{N \to \infty} r_{(n_N(X))} = 0$). Under these conditions, for $N > N^*_\varepsilon$ we can apply Proposition 7 and get

$$|\ell_{p,(n_N(X))} - \ell_{\text{unif}_{(n_N(X))}}| \leq 2\varepsilon r_{(n_N(X))}^2 X^{-1} + O(r_{(n_N(X))}^3 X^{-2}).$$  \hfill (71)

We can then turn this around: as $N \to \infty$, we have $\varepsilon \to 0$ and hence $\varepsilon = o(1)$ (as $N \to \infty$), so

$$|\ell_{p,(n_N(X))} - \ell_{\text{unif}_{(n_N(X))}}| = o(r_{(n_N(X))}^2 X^{-1}).$$  \hfill (72)

We then apply Proposition 8 (note that since $r_{(n_N(X))} \leq X/4$ and $X \leq 2\bar{y}_{(n_N(X))}$, we know automatically that $r_{(n_N(X))} \leq \bar{y}_{(n_N(X))}/2$) to get that

$$\ell_{\text{unif}_{(n_N(X))}} = \frac{1}{24} r_{(n_N(X))}^2 \bar{y}_{(n_N(X))}^{-1} + O(r_{(n_N(X))}^3 X^{-2})$$  \hfill (73)

However, since $X$ is fixed and $r_{(n_N(X))} \to 0$ as $N \to \infty$ (and $|X - \bar{y}_{(n_N(X))}| \leq r_{(n_N(X))}$ since they are both in the bin $I^{(n_N(X))}$), we know that $\bar{y}_{(n_N(X))} = X(1 + o(1))$ where $o(1)$ is in terms of $N$ (as $N \to \infty$). Hence (noting that $(1 + o(1))^{-1}$ is still $1 + o(1)$ and $O(r_{(n_N(X))}^3 X^{-2})$ is $o(1) r_{(n_N(X))}^2 X^{-1}$) we can re-write the above and combine with (72) to get

$$\ell_{\text{unif}_{(n_N(X))}} = \frac{1}{24} (1 + o(1)) r_{(n_N(X))}^2 X^{-1}$$  \hfill (74)

$$\implies \ell_{p,(n_N(X))} = \frac{1}{24} (1 + o(1)) r_{(n_N(X))}^2 X^{-1}. $$  \hfill (75)

We now split things into two cases: (i) $f'(X) > 0$; (ii) $f'(X) = 0$.

Case i ($f'(X) > 0$): For all $N$ there is a $\theta \in [0, 1]$ such that $I^{(n_N(X))} = I^{(X, \theta, 1/N)}$ (bins are pseudo-bins, see Definition 4). Thus, by Lemma 1 (which shows uniform convergence over $\theta$),

$$\lim_{N \to \infty} N r_{(n_N(X))} = f'(X)^{-1} $$  \hfill (76)
Thus, we may re-write as a little-$o$ and plug into $g_N(X)$:

$$r_{(n_N(X))} = N^{-1}f'(X)^{-1} + o(N^{-1})$$

$$= N^{-1}f'(X)^{-1}(1 + o(1))$$

$$\implies g_N(X) = N^2 \ell_{p(n_N(X))}$$

$$= N^2 \frac{1}{24}(1 + o(1))r_{(n_N(X))}^2 X^{-1}$$

$$= N^2 \frac{1}{24}(1 + o(1))N^{-2}f'(X)^{-2}X^{-1}$$

$$= \frac{1}{24}(1 + o(1))f'(X)^{-2}X^{-1}$$

implying $\lim_{N \to \infty} g_N(X) = g(X)$ as we wanted.

**Case ii** ($f'(X) = 0$): As before, for any $N$ there is some $\theta \in [0, 1]$ such that $I^{(n_N(X))} = I^{(X, \theta, 1/N)}$. Thus, by Lemma 1 and as $f'(X) = 0$, we have

$$\lim_{N \to \infty} Nr_{(n_N(X))} = \infty.$$ 

since the convergence in Lemma 1 is uniform over $\theta$. We can then re-write this as a little-$\omega$:

$$r_{(n_N(X))} = \omega(N^{-1}).$$

This implies that

$$g_N(X) = N^2 \ell_{p(n_N(X))}$$

$$= N^2 \frac{1}{24}(1 + o(1))\ell_{(n_N(X))}^2 X^{-1}$$

$$= N^2 \frac{1}{24}(1 + o(1))\omega(N^{-2}) X^{-1}$$

$$= \omega(1)$$

where $\omega(1)$ means $\lim_{N \to \infty} g_N(X) = \infty$. But since $f'(X) = 0$, by convention we have $g(X) = \frac{1}{24}f'(X)^{-2}X^{-1} = \infty$ and so $\lim_{N \to \infty} g_N(X) = g(X)$ as we wanted.

**Case (b):** $\limsup_{N \to \infty} r_{(n_N(X))} > 0$. Note that this can only happen if $f'(X) = 0$, so $g(X) = \infty$; hence our goal is to show that $\lim_{N \to \infty} g_N(X) = \infty$.

Related to the above, this only happens if $f$ is not strictly monotonic at $X$, i.e. if there is some $a < X$ or some $b > X$ such that $f(X) = f(a)$ or $f(X) = b$ (or both). If both, $[a, b] \subseteq I^{(n_N(X))}$ for all $N$. Since $p(X)$ is well-defined and positive, any nonzero-width interval containing $X$ has positive probability mass under $p$. Thus, by Lemma 2, there exists some $\alpha > 0$ such that all $J \ni [a, b]$ satisfies $\ell_{p,J} \geq \alpha$. But then $g_N(X) \geq N^2 \alpha$ and goes to $\infty$. 

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If only \( a \) exists, we divide the granularities \( N \) into two classes: first, \( N \) such that \( I^{(n_N(X))} \) has lower boundary exactly at \( X \) (which can happen if \( f(X) \) is rational), and second, \( N \) such that \( I^{(n_N(X))} \) has lower boundary below \( X \). Call the first class \( N^{(1)}(1), N^{(1)}(2), \ldots \) and the second \( N^{(2)}(1), N^{(2)}(2), \ldots \). Then as no \( b \) exists, \( \lim_{i \to \infty} r^{(n_N(\alpha_i))(X)} = 0 \), i.e. the bins corresponding to the first class shrink to 0 and the asymptotic argument applies to them, showing \( g_{N^{(1)}(i)}(X) \to \infty \).

For the second class, for any \( i \), we have \( I^{(n_N(\alpha_i))} \equiv [a, X] \) and so we have an \( \alpha > 0 \) lower bound of the interval loss, and multiplying by \( N^2 \) takes it to \( \infty \). Thus since both subsequences of \( N \) take \( g_N(X) \to \infty \), we are done. An analogous argument holds if \( b \) exists but not \( a \).

As this holds for any \( X \) under conditions 1-4, which happens almost surely, we are done.

\[ \square \]

D. Proof of Proposition 5

To finish our Dominated Convergence Theorem (DCT) argument, we to prove Proposition 5, which gives an integrable function \( h \) dominating all the local loss functions \( g_N \). As with Proposition 4, we do this in stages. We first define:

**Definition 5.** For any interval \( I \), let

\[ \ell^*_I = \sup_q \ell_{q,I} \]  

where \( q \) is a probability distribution over \([0,1] \). If \( I = I^{(n)} \) we can denote this as \( \ell^*_n \).

Since \( \ell_{q,I} \) is only affected by \( q|_I \) (i.e. what \( q \) does outside of \( I \) is irrelevant), we can restrict \( q \) to be a probability distribution over \( I \) without affecting the value of \( \ell^*_I \). The question is thus: what is the maximum single-interval loss which can be produced on interval \( I \)?

Then, we can use the upper bound

\[ g_N(x) = N^2 \ell_{p,(n_N(x))} \leq N^2 \ell^*_n(n_N(x)) \cdot \]  

This has the benefit of simplifying the term by removing \( p \). We now bound \( \ell^*_I \):

**Lemma 3.** For any interval \( I \), \( \ell^*_I \leq \frac{1}{2} \ell_{(n_N(x))}^{-1} \).

We give the proof in Section A-F. We can then add the above result to (90) in order to obtain

\[ g_N(x) \leq N^2 \ell^*_n(n_N(x)) \leq N^2 \frac{1}{2} \ell_{(n_N(x))}^{-1} \cdot \]  

However, it is hard to use this as the boundaries of \( I^{(n_N(x))} \) in relation to \( x \) are inconvenient. Instead, use an interval which is ‘centered’ at \( x \) in some way, with the help of the following:
Lemma 4. If $I \subseteq I'$, then $\ell^*_I \leq \ell^*_I$.

Proof. This follows as any $q$ over $I$ is also a distribution over $I'$ (giving 0 probability to $I \setminus I$).

Thus, if we can find some interval $J$ such that $I^{(n_N(x))} \subseteq J$ (but of the right size) and which had more convenient boundaries, we can use that instead. We define:

Definition 6. For compander $f$ at scale $N$ and $x \in [0, 1]$, define the interval

$$J_{f,N,x} = f^{-1}\left(\left[ f(x) - \frac{1}{N}, f(x) + \frac{1}{N} \right] \cap [0, 1]\right)$$  \hspace{1cm} (92)

As mentioned, we want this because it contains $I^{(n_N(x))}$:

Lemma 5. For any strictly monotonic $f$ and integer $N$,

$$I^{(n_N(x))} \subseteq J_{f,N,x}$$  \hspace{1cm} (93)

Proof. Since $f$ is strictly monotonic, it has a well-defined inverse $f^{-1}$.

By definition the bin $I^{(n_N(x))}$, when passed through the compander $f$, returns $\left[ \frac{n-1}{N}, \frac{n}{N} \right]$, i.e.

$$f(I^{(n_N(x))}) = \left[ \frac{n-1}{N}, \frac{n}{N} \right].$$ \hspace{1cm} (94)

Note that this interval has width $1/N$ and includes $f(x)$ and (by definition) it is in $[0, 1]$. Hence,

$$f(I^{(n_N(x))}) \subseteq \left[ f(x) - \frac{1}{N}, f(x) + \frac{1}{N} \right] \cap [0, 1]$$ \hspace{1cm} (95)

$$\implies f(I^{(n_N(x))}) \subseteq f(J_{f,N,x})$$ \hspace{1cm} (96)

$$\implies I^{(n_N(x))} \subseteq J_{f,N,x}$$ \hspace{1cm} (97)

and we are done.

Now we can consider the importance of $f \in \mathcal{F}^1$: by dominating a monomial $cx^\alpha$, we can ‘upper bound’ the interval $J_{f,N,x}$ by the equivalent interval with the compander $f_*(x) = cx^\alpha$ (i.e. $J_{f,N,x} \subseteq J_{f_*,N,x}$), which is then much nicer to work with.\(^9\) This also guarantees that $f$ is strictly monotonic.

Lemma 6. If $f_1, f_2 \in \mathcal{F}$ are strictly monotonic increasing companders such that $f_2 - f_1$ is also monotonically increasing (not necessarily strictly) and $f_1(0) = 0$, then for any $x \in [0, 1]$ and $N$,

$$J_{f_2,N,x} \subseteq J_{f_1,N,x}$$ \hspace{1cm} (98)

\(^9\)While $f_*(x)$ may not map to all of $[0, 1]$, it’s a valid compander (but sub-optimal as it only uses some of the $N$ labels).
The proof is given in Section A-G. Finally, we need a quick lemma concerning the guarantee that if $f \in \mathcal{F}$, the function $g(x) = \frac{1}{24} f'(x)^2 x^{-1}$ is integrable under any distribution $p$:

**Lemma 7.** Let $f \in \mathcal{F}$, and let $g(x) = \frac{1}{24} f'(x)^2 x^{-1}$. Then for any probability distribution $p$ over $[0, 1]$,

$$
\int_{[0, 1]} g \, dp < \infty.
$$

**Proof.** If $f \in \mathcal{F}$, then there is some $c > 0$ and $\alpha \in (0, 1/2]$ such that $f(x) - cx^\alpha$ is monotonically increasing. Thus (whenever it is well-defined, which is almost everywhere by Lebesgue’s differentiation theorem for monotone functions) we have $f'(x) \geq cx^{\alpha - 1}$ and since $\alpha \in (0, 1/2]$, we have $1 - 2\alpha \geq 0$. Thus, for all $x \in [0, 1]$,

$$
0 \leq g(x) \leq \frac{1}{24} c^{-2} x^{-2} x^{1-2\alpha} \leq \frac{1}{24} c^{-2} \alpha^{-2}
$$

(100)

which of course implies that $\int_{[0, 1]} g \, dp < \infty$. □

We can now prove Proposition 5, which will complete the proof of Theorem 1.

**Proof of Proposition 5.** As before, let $f_*(x) = cx^\alpha$; thus $f_*(0) = 0$ so we can apply Lemma 6. We begin, as outlined in (91), with:

$$
g_N(x) = N^2 \ell_{p_*(n_N(x))} (101)
$$

$$
\leq N^2 \ell_{p_N(x)} (102)
$$

$$
\leq N^2 \ell_{f_*,N,x} (103)
$$

$$
\leq N^2 \ell_{f_*,N,x} (104)
$$

where (102) follows from the definition of $\ell^*_I$; (103) follows from Lemmas 4 and 5; and (104) follows from Lemma 6. However, since $f_*(x) = cx^\alpha$, we have a specific formula we can work with. We have $f'_*(x) = \alpha cx^{\alpha - 1}$ and $f_*^{-1}(z) = (z/c)^{1/\alpha} = c^{-1/\alpha} z^{1/\alpha}$. Note that this means we can re-write

$$
h(x) = (2^{2/\alpha} + \alpha^2 2^{1/\alpha - 2}) f'_*(x)^{-2} x^{-1} + c^{-1/\alpha} 2^{1/\alpha - 2} (105)
$$

which sheds some light on the structure of $h(x)$. Using Lemma 7 proves that $\int_{[0, 1]} h \, dp$ is finite if $f \in \mathcal{F}$, which occurs when $\alpha \leq 1/2$.

Fix a value of $x$. Let $r_N(x)$ be the width of $J_{f_*,N,x}$. We consider two cases: (i) $cx^\alpha < 1/N$; and (ii) $cx^\alpha \geq 1/N$. 24
Case (i): This implies \( f(J^*_{x,N,x}) \subseteq [0,2/N] \) so

\[
x < c^{-1/\alpha} N^{-1/\alpha}
\]

\( \implies r_N(x) \leq c^{-1/\alpha} (N/2)^{-1/\alpha} \) \hfill (107)

Then, as \( J^*_{x,N,x} \) has lower boundary 0 in this case, \( \bar{y}_{(n_N(x))} = r_N(x)/2 \). Thus, using (91),

\[
g_N(x) \leq N^2 \frac{1}{2} r_N(x)^2 \bar{y}_{(n_N(x))^i}
\]

\( \leq c^{-1/\alpha} 2^{-1/\alpha} N^{-1/\alpha + 2} \). \hfill (109)

If \( \alpha \leq 1/2 \), then \( N^{-1/\alpha + 2} \) is maximized at \( N = 1 \), and thus

\[
g_N(x) \leq c^{-1/\alpha} 2^{-1/\alpha} \cdot \hfill (110)
\]

If \( \alpha > 1/2 \), the value \( N^{-1/\alpha + 2} \) is maximized for the largest possible \( N \) still satisfying Case (i).

Since \( c x^\alpha < 1/N \), this implies that \( N < c^{-1} x^{-\alpha} \). Then,

\[
g_N(x) \leq c^{-1/\alpha} (c^{-1} x^{-\alpha})^{-1/\alpha + 2} 2^{-1/\alpha}
\]

\( = c^{-2} x^{1-2\alpha} 2^{-1/\alpha} \) \hfill (112)

\( = \alpha^2 (c\alpha x^{\alpha-1})^{-2} x^{-1} 2^{-1/\alpha} \) \hfill (113)

\( = \alpha^2 f'_*(x)^{-2} x^{-1} 2^{-1/\alpha} \) \hfill (114)

Thus, for Case (i) we have that for any \( a \in (0,1] \),

\[
g_N(x) \leq \alpha^2 f'_*(x)^{-2} x^{-1} 2^{-1/\alpha} + c^{-1/\alpha} 2^{-1/\alpha} \cdot \hfill (115)
\]

Case (ii): When \( c x^\alpha \geq 1/N \), since \( x \in I \implies \bar{y}_I \geq x/2 \) (the midpoint of an interval cannot be less than half the largest element of the interval), we can upper-bound \( g_N(x) \) (using (104) and Lemma 3) by

\[
g_N(x) \leq N^2 \frac{1}{2} r_N(x)^2 \bar{y}_{(n_N(x))^i} \leq N^2 r_N(x)^2 x^{-1} \cdot \hfill (116)
\]

We then bound \( r_N(x) \) using the Fundamental Theorem of Calculus: since \( f \) is monotonically increasing, for any \( a \leq b \),

\[
\int_a^b f'(t) \ dt \leq f(b) - f(a) \] \hfill (117)
(any discontinuities can only make $f$ increase faster). Additionally $r_N(x) = b_1 - a_1$ where $f(b_1) = \max(f(x) + 1/N, 1)$ and $f(a_1) = f(x) - 1/N$ (since it’s Case (ii) we know $f(x) - 1/N \geq 0$ and since $f \in \mathcal{F}$ is strictly monotonic $a_1, b_1$ are unique). Thus, if we define $a_2, b_2$ such that

$$\int_{a_2}^{x} f'(t) \, dt = 1/N \quad \text{and} \quad \int_{x}^{b_2} f'(t) \, dt = 1/N \quad (118)$$

(or $a_2 = 0$ or $b_2 = 1$ if they exceed the $[0, 1]$ bounds) we have $r_N(x) \leq b_2 - a_2$. Then, because $f - f_*$ is monotonically increasing, we can define $a_3, b_3$ where

$$\int_{a_3}^{x} f'_*(t) \, dt = 1/N \quad \text{and} \quad \int_{x}^{b_3} f'_*(t) \, dt = 1/N \quad (119)$$

and get that $r_N(x) \leq b_3 - a_3$ (also allowing $b_3 \geq 1$ if necessary). This yields:

$$r_N(x) \leq c^{-1/\alpha} \int_{\max(0, cx^\alpha - 1/N)}^{\min(1, cx^\alpha + 1/N)} (f'_*)'(z) \, dz \quad (120)$$

$$= c^{-1/\alpha} \int_{\max(0, cx^\alpha - 1/N)}^{\min(1, cx^\alpha + 1/N)} \alpha^{-1} z^{\alpha - 1} \, dz \quad (121)$$

$$\leq c^{-1/\alpha} \int_{\max(0, cx^\alpha - 1/N)}^{\min(1, cx^\alpha + 1/N)} \alpha^{-1} (cx^\alpha + 1/N)^{1/\alpha - 1} \, dz \quad (122)$$

$$\leq c^{-1/\alpha} \int_{cx^\alpha - 1/N}^{cx^\alpha + 1/N} \alpha^{-1} (cx^\alpha + 1/N)^{1/\alpha - 1} \, dz \quad (123)$$

$$= (2/N) c^{-1/\alpha} \alpha^{-1} (cx^\alpha + 1/N)^{1/\alpha - 1} \quad (124)$$

$$\Rightarrow r_N(x) \leq (2/N) c^{-1/\alpha} \alpha^{-1} (cx^\alpha + 1/N)^{1/\alpha - 1} \quad (125)$$

$$\leq 2N^{-1} c^{-1/\alpha} \alpha^{-1} (2cx^\alpha)^{1/\alpha - 1} \quad (126)$$

$$= N^{-1} c^{-1/\alpha} \alpha^{-2} (2cx^\alpha)^{1/\alpha - 1} \quad (127)$$

$$= 2^{1/\alpha} N^{-1} \left( c^{-1/\alpha} x^{1 - \alpha} \right) \quad (128)$$

$$= 2^{1/\alpha} N^{-1} f'_*(x)^{-1} \quad (129)$$

Thus, we can incorporate this into our bound $(116)$

$$g_N(x) \leq N^2 r_N(x)^2 x^{-1} \quad (130)$$

$$\leq 2^{2/\alpha} f'_*(x)^{-2} x^{-1}. \quad (131)$$

So, $h(x)$, as the sum of the two cases, upper bounds $g_N(x)$ no matter what.

We can also note that if $\alpha \leq 1/2$, then $x^{1-2\alpha} \leq 1$ and hence we can upper-bound $h$ by a constant. Thus $\int_{[0,1]} h \, dp = \mathbb{E}_{X \sim p}[h(X)] < \infty$ trivially, for any $p$, and we are done. \hfill \square

This completes the proof of $(16)$ in Theorem 1.
V. Minimax Compander

Theorem 1 showed that for \( f \in \mathcal{F}^\dagger \), the asymptotic single-letter loss is equivalent to

\[
\tilde{L}(p, f) = \frac{1}{24} \int_0^1 p(x) f'(x)^{-2} x^{-1} dx.
\]  

(132)

Using this, we can analyze what is the ‘best’ compander \( f \) we can choose and what is the ‘worst’ single-letter density \( p \) in order to show Theorems 2 and 3 and their related results.

A. Optimizing for Best Compander

We show Theorem 2 and Proposition 2 together. They follow from Theorem 1 by finding \( f \in \mathcal{F} \) which minimizes \( L^\dagger(p, f) \), by optimizing over \( f' \). Since \( f : [0, 1] \rightarrow [0, 1] \) is monotonic, we use constraints \( f'(x) \geq 0 \) and \( \int_0^1 f'(x) dx = 1 \). We solve the following:

\[
\text{minimize} \quad L^\dagger(p, f) = \frac{1}{24} \int_0^1 p(x) f'(x)^{-2} x^{-1} dx
\]

subject to \( \int_0^1 f'(x) dx = 1 \) and \( f'(x) \geq 0 \) for all \( x \in [0, 1] \)

The function \( L^\dagger(p, f) \) is convex in \( f' \), and thus first order conditions show optimality. Let \( \lambda(x) \) be a function such that \( \int_0^1 \lambda(x) dx = 0 \). We derive:

\[
\frac{d}{dt} \frac{1}{24} \int_0^1 p(x) (f'(x) + t \lambda(x))^{-2} x^{-1} dx
\]

\[
= \frac{1}{24} \int_0^1 p(x) x^{-1} \frac{d}{dt} (f'(x) + t \lambda(x))^{-2} dx
\]  

(134)

\[
= -\frac{1}{12} \int_0^1 p(x) x^{-1} (f'(x) + t \lambda(x))^{-3} \lambda(x) dx
\]  

(135)

\[
= -\frac{1}{12} \int_0^1 p(x) x^{-1} f'(x)^{-3} \lambda(x) dx \quad \text{(at } t = 0) \]

(136)

\[
\lambda = -\frac{1}{12} \int_0^1 \lambda(x) dx = 0 \quad \text{if} \quad f'(x) \propto (p(x)x^{-1})^{1/3}
\]

(137)

since \( \lambda(x) \) integrates to 0. Thus, such \( f \) satisfies the first-order optimality condition under the constraint \( \int f'(x) dx = 1 \). This gives \( f'_p(x) \propto (p(x)x^{-1})^{1/3} \) and \( f(0) = 0 \) and \( f(1) = 1 \), from which (19) and (20) follow. If \( f_p \in \mathcal{F}^\dagger \), then \( f_p = \arg\min_{f} \tilde{L}(p, f) \), and for any other \( f \in \mathcal{F} \),

\[
\tilde{L}(p, f_p) = L^\dagger(p, f_p) \leq L^\dagger(p, f) \leq \lim_{N \rightarrow \infty} \inf \ N^2 \tilde{L}(p, f, N)
\]  

(138)

If \( f_p \notin \mathcal{F}^\dagger \), for any \( \delta > 0 \) define \( f_{p, \delta} = (1 - \delta) f_p + \delta x^{1/2} \) (as in (21)). Then \( f_{p, \delta} - \delta x^{1/2} = (1 - \delta) f_p \) is monotonically increasing so \( f \in \mathcal{F}^\dagger \), so Theorem 1 applies to \( f_{p, \delta} \); additionally,
\( f_{p, \delta} - (1 - \delta) f_p = \delta x^{1/2} \) is monotonically increasing as well so \( f_{p, \delta}' \geq (1 - \delta) f_p' \). Hence, plugging into the \( L^\dagger \) formula gives:

\[
\tilde{L}(p, f_{p, \delta}) = L^\dagger(p, f_{p, \delta}) \leq L^\dagger(p, f_p)(1 - \delta)^{-2}. \tag{139}
\]

Taking \( \delta \to 0 \) (and noting that \( F^\dagger \subseteq F \)) thus shows that

\[
L^\dagger(p, f_p) = \inf_{f \in F^\dagger} \tilde{L}(p, f). \tag{140}
\]

This finishes the proofs of Theorem 2 and Proposition 2.

**Remark 8.** Since we know the corresponding single-letter source \( p \) for a Dirichlet prior, using this \( p \) with Theorem 2 gives us the optimal compander for Dirichlet priors on any alphabet size. This gives us a better quantization method than EDI which was discussed in Section II-B. This optimal compander is called the beta compander and its details are given in Section B-A.

### B. Minimax and Approximate Minimax Componders

To prove Theorem 3 and Corollary 1, we first consider what density \( p \) maximizes

\[
\frac{1}{24} \left( \int_0^1 (p(x)x^{-1})^{1/3} dx \right)^3 \tag{141}
\]

(equation (19)), i.e. is most difficult to quantize with a compander.

Using calculus of variations to maximize

\[
\int_0^1 (p(x)x^{-1})^{1/3} dx \tag{142}
\]

(which of course maximizes (19)) subject to \( p(x) \geq 0 \) and \( \int_0^1 p(x) dx = 1 \), we find that maximizer is \( p(x) = \frac{1}{2} x^{1/2} \). However, while interesting, this is only for a single letter; and because \( \mathbb{E}[X] = 1/3 \) under this distribution, it is clearly impossible to construct a prior over the simplex (whose output vector must sum to 1) with this marginal (unless \( K = 3 \)).

Hence, we add an expected value constraint to the problem of maximizing (142), giving:

\[
\text{maximize } \int_0^1 (p(x)x^{-1})^{1/3} dx \tag{143}
\]

subject to

\[
\int_0^1 p(x) dx = 1; \tag{144}
\]

\[
\int_0^1 p(x) dx = \frac{1}{K}; \tag{145}
\]

and \( p(x) \geq 0 \) for all \( x \) \( \tag{146} \)
We can solve this again using variational methods (we are maximizing a concave function so satisfying first order conditions are enough to ensure optimality). A function $p(x) > 0$ is optimal if, for any $\lambda(x)$ where

$$\int_0^1 \lambda(x) \, dx = 0 \quad \text{and} \quad \int_0^1 \lambda(x) x \, dx = 0$$

(147)

the following holds:

$$\frac{d}{dt} \int_0^1 x^{-1/3} (p(x) + t \lambda(x))^{1/3} \, dx = 0.$$  

(148)

We have by the same logic as before:

$$\frac{d}{dt} \int_0^1 x^{-1/3} (p(x) + t \lambda(x))^{1/3} \, dx = \frac{1}{3} \int_0^1 x^{-1/3} (p(x) + t \lambda(x))^{-2/3} \lambda(x) \, dx$$

(149)

$$= \frac{1}{3} \int_0^1 x^{-1/3} p(x)^{-2/3} \lambda(x) \, dx \quad \text{(at } t = 0)$$

(150)

Thus, if we can arrange things so that there are constants $a_K, b_K$ such that

$$x^{-1/3} p(x)^{-2/3} = a_K + b_K x$$

(151)

this ensures (150) equals zero. In that case,

$$x^{-1/3} p(x)^{-2/3} = a_K + b_K x$$

(152)

$$\iff \quad p(x)^{-2/3} = a_K x^{1/3} + b_K x^{4/3}$$

(153)

$$\iff \quad p(x) = (a_K x^{1/3} + b_K x^{4/3})^{-3/2}$$

(154)

This yields the maximin density $p_K^e$ (23) from Theorem 3, where $a_K, b_K$ are set to meet the constraints (144) and (145). Exact formulas for $a_K, b_K$ are difficult to find. We will give more details on $a_K, b_K$ after the next step.

We want to determine the optimal compander for the maximin density (154). We know from (137) that we need to first compute

$$\phi(x) = \int_0^x z^{-1/3} (a_K z^{1/3} + b_K z^{4/3})^{-1/2} \, dz = \frac{2 \text{ArcSinh} \left( \sqrt{\frac{b_K x}{a_K}} \right)}{\sqrt{b_K}}.$$  

(155)

The best compander $f(x)$ is proportional to (155) and is exactly given by $f(x) = \phi(x)/\phi(1)$. The resulting compander, which we call the minimax compander, is

$$f(x) = \frac{\text{ArcSinh} \left( \sqrt{\frac{b_K x}{a_K}} \right)}{\text{ArcSinh} \left( \sqrt{\frac{b_K}{a_K}} \right)}.$$  

(156)
Given the form of \( f(x) \), it is natural to determine an expression for the ratio \( b_K/a_K \). We can parameterize both \( a_K \) and \( b_K \) by \( b_K/a_K \) and then examine how \( b_K/a_K \) behaves as a function of \( K \). The constraints on \( a_K \) and \( b_K \) give that

\[
a_K = 4^{1/3}(b_K/a_K + 1)^{-1/3} \tag{157}
\]

\[
b_K = 4a_K^{-2} - a_K \tag{158}
\]

The ratio \( b_K/a_K \) grows approximately as \( K \log K \). Hence, we choose to parameterize

\[
b_K/a_K = c_K K \log K \tag{159}
\]

To satisfy the constraints, we get that \( .25 < c_K < .75 \) so long as \( K > 24 \) (the details are in Section C-A), and Lemma 11 in Section C-B shows that \( c_K \to 1/2 \) as \( K \to \infty \).

We can express \( a_K, b_K \) in terms of \( c_K \):

\[
a_K = 4^{1/3}(c_K K \log K + 1)^{-1/3} \tag{160}
\]

\[
b_K = 4a_K^{-2} - a_K \tag{161}
\]

\[
= 4^{1/3}(c_K K \log K + 1)^{2/3} - 4^{1/3}(c_K K \log K + 1)^{-1/3} \tag{162}
\]

\[
= 4^{1/3}(c_K K \log K)^{2/3} (1 + o(1)) \tag{163}
\]

We use here that for large \( K \), the second term in (162) is negligible compared to the first.

Thus we get the minimax compander and approximate minimax compander, respectively:

\[
f_K^*(x) = \frac{\text{ArcSinh} \left( \sqrt{(c_K K \log K)} x \right)}{\text{ArcSinh} (\sqrt{c_K K \log K})} \tag{164}
\]

\[
\approx f_K^{**}(x) = \frac{\text{ArcSinh} \left( \sqrt{(1/2) K \log K} x \right)}{\text{ArcSinh} (\sqrt{(1/2) K \log K})} \tag{165}
\]

The minimax compander minimizes the maximum (raw) loss against all densities in \( \mathcal{P}_{1/K} \), while the approximate minimax compander performs very similarly but is more applicable since it can be used without computing \( c_K \).

To compute the loss of the minimax compander, we can use (19) to get

\[
L^1(p_K^*, f_K^*) = \frac{1}{24} \left( \frac{2 \text{ArcSinh} (\sqrt{c_K K \log K})}{\sqrt{b_K}} \right)^3 \tag{166}
\]
Substituting we get

\[
L^l(p^*_K, f^*_K) = \frac{1}{24} \left( \log \left( \sqrt{c_K K \log K} + \sqrt{c_K K \log K + 1} \right) \right)^3
\]

which is concave (actually linear) in \( p \) and convex in \( f' \), and we can show that the pair \((f^*_K, p^*_K)\) form a saddle point, thus proving (25)-(26) from Theorem 3.

We can compute that

\[
(f^*_K)'(x) \propto (p^*_K(x)x^{-1})^{1/3}
\]

which is

\[
= x^{-1/3}(a_K x^{1/3} + b_K x^{4/3})^{-1/2}
\]

and

\[
= \frac{1}{\sqrt{a_K x + b_K x^2}}.
\]

Assume we set \( a_K \) and \( b_K \) to the appropriate values for \( K \). For any \( p \in \mathcal{P}_{1/K} \),

\[
L^l(p, f^*_K) = \int_0^1 p(x)x^{-1}((f^*_K)'(x))^{-2}dx
\]

which is

\[
= \int_0^1 p(x)x^{-1}(a_K x + b_K x^2)dx
\]

and

\[
= \int_0^1 p(x)(a_K + b_K x)dx
\]

which is

\[
= a_K + b_K \frac{1}{K}
\]

i.e. \( L^l(p, f^*_K) \) doesn’t depend on \( p \). Since \( f^*_K \) is the optimal compander against the maximin compander \( p^*_K \) we can therefore conclude:

\[
\sup_{p \in \mathcal{P}_{1/K}} \sup_{f \in \mathcal{F}} L^l(p, f^*_K) = \sup_{p \in \mathcal{P}_{1/K}} L^l(p^*_K, f^*_K)
\]

\[
= \inf_{f \in \mathcal{F}} L^l(p^*_K, f) = \sup_{p \in \mathcal{P}_{1/K}} \inf_{f \in \mathcal{F}} L^l(p, f).
\]

Since it is always true that

\[
\sup_{p \in \mathcal{P}_{1/K}} \inf_{f \in \mathcal{F}} L^l(p, f) \leq \inf_{f \in \mathcal{F}} \sup_{p \in \mathcal{P}_{1/K}} L^l(p, f),
\]
this completes showing that \((f^*_K, p^*_K)\) is a saddle point.

Furthermore, \(f^*_K \in \mathcal{F}^+\) (specifically it behaves as a multiple of \(x^{1/2}\) near 0), so \(\tilde{L}(p, f^*_K) = L^1(p, f^*_K)\) for all \(p\), thus showing that \(f^*_K\) performs well against any \(p \in \mathcal{P}_{1/K}\). Using (13) with the expressions for \(p^*_K\) and \(f^*_K\) and (169) gives (27). This completes the proof of Theorem 3.

**Remark 9.** While the power compander \(f(x) = x^{1/\log K}\) is not minimax optimal, it has similar properties to the minimax compander and differs in loss by at most a constant factor. We analyze the power compander in Section B-B.

### C. Existence of Priors with Given Marginals

While \(p^*_K\) is the worst density in \(\mathcal{P}_{1/K}\), it is unclear whether a prior \(P^*\) on \(\triangle_{K-1}\) exists with marginals \(p^*_K\); even though \(p^*_K\) has the correct expectation so that \(K\) copies will sum to 1 in expectation, it may not be possible to correlate them to guarantee they sum to 1. However, it is possible to construct a prior \(P^*\) whose marginals are as hard to quantize, up to a constant, as \(p^*_K\), by use of clever correlation between the letters. We start with a lemma:

**Lemma 8.** Let \(p \in \mathcal{P}_{1/K}\). Then there exists a joint distribution of \((X_1, \ldots, X_K)\) such that (i) \(X_i \sim p\) for all \(i \in [K]\) and (ii) \(\sum_{i \in [K]} X_i \leq 2\), guaranteed.

**Proof.** Let \(F\) be the cumulative distribution function of \(p\). Define the quantile function \(F^{-1}\) as

\[
F^{-1}(u) = \inf\{x : F(x) \geq u\}. \tag{180}
\]

We break \([0, 1]\) into \(K\) uniform sub-intervals \(I_i = ((i-1)/K, i/K]\) (let \(I_1 = [0, 1/K]\)). We then generate \(X_1, X_2, \ldots, X_K\) jointly by the following procedure:

1) Choose a permutation \(\sigma : [K] \to [K]\) uniformly at random (from \(K!\) possibilities).
2) Let \(U_k \sim \text{unif}_{I_{\sigma(k)}}\) independently for all \(k\).
3) Let \(X_k = F^{-1}(U_k)\).

Now we consider \(\sum_k X_k\). Let \(b_i = F^{-1}(i/k)\) for \(i = 0, 1, \ldots, K\). Note that if \(\sigma(k) = i\) then \(U_k \in ((i-1)/K, i/K]\) and hence \(X_k = F^{-1}(U_k) \in [b_{i-1}, b_i]\). Therefore \(X_{\sigma^{-1}(i)} \in [b_{i-1}, b_i]\) and
Thus for any permutation $\sigma$, 
\[
\sum_{i=1}^{K} b_{i-1} \leq \sum_{i=1}^{K} X_{\sigma^{-1}(i)} \leq \sum_{i=1}^{K} b_i \tag{181}
\]
\[
= \left( \sum_{i=1}^{K} b_{i-1} \right) + b_K - b_0 \tag{182}
\]
\[
\leq \left( \sum_{i=1}^{K} b_{i-1} \right) + 1 \leq 2 \tag{183}
\]

Note that we used
\[
\sum_{i=1}^{K} b_{i-1} \leq \sum_{i=1}^{K} \mathbb{E}[X_{\sigma^{-1}(i)}] = K \mathbb{E}_{X \sim p}[X] = 1. \tag{184}
\]

Lemma 8 shows a joint distribution of $Z_1, \ldots, Z_{K-1}$ such that $Z_i \sim p_K^*$ for all $i$ and $\sum_{i=1}^{K-1} Z_i \leq 2$ (guaranteed) exists. Then, if $X_i = Z_i/2$ for all $i \in [K-1]$, we have $\sum_{i=1}^{K-1} X_i \leq 1$. Then setting $X_K = 1 - \sum_{i=1}^{K-1} X_i \geq 0$ ensures that $(X_1, \ldots, X_K)$ is a probability vector. Denoting this prior $P_{\text{hard}}^*$ and letting $p_{K}^{**}(x) = 2p_K^*(2x)$ (so $Z_i \sim p_K^* \implies X_i \sim p_{K}^{**}$) we get that
\[
\inf_{f \in \mathcal{F}} \tilde{L}_K(P_{\text{hard}}^*, f) \geq (K-1) \inf_{f \in \mathcal{F}} \tilde{L}(p_K^{**}, f) \tag{185}
\]
\[
= (K-1) \frac{1}{2} L^\dagger(p_K^*, f^*_K) \geq \frac{1}{2} \frac{K-1}{K} \sup_{P \in \mathcal{P}_K^*} \tilde{L}_K(P, f^*_K). \tag{186}
\]

The last inequality holds because $p_K^*$ is the maximin density (under expectation constraints). To make $P_{\text{hard}}^*$ symmetric, we permute the letter indices randomly without affecting the raw loss; thus we get Corollary 1. To show how to get (186) from (185), we have
\[
\inf_{f \in \mathcal{F}} \tilde{L}(2p_K^*(2x), f) = \frac{1}{24} \left( \int_0^1 (2p_K^*(2x)^{1/3} dx \right)^3 \tag{187}
\]
\[
= \frac{1}{24} \left( \int_0^1 (2p_K^*(u)^{1/3} du \right)^3 \tag{188}
\]
\[
= \frac{1}{2} L^\dagger(p_K^*, f^*_K) \tag{189}
\]

In Figure 3, we validate the distribution $P_{\text{hard}}^*$ by showing the performance of each compander when quantizing random distributions drawn from $P_{\text{hard}}^*$. For the minimax compander, the KL divergence loss on the worst-case prior looks to be within a constant of that for the other datasets.
VI. COMPANDERS FOR OTHER DISTANCES AND SPACES

While our primary focus has been KL divergence over the simplex, for context we compare our results to what the same compander analysis would give for other loss functions like squared Euclidean distance ($L_2^2$) and absolute distance ($L_1$). For a vector $x$ and its representation $y$ let

\[ L_2^2(x, y) = \sum_i (x_i - y_i)^2 \]  
\[ L_1(x, y) = \sum_i |x_i - y_i| \]

For squared Euclidean distance, asymptotic loss was already given by (35) in [2]. Note that the formula scales as $1/N^2$. It turns out that the maximin single-letter distribution over a bounded interval is the uniform distribution. Thus, the minimax compander for $L_2^2$ is simply the identity function. Uniform quantization is indeed the ‘right’ choice for quantizing a hypercube in high-dimensional space. (For unbounded spaces, $L_2^2$ loss does not scale with $1/N^2$.)

If we add the expected value constraint to the $L_2^2$ compander optimization problem, we can derive the best square distance compander for the probability simplex. For alphabet size $K$, we get that the minimax compander for $L_2^2$ is given by

\[ f_{L_2^2, K}(x) = \frac{\sqrt{1 + K(K - 2)x} - 1}{K - 2} \]
<table>
<thead>
<tr>
<th>Loss</th>
<th>Space</th>
<th>Optimal Compander</th>
<th>Asymp. UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>KL</td>
<td>Simplex</td>
<td>$f^R_K(x) = \frac{\text{ArcSinh}(\sqrt{c_K(K \log K)} x)}{\text{ArcSinh}(\sqrt{c_K K \log K})}$</td>
<td>$(\log K)^2$</td>
</tr>
<tr>
<td>$L_2^2$</td>
<td>Simplex</td>
<td>$f_{L_2,K}(x) = \frac{\sqrt{1+K(K-2)x-1}}{K-2}$</td>
<td>$\frac{1}{N^2}$</td>
</tr>
<tr>
<td>$L_2^2$</td>
<td>Hypercube</td>
<td>$f_{L_2}(x) = x$ (uniform quantizer)</td>
<td>$\frac{K}{N^2}$</td>
</tr>
<tr>
<td>$L_1$</td>
<td>Simplex</td>
<td>$f_{L_1,K}(x) = \frac{\log(\gamma_K x+1)}{\log(\gamma_K +1)}$</td>
<td>$(\log K)^3$</td>
</tr>
<tr>
<td>$L_1$</td>
<td>Hypercube</td>
<td>$f_{L_1}(x) = x$ (uniform quantizer)</td>
<td>$\frac{K}{N}$</td>
</tr>
</tbody>
</table>

Fig. 4. Summary of results for various losses and spaces. Asymp. UB is the asymptotic upper bound, i.e an upper bound on how we expect the loss of the optimal compander to scale with $N$ and $K$ (constant terms are neglected).

and the total $L_2^2$ loss for probability vector $x$ and its quantization $y$ has the relation

$$
\lim_{N \to \infty} N^2 L_2^2(x, y) \leq \frac{1}{3}.
$$

(193)

For $L_1$, unlike KL divergence and $L_2^2$, the loss scales as $1/N$. Like $L_2^2$, the minimax single-letter compander for $L_1$ loss in the hypercube $[0, 1]^K$ is the identity function, i.e. uniform quantization. In general, the derivative of the optimal compander for single-letter density $p(x)$ has the form

$$
f'_{L_1,K}(x) \propto \sqrt{p(x)}.
$$

(194)

On the probability simplex for alphabet size $K$, the worst case prior $p(x)$ has the form

$$
p(x) = (\alpha_K x + \beta_K)^{-2}
$$

(195)

where $\alpha_K, \beta_K$ are constants scaling to allow $\int_{[0,1]} dp = 1$ (i.e. $p$ is a valid probability density) and $\int_{[0,1]} x dp = 1/K$ (i.e. $E_{X \sim p}[X] = 1/K$ so $K$ copies of it are expected to sum to 1).

Thus, the minimax compander on the simplex for $L_1$ loss (and letting $\gamma_K = \alpha_K / \beta_K$) satisfies

$$
f'_{L_1,K}(x) \propto (\alpha_K x + \beta_K)^{-1}
$$

(196)

$$
\implies f_{L_1,K}(x) \propto \log((\alpha_K / \beta_K) x + 1)
$$

(197)

$$
\implies f_{L_1,K}(x) = \frac{\log(\gamma_K x + 1)}{\log(\gamma_K + 1)}
$$

(198)

since $f_{L_1,K}(x)$ has to be scaled to go from 0 to 1.

The asymptotic $L_1$ loss for probability vector $x$ and its quantization $y$ is bounded by

$$
\lim_{N \to \infty} NL_1(x, y) \leq \frac{(\log K)^3}{4}.
$$

(199)
VII. ACKNOWLEDGEMENTS

We would like to thank Anthony Philippakis for his guidance on the DNA \(k\)-mer experiments.

REFERENCES


Appendix Organization

Appendix A: We fill in the details on the lemmas and propositions used in the proof of Theorem 1. In Sections A-A to A-E we cover the results needed to prove Proposition 4, while in Sections A-F and A-G we cover the results needed to prove Proposition 5.

Appendix B: We develop and analyze other types of companders, specifically beta companders, which are optimized to quantize vectors from Dirichlet priors (Section B-A), and power companders, which have the form \( f(x) = x^s \) and have properties similar to the minimax compander (Section B-B). Supplemental experimental results are also provided.

Appendix C: We analyze the minimax compander and approximate minimax compander more deeply, showing that \( c_K \in [1/4, 3/4] \) (Section C-A) and \( \lim_{K \to \infty} c_K = 1/2 \) (Section C-B). We also show that when \( c_K \approx 1/2 \), the approximate minimax compander has performance close to the minimax compander against all priors \( p \in \mathcal{P} \) (Section C-C). Supplemental experimental results are also provided.

Appendix D: We prove Theorem 5, showing bounds on the worst-case loss (adversarially selected \( x \), rather than from a prior) for the power, minimax, and approximate minimax companders.

Appendix A

Asymptotic Single-Letter Loss Proofs

In this appendix, we fill in the details on the lemmas and propositions used in the proof of Proposition 4 (showing that the local loss functions \( g_N \) converge to the asymptotic local loss function \( g \) a.s. when the input \( X \) is distributed according to \( p \in \mathcal{P} \)), including proofs for all results from Section IV-B (specifically Lemmas 1 and 2 and Propositions 6 to 8). This is covered in Sections A-A to A-E.

We then fill in the details of the lemmas for the proof of Proposition 5 (showing the existence of an integrable \( h \) dominating \( g_N \) when the compander \( f \) is from the ‘well-behaved’ set \( \mathcal{F}^+ \)), specifically Lemmas 3 and 6.

A. Proof of Lemma 1

Proof. Note that for fixed \( \theta \) and \( x \), \( r^{(x, \theta, \varepsilon)} \) is nonnegative and monotonically decreases as \( \varepsilon \) decreases. Thus \( \lim_{\varepsilon \to 0} r^{(x, \theta, \varepsilon)} \geq 0 \) is well defined.
We first assume that \( \lim_{\varepsilon \to 0} r^{(x, \theta, \varepsilon)} = 0 \) for all \( \theta \in [0, 1] \). Let \( s_\theta(r) \) be defined as
\[
s_\theta(r) := \frac{f(x + (1 - \theta)r) - f(x - \theta r)}{r}. \tag{200}
\]
We want to show that \( \lim_{r \to 0} s_\theta(r) = f'(x) \) for all \( \theta \in [0, 1] \), and that this limit is uniform over \( \theta \in [0, 1] \). For \( \theta \in \{0, 1\} \) we get respectively the right and left derivatives and since \( f \) is differentiable at \( x \) we are done for those cases. For \( \theta \in (0, 1) \) we write:
\[
s_\theta(r) = \frac{f(x + (1 - \theta)r) - f(x - \theta r)}{r} \tag{201}
\]
\[
= \frac{f(x + (1 - \theta)r) - f(x) + f(x) - f(x - \theta r)}{r} \tag{202}
\]
\[
= (1 - \theta) \frac{f(x + (1 - \theta)r) - f(x)}{(1 - \theta)r} + \theta \frac{f(x - \theta r) - f(x)}{-\theta r}. \tag{203}
\]
This implies
\[
\lim_{r \to 0} s_\theta(r) = \lim_{r \to 0} \left( (1 - \theta) \frac{f(x + (1 - \theta)r) - f(x)}{(1 - \theta)r} + \theta \frac{f(x - \theta r) - f(x)}{-\theta r} \right) \tag{204}
\]
\[
= (1 - \theta) f'(x) + \theta f'(x) = f'(x). \tag{205}
\]
Furthermore we note that the convergence is uniform over \( \theta \in [0, 1] \). This is because for any \( \alpha > 0 \), there is a \( \delta > 0 \) such that for \( \|r\| \leq \delta \),
\[
\left| \frac{f(x + r) - f(x)}{r} - f'(x) \right| \leq \alpha. \tag{206}
\]
But \( |r| \leq \delta \implies | - \theta r| \leq \delta \) and \( |(1 - \theta)r| \leq \delta \). Thus,
\[
|s_\theta(r) - f'(x)| = \left| (1 - \theta) \frac{f(x + (1 - \theta)r) - f(x)}{(1 - \theta)r} + \theta \frac{f(x - \theta r) - f(x)}{-\theta r} - f'(x) \right| \tag{207}
\]
\[
\leq \left| (1 - \theta) \frac{f(x + (1 - \theta)r) - f(x)}{(1 - \theta)r} - (1 - \theta) f'(x) \right| + \left| \theta f(x - \theta r) - f(x) - \theta f'(x) \right| \tag{208}
\]
\[
\leq (1 - \theta) \alpha + \theta \alpha \tag{209}
\]
\[
= \alpha. \tag{210}
\]
Thus we have uniform convergence of \( s_\theta(r) \) to \( f'(x) \) over all \( \theta \in [0,1] \) as \( r \to 0 \). Since \( r(x,\theta,\varepsilon) \to 0 \) as \( \varepsilon \to 0 \),

\[
f'(x) = \lim_{\varepsilon \to 0} s_\theta(r(x,\theta,\varepsilon)) = \lim_{\varepsilon \to 0} \frac{f(x + (1 - \theta)r(x,\theta,\varepsilon)) - f(x - \theta r(x,\theta,\varepsilon))}{r(x,\theta,\varepsilon)}
\]

\[
= \lim_{\varepsilon \to 0} \frac{\varepsilon}{r(x,\theta,\varepsilon)}
\]

\[
\implies \lim_{\varepsilon \to 0} \varepsilon^{-1} r(x,\theta,\varepsilon) = f'(x)^{-1}
\]

as we wanted. The third equality comes from the definition of \( r(x,\theta,\varepsilon) \) (61) and the fact that \( f'(x) \) is well-defined.

Now we need to consider what happens if \( \lim_{\varepsilon \to 0} r(x,\theta,\varepsilon) \neq 0 \) for some values of \( \theta \); this can either be because the limit is positive or because the limit doesn’t exist, but in either case it is clearly only possible if \( f \) is not strictly monotonic at \( x \) and hence only if \( f'(x) = 0 \). Additionally, it can only happen if \( f \) is flat at \( x \), i.e. there is either some \( a < x \) or some \( a > x \) such that \( f(a) = f(x) \) (or both). In this case, for any \( 0 < \theta < 1 \), \( I(x,\theta,\varepsilon) \) contains the interval between \( a \) and \( x \) and hence \( r(x,\theta,\varepsilon) \geq |x - a| \). For \( \theta = 0 \) and \( \theta = 1 \), either \( r(x,\theta,\varepsilon) \) is bounded away from 0, or it approaches 0; in the first case, \( \varepsilon^{-1} p(x,\theta,\varepsilon) \to \infty \) by default, while in the second the proof for the \( \lim_{\varepsilon \to 0} r(x,\theta,\varepsilon) = 0 \) case holds.

Thus, for all values of \( \theta \in [0,1] \), we know that \( \lim_{\varepsilon \to 0} \varepsilon^{-1} r(x,\theta,\varepsilon) = \infty \) as we need; and this is uniform over \( \theta \) because for any \( \theta \in (0,1) \) we have \( \varepsilon^{-1} r(x,\theta,\varepsilon) \geq \varepsilon^{-1} |x - a| \), meaning that for any large \( \alpha > 0 \), we can choose \( \varepsilon^* \) small enough so that for all \( \varepsilon < \varepsilon^* \) all of the following hold: (i) \( \varepsilon^{-1} |x - a| > \alpha \); (ii) \( \varepsilon^{-1} p(x,0,\varepsilon) \geq \alpha \); and (iii) \( \varepsilon^{-1} p(x,0,\varepsilon) \geq \alpha \). Thus, we have uniform convergence and we are done.

B. Proof of Proposition 6

Proof. We can assume that \( \varepsilon \leq \frac{1}{2} \) (if not, just use the value of \( \delta \) corresponding to \( \varepsilon = \frac{1}{2} \)). Let \( \delta > 0 \) be such that for all \( x' \) such that \( |x' - x| \leq \delta \),

\[
\left| \frac{F_p(x') - F_p(x)}{x' - x} - p(x) \right| \leq p(x)\varepsilon/8
\]

Since the derivative \( p(x) = F'_p(x) \) is well-defined, this \( \delta \) must exist. Then for \( x' \in I \),

\[
\left| (F_p(x') - F_p(x)) - (x' - x)p(x) \right| \leq |x' - x|p(x)\varepsilon/8 \]

\[
\leq r_I p(x)\varepsilon/8
\]
Now let $x''$ also be such that $|x'' - x| \leq \delta$. Then

\[
|F_p(x'') - (x'' - x)p(x)| = \left|((F_p(x'') - F_p(x)) - (x'' - x)p(x)) - ((F_p(x) - F_p(x)) - (x' - x)p(x))\right| \leq r_I p(x)\varepsilon/4
\]  

(218)

Let $x'$ be the lower boundary of $I$, so $x' + r_I$ is the upper boundary of $I$ (for which the above of course applies). Then we get

\[
\left|\frac{F_p(x' + r_I) - F_p(x')}{{r_I}p(x)}\right| \leq \varepsilon/4.
\]  

(222)

Then we know that for any $x'' \in I$,

\[
F_{p|I}(x'') = \frac{F_p(x'') - F_p(x')}{{F_p(x') + r_I} - F_p(x')}.
\]  

(224)

By (221) we know that

\[
(x'' - x')p(x) - r_I p(x)\varepsilon/4 \leq F_p(x'') - F_p(x') \leq (x'' - x')p(x) + r_I p(x)\varepsilon/4
\]  

(255)

\[
\Rightarrow r_I p(x)((x'' - x')/r_I - \varepsilon/4) \leq F_p(x'') - F_p(x') \leq r_I p(x)((x'' - x')/r_I + \varepsilon/4)
\]  

(226)

and by (223) we know that

\[
r_I p(x) - r_I p(x)\varepsilon/4 \leq F_p(x + r_I) - F_p(x') \leq r_I p(x) + r_I p(x)\varepsilon/4
\]  

(227)

\[
\Rightarrow r_I p(x)(1 - \varepsilon/4) \leq F_p(x + r_I) - F_p(x') \leq r_I p(x)(1 + \varepsilon/4).
\]  

(228)
Noting that $(x'' - x')/r_I = F_{\text{unif}_I}(x'') \in [0, 1]$ is the CDF of the uniform distribution on $I$, we get that

$$F_{p|I}(x'') \geq \frac{r_I p(x)((x'' - x')/r_I - \varepsilon/4)}{r_I p(x)(1 + \varepsilon/4)}$$

$$= \frac{(x'' - x')/r_I - \varepsilon/4}{1 + \varepsilon/4}$$

$$\geq F_{\text{unif}_I}(x'') - \varepsilon$$

and similarly that

$$F_{p|I}(x'') \leq \frac{r_I p(x)((x'' + x')/r_I - \varepsilon/4)}{r_I p(x)(1 - \varepsilon/4)}$$

$$= \frac{(x'' - x')/r_I + \varepsilon/4}{1 - \varepsilon/4}$$

$$\leq F_{\text{unif}_I}(x'') + \varepsilon$$

and hence for such a $\delta > 0$ we have for all $I$ containing $x$ and such that $r_I \leq \delta$ we have

$$|F_{p|I}(x'') - F_{\text{unif}_I}(x'')| \leq \varepsilon$$

for all $x'' \in I$. For $x'' \notin I = [x', x' + r_I]$ we then observe that

$$F_{p|I}(x'') = F_{\text{unif}_I}(x'') = \begin{cases} 0 & \text{if } x'' < x' \\ 1 & \text{if } x'' > x' + r_I \end{cases}$$

thus finishing the proof. \qed

C. Proof of Proposition 7

Proof. Let $\xi = \tilde{y}_{p,I} - \tilde{y}_I$. Then:

$$|\xi| = \left| \int_I \left( \mathbb{P}_{X \sim p|I}[X \geq x] - \mathbb{P}_{X \sim \text{unif}_I}[X \geq x] \right) dx \right|$$

$$\leq \int_I |\mathbb{P}_{X \sim p|I}[X \geq x] - \mathbb{P}_{X \sim \text{unif}_I}[X \geq x]| dx$$

$$\leq r_I \varepsilon.$$ 

For any distribution $q$ and any fixed value $z$, define the shift operator $T_z(q)$ to denote the distribution of $X - z$ where $X \sim q$ (i.e. just shift it by $z$). Note that $T_{\tilde{y}_{p,I}}(p|I)$ and $T_{\tilde{y}_I}(\text{unif}_I)$ are
both constructed to have expectation 0, and in particular $T_{g_i}(\text{unif}_I)$ is the uniform distribution over an interval of width $r_I$ centered at 0. Additionally,

$$d_{KS}(T_{g_i}(p|I), T_{g_i}(\text{unif}_I)) \leq d_{KS}(T_{g_i}(p|I), T_{g_i}(\text{unif}_I))$$

$$+ d_{KS}(T_{g_i}(\text{unif}_I), T_{g_i}(\text{unif}_I)) \leq 2\varepsilon$$

since $d_{KS}(<, \cdot )$ is a metric, $d_{KS}(q_1, q_2) = d_{KS}(T_z(q_1), T_z(q_2))$ for any $q_1, q_2$ and $z$, and

$$d_{KS}(T_{z_1}(\text{unif}_I), T_{z_2}(\text{unif}_I)) \leq |z_2 - z_1|/r_I .$$

For convenience, let $q_1 = T_{g_i}(p|I)$ and $q_2 = T_{g_i}(\text{unif}_I)$, and let $Z_1 \sim q_1$ and $Z_2 \sim q_2$. We know the following: $E[Z_1] = E[Z_2] = 0$; $d_{KS}(q_1, q_2) \leq 2\varepsilon$; and $q_1, q_2$ have support on $[-r_I, r_I]$.

Let $\eta_i = E[Z_1] - E[Z_2]$. Then we can compute the following:

$$|\eta_i| = \left| \int_0^{r_I} (\mathbb{P}[Z_1 \geq x] - \mathbb{P}[Z_2 \geq x]) \, dx - \int_0^{-r_I} (\mathbb{P}[Z_1 \leq -x] - \mathbb{P}[Z_2 \leq -x]) \, dx \right|$$

If $i$ is odd, then we do a $u$-substitution with $u = x^{1/i}$ and get

$$|\eta_i| = \left| \int_0^{r_I} (\mathbb{P}[Z_1 \geq x^{1/i}] - \mathbb{P}[Z_2 \geq x^{1/i}]) \, dx - \int_{-r_I}^0 (\mathbb{P}[Z_1 \leq -x^{1/i}] - \mathbb{P}[Z_2 \leq -x^{1/i}]) \, dx \right|

= i \left| \int_0^{r_I} u^{-1}(\mathbb{P}[Z_1 \geq u] - \mathbb{P}[Z_2 \geq u]) \, du - \int_{-r_I}^0 u^{-1}(\mathbb{P}[Z_1 \leq u] - \mathbb{P}[Z_2 \leq u]) \, du \right|

\leq 2 \int_0^{r_I} iu^{-1}2\varepsilon \, du = 4\varepsilon r_I$$

Similarly if $i$ is even we get

$$|\eta_i| = \left| \int_0^{r_I} (\mathbb{P}[Z_1 \geq x^{1/i}] - \mathbb{P}[Z_2 \geq x^{1/i}]) \, dx + \int_{-r_I}^0 (\mathbb{P}[Z_1 \leq -x^{1/i}] - \mathbb{P}[Z_2 \leq -x^{1/i}]) \, dx \right|

= i \left| \int_0^{r_I} u^{-1}(\mathbb{P}[Z_1 \geq u] - \mathbb{P}[Z_2 \geq u]) \, du + \int_{-r_I}^0 u^{-1}(\mathbb{P}[Z_1 \leq u] - \mathbb{P}[Z_2 \leq u]) \, du \right|

\leq 2 \int_0^{r_I} iu^{-1}2\varepsilon \, du = 4\varepsilon r_I$$

and we can conclude that $|\eta_i| \leq 4\varepsilon r_I$ in general.
Then we can take the respective Taylor expansions: let \( X_1 \sim p \mid I \) and \( X_2 \sim \text{unif}_I \) (and \( Z_1 \sim q_1, Z_2 \sim q_2 \) as above). We get

\[
\ell_{p,I} = \mathbb{E}[X_1 \log(X_1/\tilde{y}_{p,I})] = \tilde{y}_{p,I} \mathbb{E}[(Z_1/\tilde{y}_{p,I} + 1) \log(Z_1/\tilde{y}_{p,I} + 1)] = \tilde{y}_{p,I} \mathbb{E} \left[ Z_1/\tilde{y}_{p,I} + \frac{(Z_1/\tilde{y}_{p,I})^2}{2} - \frac{(Z_1/\tilde{y}_{p,I})^3}{6(1 + \eta)^2} \right] \tag{253}
\]

where \( \eta \) is a number between 0 and \( Z_1/\tilde{y}_{p,I} \) (we get this using Lagrange’s formula for the error).

Since \( Z_1 + \tilde{y}_{p,I} \in I \), we know that

\[
\tilde{y}_{p,I} - r_I \leq z + \tilde{y}_{p,I} \leq \tilde{y}_{p,I} + r_I \tag{254}
\]

Since \( r_I < x/4 \) and \( \tilde{y}_{p,I} \geq x - r_I \) (as \( x, \tilde{y}_{p,I} \) share the width-\( r_I \) interval \( I \)), we get that \( \tilde{y}_{p,I} > 3r_I \), and therefore

\[
\frac{2}{3} \tilde{y}_{p,I} < Z_1 + \tilde{y}_{p,I} < \frac{4}{3} \tilde{y}_{p,I} \tag{255}
\]

\[
\implies -\frac{1}{3} < Z_1/\tilde{y}_{p,I} < \frac{1}{3} \tag{256}
\]

This gives that \(|\eta| < 1/3\). Using this and the fact that \( \mathbb{E}[Z_1] = 0 \) by construction, we can write (253) as

\[
\ell_{p,I} \leq \frac{1}{2} \mathbb{E}[Z_1^2]/\tilde{y}_{p,I} + \frac{|\mathbb{E}[Z_1^2]|}{8/3} (\tilde{y}_{p,I})^{-2} \tag{257}
\]

\[
\leq \frac{1}{2} \mathbb{E}[Z_1^2]/\tilde{y}_{p,I} + \frac{r_I^3}{8/3(x - r_I)^2} \tag{258}
\]

Since \( r_I < x/4 \), we know that \( x - r_I > (3/4)x \), and hence

\[
\ell_{p,I} \leq \frac{1}{2} \mathbb{E}[Z_1^2]/\tilde{y}_{p,I} + (2/3)r_I^3 x^{-2} \tag{259}
\]

Hence we get

\[
\ell_{p,I} = \frac{1}{2} \mathbb{E}[Z_1^2]/\tilde{y}_{p,I} + O(r_I^3 x^{-2}) \tag{260}
\]

Because \( x - r_I \leq \bar{y}_I \) as well (and \( Z_2 \) has support on \([-r_I, r_I]\)) we can repeat the above arguments to conclude similarly that

\[
\ell_{\text{unif}_I} = \frac{1}{2} \mathbb{E}[Z_2^2]/\bar{y}_I + O(r_I^3 x^{-2}) \tag{261}
\]

Hence their difference is

\[
|\ell_{p,I} - \ell_{\text{unif}_I}| \leq \frac{1}{2} |\mathbb{E}[Z_1^2]/\tilde{y}_{p,I} - \mathbb{E}[Z_2^2]/\bar{y}_I| + O(r_I^3 x^{-2}) \tag{262}
\]
Taking the main term, we split it into three parts:

\[
\left| \mathbb{E}[Z_1^2]/\tilde{y}_{p,I} - \mathbb{E}[Z_2^2]/\tilde{y}_I \right| \leq \left| \mathbb{E}[Z_1^2]/\tilde{y}_{p,I} - \mathbb{E}[Z_2^2]/x \right| + \left| \mathbb{E}[Z_2^2]/\tilde{y}_I - \mathbb{E}[Z_2^2]/x \right| + \left| \mathbb{E}[Z_2^2]/x - \mathbb{E}[Z_2^2]/x \right|. \tag{263}
\]

The first part (263) can be bounded by

\[
\left| \mathbb{E}[Z_1^2]/\tilde{y}_{p,I} - \mathbb{E}[Z_2^2]/x \right| \leq \left| \mathbb{E}[Z_1^2]/x - 1/\tilde{y}_{p,I} - 1/x \right| \leq r_I^2 \frac{|x - \tilde{y}_{p,I}|}{\tilde{y}_{p,I} x} \leq (4/3) r_I^3 x^{-2} = O(r_I^3 x^{-2}). \tag{266}
\]

An analogous argument bounds (264), giving

\[
\left| \mathbb{E}[Z_2^2]/\tilde{y}_I - \mathbb{E}[Z_2^2]/x \right| = O(r_I^3 x^{-2}). \tag{270}
\]

Finally, (265) follows from

\[
\left| \mathbb{E}[Z_2^2]/x - \mathbb{E}[Z_2^2]/x \right| = |\eta_2| x^{-1} \leq 4 \varepsilon r_I^2 x^{-1}. \tag{271}
\]

Thus, plugging it all into (262) we get

\[
|\ell_{p,I} - \ell_{\text{unif},I}| \leq 2\varepsilon r_I^2 x^{-1} + O(r_I^3 x^{-2}). \tag{272}
\]

\[\square\]

D. Proof of Proposition 8

Proof. Let \(i^*\) be such that \(r_{I,i^*} \leq x/4\) for all \(i \geq i^*\) (since \(\lim_{i \to \infty} r_{I,i} = 0\) this exists) and WLOG consider the sequence of \(i \geq i^*\). The result then follows from the Taylor series of \(\ell_{\text{unif},I}\), as shown by (261) (see proof of Proposition 7 in Section A-C). Keeping the definition from the proof of Proposition 7, we let \(Z_2 \sim T_{\tilde{y}_I}(\text{unif}_{I})\), i.e. uniform over a width-\(r_{I,i}\) interval centered at 0. Thus we have \(\mathbb{E}[Z_2^2] = \frac{1}{12} r_{I,i}^2\) and hence (261) yields

\[
\ell_{\text{unif},I} = \frac{1}{2} \mathbb{E}[Z_2^2]/\tilde{y}_I + O(r_I^3 x^{-2}) = \frac{1}{24} r_{I,i}^2 \tilde{y}_I^{-1} + O(r_I^3 x^{-2}).
\]

\[\square\]
But $\tilde{y}_I$ and $x$ share the interval $I_i$ and hence as $r_I \to 0$,

$$\tilde{y}_I = x + O(r_I)$$  \hspace{2cm} (275)

$$= x(1 + O(r_Ix^{-1}))$$  \hspace{2cm} (276)

$$\implies \tilde{y}_I^{-1} = x^{-1}(1 + O(r_Ix^{-1}))$$  \hspace{2cm} (277)

since when $r_I$ is very small, $O(r_Ix^{-1})$ is very small so $(1 + O(r_Ix^{-1})^{-1} = 1 + O(r_Ix^{-1})$ (the inverse of a value close to 1 is also close to 1). Thus, we can replace $\tilde{y}_I^{-1}$ in (274) to get

$$\ell_{unif_I} = \frac{1}{24} r_I^2 x^{-1} + O(r_I^3x^{-2})$$  \hspace{2cm} (278)

as we wanted. \hfill \square

**E. Single-Interval Loss Function Properties and Proof of Lemma 2**

We prove Lemma 2 here; to do so, we show a few lemmas concerning the single-interval loss function $\ell_{p,I}$. First, we show an alternative formula for $\ell_{p,I}$ which sheds some light on it:

**Lemma 9.** For any $p, I$,

$$\ell_{p,I} = \mathbb{E}_{X \sim p|I}[X \log X] - \tilde{y}_{p,I} \log(\tilde{y}_{p,I})$$  \hspace{2cm} (279)

**Proof.** We compute $\ell_{p,I}$ as follows:

$$\ell_{p,I} = \mathbb{E}_{X \sim p}[X \log(X/\tilde{y}_{p,I}) | X \in I]$$  \hspace{2cm} (280)

$$= \mathbb{E}_{X \sim p|I}[X \log(X/\tilde{y}_{p,I})]$$  \hspace{2cm} (281)

$$= \mathbb{E}_{X \sim p|I}[X \log(X) - X \log(\tilde{y}_{p,I})]$$  \hspace{2cm} (282)

$$= \mathbb{E}_{X \sim p|I}[X \log X] - \mathbb{E}_{X \sim p|I}[X] \log(\tilde{y}_{p,I})$$  \hspace{2cm} (283)

$$= \mathbb{E}_{X \sim p|I}[X \log X] - \tilde{y}_{p,I} \log(\tilde{y}_{p,I})$$  \hspace{2cm} (284)

since $\tilde{y}_{p,I} = \mathbb{E}_{X \sim p|I}[X]$. \hfill \square

We now want to show that it really does represent something resembling a loss function: first, that it is nonnegative, and second that it achieves equality if and only if $X \sim p$ on $I$ is known for sure (so the decoded value can be guaranteed to equal $X$).

**Lemma 10.** For any $p$ and $I \subseteq [0, 1]$ (even $p$ is not continuous),

$$\ell_{p,I} \geq 0$$  \hspace{2cm} (285)
with equality if and only if there is some $z \in I$ s.t.

$$\mathbb{P}_{X \sim p}[X = z \mid X \in I] = 1.$$  

(286)

Proof. Using Lemma 9, if we define the function $h(t) = t \log t$ then since $h$ is strictly convex, by Jensen’s Inequality (where all expectations are over $X \sim p|_I$)

$$\ell_{p, I} = \mathbb{E}[h(X)] - h(\mathbb{E}[X]) \geq 0$$

(287)

with equality if and only if $X \sim p|_I$ is fixed with probability 1.

This yields the following corollary:

**Corollary 2.** If $p \in \mathcal{P}$ and $I$ has nonzero width,

$$\ell_{p, I} > 0.$$  

(288)

This follows because $p \in \mathcal{P}$ is continuous and so cannot have all its mass on a particular value in any nonzero-width $I$. If $I$ has zero probability mass under $p$, then $\ell_{p, I}$ defaults to the interval loss under a uniform distribution.

Finally, we can prove Lemma 2. Recall that it shows that if $I$ has nonzero probability mass under $p$, one cannot get the interval loss to approach 0 by choosing $J \supseteq I$, i.e. if $p \in \mathcal{P}$ and $I$ is such that $\mathbb{P}_{X \sim p}[X \in I] > 0$, then there is some $\alpha > 0$ (which can depend on $I$) such that

$$\ell_{p, J} \geq \alpha \text{ for all } J \supseteq I.$$  

(289)

Proof of Lemma 2. We can re-write $\ell_{p, J}$ as

$$\ell_{p, J} = \mathbb{E}_{X \sim p}[X \log(X/\widehat{y}_{p, J}) \mid X \in J]$$

(290)

$$= \int_J p(x) \frac{x \log(x/\widehat{y}_{p, J})}{\int_J dp} dx$$

(291)

where $\int_J dp$ is just the integral representation of $\mathbb{P}_{X \sim p}[X \in J]$.

Therefore, since $p \in \mathcal{P}$, $\ell_{p, J}$ is continuous at $J$ with respect to the boundaries of $J$ (the inverse probability mass $(\int_J dp)^{-1}$ is continuous since $\int_J dp \geq \int_I dp > 0$).

Thus, we can consider $\ell_{p, J}$ as a continuous function over the boundaries of $J$ on the domain where $I \subseteq J \subseteq [0, 1]$; this domain can be represented as a closed subset of $[0, 1]^2$ and hence is compact. Thus, by the Weierstrass extreme value theorem, $\ell_{J, p}$ achieves its minimum $\alpha$ on this domain, and by Corollary 2 it must be positive.

Hence, we have shown that there is an $\alpha > 0$ such that for any $J \supseteq I$, $\ell_{p, J} > \alpha$.  

\qed
F. Proof of Lemma 3

Proof. We WLOG restrict ourselves to $q$ which are probability distributions over $I$. Let $\mathcal{P}_I$ denote the set of probability distributions over $I$ (not necessarily continuous) and $\mathcal{P}'_I$ denote the set of probability distributions over $I$ which place all the probability mass on the boundaries $\bar{y}_I - r_I/2$ and $\bar{y}_I + r_I/2$, i.e. for all $q' \in \mathcal{P}'_I$ we have

$$\mathbb{P}_{X \sim q}[X \in \{\bar{y}_I - r_I/2, \bar{y}_I + r_I/2\}] = 1. \quad (292)$$

We then make the following claim:

Claim 1: For all $q \in \mathcal{P}_I$, exists $q' \in \mathcal{P}'_I$ such that $\ell_{q,I} \leq \ell_{q',I}$.

This follows from the convexity of the function $x \log(x)$ and the definition of $\ell_{q,I}$, i.e.

$$\ell_{q,I} = \mathbb{E}_{X \sim q}[X \log(X/\bar{y}_{q,I})] \quad (293)$$

(since $q$ in this case is a distribution over $I$, we removed the condition $X \in I$ as it is redundant). In particular, if $q'$ is the (unique) distribution in $\mathcal{P}'_I$ such that $\mathbb{E}_{X \sim q}[X] = \bar{y}_{q,I}$ (i.e. we move all the probability mass to the boundary but keep the expected value the same), then $\ell_{q',I}$ can be computed by considering the average over the linear function which connects the end points of $X \log(X/\bar{y}_{q,I})$ over $I$. Because of convexity, this linear function is always greater than or equal to $X \log(X/\bar{y}_{q,I})$ on $I$, and therefore $\ell_{q,I} \leq \ell_{q',I}$. Thus, Claim 1 holds and we can restrict our attention to $\mathcal{P}'_I$.

For simplicity we introduce a linear mapping $z$ from $[-1/2, 1/2]$ to $I$: for $\theta \in [-1/2, 1/2]$, let $z(\theta) = \bar{y}_I + \theta r_I$ (so $z(-1/2) = \bar{y}_I - r_I/2$ is the lower boundary of $I$, $z(1/2) = \bar{y}_I + r_I/2$ is the upper boundary, and $z(0) = \bar{y}_I$ is the midpoint). We also specially denote $a = z(-1/2)$ to be the lower boundary and $b = z(1/2)$ to be the upper boundary. Then, since any $q \in \mathcal{P}'_I$ can only assign probabilities to $a$ and $b$, we can parametrize all $q \in \mathcal{P}'_I$: let $q(\theta)$ denote the distribution assigning probability $1/2 + \theta$ to the upper boundary $b$ and $1/2 - \theta$ to the lower boundary $a$. Then this gives the nice formula:

$$\bar{y}_{q(\theta),I} = \bar{y}_I + \theta r_I = z(\theta) \quad (294)$$

i.e. $q(\theta)$ is the unique distribution in $\mathcal{P}'_I$ with expectation $z(\theta)$. This brings us to our next claim:

Claim 2: $\ell_{q(\theta),I} \leq 2\ell_{q(0),I}$ for any $\theta \in [-1/2, 1/2]$. Ignoring the redundant condition $X \in I$, we use

$$\ell_{q,I} = \mathbb{E}_{X \sim q}[X \log(X)] - \bar{y}_{q,I} \log(\bar{y}_{q,I}) \quad (295)$$
to re-write \( \ell_{q(\theta),I} \) as follows:

\[
\ell_{q(\theta),I} = (1/2 - \theta)a \log(a) + (1/2 + \theta)b \log(b) - z(\theta) \log(z(\theta))
\]  

(296)

This implies that

\[
\ell_{q(\theta),I} \leq \ell_{q(\theta),I} + \ell_{q(-\theta),I} = (a \log(a) + b \log(b)) - (z(\theta) \log(z(\theta)) + z(-\theta) \log(z(-\theta)))
\]  

(297)

\[
\leq (a \log(a) + b \log(b)) - 2\bar{y}_I \log(\bar{y}_I)
\]  

(298)

\[
= 2\ell_{q(0),I}
\]  

(299)

where the inequality follows because \( x \log(x) \) is convex and the mean of \( z(\theta) \) and \( z(-\theta) \) is \( z(0) = \bar{y}_I \), showing Claim 2.

**Claim 3:** \( 2\ell_{q(0),I} \leq \frac{1}{2T_I^2}\bar{y}_I^{-1} \).

This comes from rewriting according to (295) and then applying the Taylor series expansion of \((1 + t) \log(1 + t)\). Define \( t = r_I/(2\bar{y}_I) \leq 1 \) (otherwise \( I \notin [0,1] \)), we get:

\[
2\ell_{q(0),I}
\]  

(300)

\[
= (a \log(a) + b \log(b)) - 2\bar{y}_I \log(\bar{y}_I)
\]  

(301)

\[
= (\bar{y}_I - r_I/2) \log(\bar{y}_I - r_I/2) + (\bar{y}_I + r_I/2) \log(\bar{y}_I + r_I/2) - 2\bar{y}_I \log(\bar{y}_I)
\]  

(302)

\[
= (\bar{y}_I - r_I/2)(\log(\bar{y}_I - r_I/2) - \log(\bar{y}_I)) + (\bar{y}_I + r_I/2)(\log(\bar{y}_I + r_I/2) - \log(\bar{y}_I))
\]  

(303)

\[
= \bar{y}_I ((1 - t) \log(1 - t) + (1 + t) \log(1 + t))
\]  

(304)

\[
We can use the inequality that \((1 - t) \log(1 - t) + (1 + t) \log(1 + t) \leq 2t^2\) for \(|t| \leq 1\), to get

\[
2\ell_{q(0),I} \leq 2\bar{y}_I t^2 = \frac{1}{2}r_I^2\bar{y}_I^{-1}
\]  

(305)

(306)

This resolves Claim 3.

The lemma then follows from Claims 1, 2, and 3. \(\square\)

**G. Proof of Lemma 6**

**Proof.** First, note that the above conditions imply that \( f_2(x) \geq f_1(x) \) and that \( f'_2(x) \geq f'_1(x) \) for all \( x \) where both are defined (almost everywhere).

Let \( J^{f_i,N}_i = [a_i, b_i] \) for \( i = 1, 2 \). We will prove that \( a_1 \leq a_2 \) and \( b_1 \geq b_2 \). Note that by definition if \( f_1(x) - 1/N \leq 0 \) then \( a_1 = 0 \) and \( a_1 \leq a_2 \) happens by default; thus this is also
the case if \( f_2(x) - 1/N \leq 0 \) since \( f_2 \geq f_1 \) means this implies \( f_1(x) - 1/N \leq 0 \). Meanwhile, if \( f_2(x) + 1/N \geq 1 \) we have

\[
1/N \geq 1 - f_2(x) \geq f_2(1) - f_2(x) \geq f_1(1) - f_1(x)
\]

meaning that \( b_1 = 1 \) (and \( b_2 = 1 \)) so \( b_1 \geq b_2 \); and similarly \( f_1(x) + 1/N \geq 1 \) simply implies \( b_1 = 1 \geq b_2 \).

Thus we don’t need to worry about the boundaries hitting 0 or 1 (i.e. we can ignore the ‘\( \cap [0, 1] \)’ in the definition), as the needed result easily holds whenever it happens.

Then \( a_1 \) and \( a_2 \) are the values for which

\[
\int_{a_2}^x f'_2(t) \, dt = \int_{a_1}^x f'_1(t) \, dt = 1/N
\]

But since \( 0 \leq f'_1(t) \leq f'_2(t) \), we know that

\[
\int_{a_2}^x f'_2(t) \, dt = 1/N = \int_{a_1}^x f'_1(t) \, dt \leq \int_{a_1}^x f'_2(t) \, dt
\]

which implies that \( a_2 \geq a_1 \). An analogous proof on the opposite side proves \( b_1 \geq b_2 \) and hence

\[
J_{f_2,N,x} = [a_2, b_2] \subseteq [a_1, b_1] = J_{f_1,N,x}
\]

as we needed.

\[\square\]

**APPENDIX B**

**BETA AND POWER COMPANDERS**

In this appendix, we analyze beta companders, which are optimal companders for symmetric Dirichlet priors and are based on the normalized incomplete beta function (Section B-A) and power companders, which have the form \( f(x) = x^s \) and which have properties similar to the minimax compander when \( s = 1/\log K \) (Section B-B).

We also add supplemental experimental results. First, we compare the beta compander with truncation (identity compander) and the EDI (Exponential Density Interval) compander we developed in [1] in the case of the uniform prior on \( \bigtriangleup_{K-1} \) (which is equivalent to a Dirichlet prior with all parameters set to 1), on book word frequencies, and on DNA k-mer frequencies. EDI was, in a sense, developed to minimize the expected KL divergence loss for the uniform prior (specifically to remove dependence on \( K \)) as a means of proving a result on rate distortion; beta was then directly developed for all Dirichlet priors.
Second, we compare the theoretical prediction for the power compander against various data sets; this demonstrates a close match to the theoretical performance for synthetic (uniform on $\triangle_{K-1}$) data and DNA $k$-mer frequencies, while the power compander performs better on book word frequencies. Note that this is not a contradiction, as the theoretical prediction is for its performance on the worst possible prior – it instead indicates that book word frequencies are somehow more suited to power companders than the uniform distribution or DNA $k$-mer frequencies.

Finally, we compare how quickly the beta and power companders converge to their theoretical limits (with uniform prior); specifically how quickly $N^2\tilde{L}(p, f, N)$ converges to $\tilde{L}(p, f)$. The results show that for large $K (\approx 10^5)$, both are already very close by $N = 2^8 = 256$; while for smaller values of $K$, power companders still converge very quickly while beta companders may take even until $N = 2^{16} = 65536$ or beyond to be close.

A. Beta Companders for Symmetric Dirichlet Priors

**Definition 7.** When $X$ is drawn from a Dirichlet distribution with parameters $\alpha = \alpha_1, \ldots, \alpha_K$, we use the notation $X \sim \text{Dir}(\alpha)$. When $\alpha_1 = \cdots = \alpha_K = \alpha$, then $X$ is drawn from a symmetric Dirichlet with parameter $\alpha$ and we use the notation $X \sim \text{Dir}_K(\alpha)$.

As a corollary to Theorem 2, we get that the optimal compander for the symmetric Dirichlet distribution is the following:

**Corollary 3.** When $x \sim \text{Dir}_K(\alpha)$, let $p(x)$ be the associated single-letter density (same for all elements due to symmetry). The optimal compander for $p$ satisfies

$$f'(x) = B\left(\frac{\alpha + 1}{3}, \frac{(K - 1)\alpha + 2}{3}\right)^{-1} x^{(\alpha-2)/3} (1 - x)^{(\alpha-1)/3}$$

(311)

where $B(a, b)$ is the Beta function. Therefore, $f(x)$ is the normalized incomplete Beta function $I_x((\alpha + 1)/3, ((K - 1)\alpha + 2)/3)$.

Then

$$\tilde{L}(p, f) = \frac{1}{2} B\left(\frac{\alpha + 1}{3}, \frac{(K - 1)\alpha + 2}{3}\right)^3 B(\alpha, (K - 1)\alpha)^{-1}$$

(312)

This result uses the following fact:
Fig. 5. Comparing the beta compander and the EDI method. The random data is drawn with Dir$_K(1)$ (i.e. uniform).

**Fact 1.** For $X \sim \text{Dir}(\alpha_1, \ldots, \alpha_K)$, the marginal distribution on $X_k$ is $X_k \sim \text{Beta}(\alpha_k, \beta_k)$, where $\beta_k = \sum_{j \neq k} \alpha_j$. When the prior is symmetric with parameter $\alpha$, we get that all $X_k$ are distributed according to $\text{Beta}(\alpha, (K-1)\alpha)$.

**Remark 10.** Since (312) scales with $K^{-1}$, this means that $\hat{L}_K(\text{Dir}_K(\alpha), f)$ is constant with respect to $K$. This is consistent with what we get with the EDI compander (see [1]).

We will call the compander $f$ derived from integrating (311) the *beta compander*. (This is because integrating (311) gives an incomplete beta function.) The beta compander naturally performs better than the EDI method since this compander is optimized to do so. We can see the comparison in Figure 5 that on random uniform distributions, the beta compander is better than the EDI method by a constant amount for all $K$.

The beta compander is not the easiest algorithm to implement however. It is necessary to compute an incomplete beta function in order to find the compander function $f$, which is not known to have a closed form expression. We reiterate Remark 4 that it is indeed interesting that the minimax compander, on the other hand, does have a closed form.
Fig. 6. Comparing theoretical performance (315) of the power compander to experimental results.

B. Analysis of the Power Compander

Starting with Theorem 1, we can use the asymptotic analysis to understand why the power compander works well for all distributions.

**Proposition 9.** Let single-letter density \( p \) be the marginal probability of one letter on any symmetric probability distribution \( P \) over \( K \) letters. For the power compander \( f(x) = x^s \) where \( s \leq \frac{1}{2} \),

\[
\tilde{L}(p, f) \leq \frac{1}{K} \frac{1}{24} s^{-2} K^{2s}
\]

and for any prior \( P \in \mathcal{P}_K^\Delta \),

\[
\tilde{L}_K(P, x^s) \leq \frac{1}{24} s^{-2} K^{2s}.
\]

Optimizing over \( s \) gives

\[
\tilde{L}_K(P, f) \leq \frac{e^2}{24} (\log K)^2.
\]

*Proof.* Since \( f(x) = x^s \) we have that \( f'(x) = sx^{s-1} \). Using Theorem 1, this gives

\[
\tilde{L}(p, f) = \frac{1}{24} s^{-2} \int_0^1 x^{1-2s} p(x) dx = \frac{1}{24} s^{-2} \mathbb{E}_{X \sim p}[X^{1-2s}].
\]

The function \( x^{1-2s} \) is increasing and also a concave function. We want to find the maximin prior distribution \( P \in \mathcal{P}_K^\Delta \) (with marginals \( p \)) with the constraint

\[
\sum_i \mathbb{E}_{X_i \sim p}[X_i] = 1
\]
(another constraint is that values of \( p \) are such that must sum to one, but we give a weaker constraint here).

We want to choose \( P \) to maximize

\[
\sum_i \mathbb{E}_{X_i \sim p}[X_i^{1-2s}] = \mathbb{E}_{(X_1,\ldots,X_K) \sim P} \left[ \sum_i X_i^{1-2s} \right].
\]  

(318)

By concavity (even ignoring any constraint that \( P \) is symmetric), the maximum solution is given when \( X_1 = \cdots = X_K \). Therefore, the maximin \( P \) is such that the marginal on one letter \( p \) is

\[
p \left( \frac{1}{K} \right) = 1.
\]  

(319)

The probability mass function where \( 1/K \) occurs with probability 1 is a limit point of a continuous density which is of the form

\[
p(x) = \frac{1}{2 \epsilon} \text{ on } x \in \left[ \frac{1}{K} - \epsilon, \frac{1}{K} + \epsilon \right]
\]  

and 0 otherwise. We use this since we are restricting to continuous probability distribution.

Evaluating with this gives

\[
\tilde{L}(p, f) = \frac{1}{24} s^{-2} \mathbb{E}_{X \sim p}[X^{1-2s}] \leq \frac{1}{24} s^{-2} \left( \frac{1}{K} \right)^{1-2s} \leq \frac{1}{K} \frac{1}{24} s^{-2} K^{2s}
\]  

(320)

which shows (315). Multiplying by \( K \) gives \( \tilde{L}_K(P, f) \) for symmetric \( P \).

Note that for any non-symmetric \( P \), we can always symmetrize \( P \) to a symmetric prior \( P_{sym} \) by averaging over all random permutations of the indices. Because the loss \( \tilde{L}_K(P, f) \) is concave in \( P \), the symmetrized prior \( P_{sym} \) will give an higher value, that is \( \tilde{L}_K(P, f) \leq \tilde{L}_K(P_{sym}, f) \). Hence \( \tilde{L}_K(P, f) \leq \frac{1}{24} s^{-2} K^{2s} \) holds for all priors.

Finding the \( s \) which minimizes \( \frac{1}{24} s^{-2} K^{2s} \) is equivalent to finding \( s \) which minimizes \( s \log K - \log s \).

\[
0 = \frac{d}{ds} \log K - \log s = \log K - \frac{1}{s}
\]  

(321)

\[
\implies s = \frac{1}{\log K}.
\]  

(322)

(323)

We can plug this back into our equation, using the fact that \( e^{\log K} = K \) implies that \( K^{1/\log K} = e \).
Our final result is that using \( f(x) = x^{1/\log K} \) gives that for any prior \( P \)
\[
\mathcal{L}_K(P, f) \leq \frac{c^2}{24} (\log K)^2.
\]

The power compander turns out to give guarantees bounds on the value on \( \mathcal{L}_K(P, f) \) when \( f \) is chosen so that \( s = 1/\log K \). We show the comparison between this theoretical result on raw loss with the experimental results in Figure 6.

C. Converging to Theoretical

For both the power compander and the beta compander, we show in Figure 7 how quickly the experimental results converge to the theoretical results. Experimental results have a fixed granularity \( N \) whereas the theoretical results assume that \( N \to \infty \). The plots show that by \( N = 2^{16} \) (each value gets 16 bits), the experimental results for the power compander are very close to the theoretical results, and even for \( N = 2^8 \) they are not so far. For the beta compander, the experimental results are close to the theoretical when \( K \) is large. When \( K = 100 \), the results for \( N = 2^{16} \) is not that close to the theoretical result, which demonstrates the effect of using unnormalized (or raw) values. The difference between normalizing and not normalizing gets smaller as \( K \) increases.

APPENDIX C

MINIMAX AND APPROXIMATE MINIMAX COMPANDERS

In this appendix, we analyze the minimax compander and approximate minimax compander. Specifically, we analyze the constant \( c_K \), to show that it falls in \([1/4, 3/4] \) (Section C-A) and that \( \lim_{K \to \infty} c_K = 1/2 \) (Section C-B). We also show that when \( c_K \) is close to \( 1/2 \), the approximate minimax compander (which is the same as the minimax compander except it replaces \( c_K \) with \( 1/2 \)) has performance close to the minimax compander against all priors \( p \in \mathcal{P} \) (Section C-C).

A. Analysis of Minimax Companding Constant

1) Determining bounds on \( c_K \): If \( a_K, b_K \geq 0 \), then \( p(x) \) is well-behaved (and bigger than 0).
Fig. 7. Comparing theoretical expression $\tilde{L}(p, f)$ with experimental result. The KL divergence value of the experimental results are multiplied to $N^2$ in order to be comparable to $\tilde{L}(p, f)$.

Fig. 8. Comparing theoretical performance (169) of the approximate minimax compander to experimental results.
We need \( a_K \) and \( b_K \) to be such that \( p(x) \) is a density that integrates to 1 and also that \( p(x) \) has expected value of \( 1/K \). To do this, first we compute that

\[
E_{X \sim p}[X] = \int_0^1 x \left( a_K x^{1/3} + b_K x^{4/3} \right)^{-3/2} dx
\]

(327)

\[
= \frac{-2}{b_K \sqrt{a_K + b_K}} + \frac{2 \text{ArcSinh} \left( \sqrt{\frac{b_K}{a_K}} \right)}{b_K^{3/2}}
\]

(328)

The constraint that \( \int_0^1 p(x) dx = 1 \) requires that \( a_K \sqrt{a_K + b_K} = 2 \). We can use this to get

\[
E_{X \sim p}[X] = \frac{-a_K}{b_K} + \frac{a_K \sqrt{\frac{a_K}{b_K}} + 1 \text{ArcSinh} \left( \sqrt{\frac{b_K}{a_K}} \right)}{b_K}
\]

(329)

\[
= \frac{-1}{r} + \frac{\sqrt{\frac{1}{r}} + 1 \text{ArcSinh} \left( \sqrt{\frac{1}{r}} \right)}{r}
\]

(330)

\[
= \frac{-1}{r} + \frac{\sqrt{\frac{1}{r}} + 1 \log \left( \sqrt{r} + \sqrt{r + 1} \right)}{r}
\]

(331)

where we use \( r = b_K/a_K \). We will find upper and lower bounds in order to approximate what \( r \) should be. Using (331), we can get

\[
E_{X \sim p}[X] \leq \frac{1 \log r}{2 \frac{1}{r}}
\]

(332)

so long as \( r > 3 \). If we choose \( r = c_1 K \log K \) and set \( c_1 = .75 \), then

\[
E_{X \sim p}[X] \leq \frac{1}{2 \frac{1}{c_1 K \log K}} \log(c_1 K \log K)
\]

\[
\leq \frac{1}{2c_1 K} + \frac{\log \log K}{2c_1 K \log K} + \frac{\log c_1}{2c_1 K \log K} \leq \frac{1}{K}
\]

(333)

(334)

so long as \( K > 4 \). Similarly, we have

\[
E_{X \sim p}[X] \geq \frac{1 \log r}{3 \frac{1}{r}}
\]

(335)

for all \( r \). If we choose \( r = c_2 K \log K \) and set \( c_2 = .25 \), then

\[
E_{X \sim p}[X] \geq \frac{1}{3 \frac{1}{c_2 K \log K}} \log(c_2 K \log K)
\]

\[
\geq \frac{1}{K}
\]

(336)

so long as \( K > 24 \).

Changing the value of \( c \) changes the value of \( E_{X \sim p}[X] \) continuously. Hence, for each \( K > 24 \), there exists a \( c_K \) so that if \( r = c_K K \log K \), then

\[
E_{X \sim p}[X] = \frac{1}{K}.
\]

(337)

The analysis above gives that \( .25 < c_K < .75 \).
B. Limiting value of $c_K$

**Lemma 11.** In the limit, $c_K \to 1/2$. 

**Proof.** We start with $r = \frac{b_K}{a_K} = c_K K \log K$, and we need to meet the condition that

$$\frac{-1}{r} + \frac{1}{r} \log \left( \sqrt{r} + \sqrt{r + 1} \right) = \frac{1}{K}.$$

(338)

Substituting we get

$$\frac{1}{K} = \frac{-1}{c_K K \log K} + \frac{1}{c_K K \log K} \log \left( \sqrt{c_K K \log K} + \sqrt{c_K K \log K + 1} \right)$$

(339)

$$\implies c_K = \frac{-1}{\log K} + \frac{1}{\log K} \log \left( \sqrt{c_K K \log K} + \sqrt{c_K K \log K + 1} \right)$$

(340)

Let $c = \lim_{K \to \infty} c_K$. Since $c_K$ is bounded, we know that $\lim_{K \to \infty} c_K K \log K \to \infty$ since $c_K$ is bounded above by $3/4$. 

$$c = \lim_{K \to \infty} \frac{-1}{\log K} + \frac{1}{\log K} \log \left( \sqrt{c_K K \log K} + \sqrt{c_K K \log K + 1} \right)$$

(341)

$$= 0 + \lim_{K \to \infty} \frac{\log \left( 2 \sqrt{c_K K \log K} \right)}{\log K}$$

(342)

$$= \lim_{K \to \infty} \log 2 + \frac{1}{2} \log c_K + \frac{1}{2} \log K + \frac{1}{2} \log \log K$$

(343)

$$= \frac{1}{2}$$

(344)

\[ \square \]

C. Approximate Minimax Compander vs. Minimax Compander

For any $K$, $c_K$ can be approximated numerically. To simplify the quantizer, recall we can use $c_K \approx \frac{1}{2}$ for large $K$ to get the approximate minimax compander (30).

This is close to optimal without needing to compute $c_K$. Here we prove Theorem 4.

**Proof.** Since $f_K^*, f_K^{**} \in \mathcal{F}'$, we know that

$$\tilde{L}(p, f_K^*) = L'(p, f_K^*) \quad \text{and} \quad \tilde{L}(p, f_K^{**}) = L'(p, f_K^{**}).$$

(345)
We define the corresponding asymptotic local loss functions
\[
g^*(x) = \frac{1}{24} (f^*_K(x))^{-2}x^{-1} \quad \text{and} \quad g^{**}(x) = \frac{1}{24} (f^{**}_K(x))^{-2}x^{-1}
\] (346)
so that our goal is to prove
\[
\int g^{**} \, dp \leq (1 + \varepsilon) \int g^* \, dp .
\] (347)

Let \( \gamma^* = c_K(K \log K) \) and \( \gamma^{**} = \frac{1}{2}(K \log K) \) (the constants in \( f^*_K \) and \( f^{**}_K \) respectively) and let \( \phi^*(x) = \text{ArcSinh}(\sqrt{\gamma^* x}) \) and \( \phi^{**}(x) = \text{ArcSinh}(\sqrt{\gamma^{**} x}) \). Then
\[
(\phi^*)'(x) = \frac{\sqrt{\gamma^*}}{2\sqrt{x}\sqrt{\gamma^* x + 1}} \quad \text{and} \quad (\phi^{**})'(x) = \frac{\sqrt{\gamma^{**}}}{2\sqrt{x}\sqrt{\gamma^{**} x + 1}} .
\] (348)
Note that \( f^*_K(x) = \phi^*(x)/\phi^*(1) \) and \( f^{**}_K(x) = \phi^{**}(x)/\phi^{**}(1) \). We now split into two cases: (i) \( c_K > 1/2 \) and (ii) \( c_K < 1/2 \).

In case (i) (which implies \( \gamma^* > \gamma^{**} \), and note that \( \gamma^*/\gamma^{**} = 2c_K \leq 1 + \varepsilon \)), we get for all \( x \in [0, 1] \),
\[
\frac{(\phi^*)'(x)}{(\phi^{**})'(x)} = \sqrt{\gamma^*/\gamma^{**}} \sqrt{\gamma^{**} x + 1} \in [1, \sqrt{\gamma^*/\gamma^{**}}] \subseteq [1, \sqrt{1 + \varepsilon}]
\] (349)
since \( \sqrt{\gamma^{**} x + 1} \in [\sqrt{\gamma^{**} / \gamma^*}, 1] \). Because \( \gamma^* \geq \gamma^{**} \) and ArcSinh is an increasing function, we know that \( \phi^*(1) \geq \phi^{**}(1) \). Thus, for any \( x \in [0, 1] \),
\[
(f^{**}_K)'(x) = \frac{(\phi^{**})'(x)}{\phi^{**}(1)} \geq \frac{1}{\sqrt{1 + \varepsilon}} (\phi^*)'(x)
\] (350)
\[
= \frac{1}{\sqrt{1 + \varepsilon}} (f^*_K)'(x)
\] (351)

\[\implies (f^{**}_K)'(x)^{-2} \leq (1 + \varepsilon)(f^*_K)'(x)^{-2}
\] (352)

\[\implies g^{**}(x) \leq (1 + \varepsilon)g^*(x)
\] (353)

\[\implies \int g^{**} \, dp \leq (1 + \varepsilon) \int g^* \, dp
\] (354)

which is what we wanted to prove.

Case (ii), where \( c_K < 1/2 \) (implying \( \gamma^{**} > \gamma^* \)) can be proved analogously:
\[
\frac{(\phi^{**})'(x)}{(\phi^*)'(x)} = \sqrt{\gamma^{**}/\gamma^*} \sqrt{\gamma^* x + 1} \in [1, \sqrt{\gamma^{**} / \gamma^*} \subseteq [1, \sqrt{1 + \varepsilon}]
\] (356)
which then gives us \((\phi^*)'(x) \geq (\phi^*)'(x)\) and
\[
\phi^{**}(1) = \int_0^1 (\phi^{**})'(t) \, dt
\]
\[
\leq \sqrt{1 + \varepsilon} \int_0^1 (\phi^*)'(t) \, dt
\]
\[
\leq (\sqrt{1 + \varepsilon})\phi^*(1).
\]
Thus, for any \(x \in [0, 1]\),
\[
(f_K^{**})'(x) = \frac{(\phi^{**})'(x)}{\phi^{**}(1)}
\]
\[
\geq \frac{(\phi^*)'(x)}{(\sqrt{1 + \varepsilon})\phi^*(1)}
\]
\[
= \frac{1}{\sqrt{1 + \varepsilon}} (f_K^*)'(x)
\]
\[
\Rightarrow (f_K^{**})'(x)^{-2} \leq (1 + \varepsilon)(f_K^*)'(x)^{-2}
\]
\[
\Rightarrow g^{**}(x) \leq (1 + \varepsilon)g^*(x)
\]
\[
\Rightarrow \int g^{**} \, dp \leq (1 + \varepsilon) \int g^* \, dp
\]
completing the proof for both cases. \(\square\)

We show the comparison of the theoretical (asymptotic in \(K\) result) of the approximate minimax compander with the experimental results in Figure 8.

**APPENDIX D**

**WORST-CASE ANALYSIS**

In this section, we prove Theorem 5 which applies both to the minimax compander and the power compander. Since we are dealing with worst-case (i.e. not a random \(x\)) the centroid is not defined; therefore this theorem works with the *midpoint decoder*. Thus, the (raw) decoded value of \(x\) is \(\bar{y}_{(n,N(x))}\).

Additionally, we are not using the raw reconstruction but the normalized reconstruction, and hence it does not suffice to deal with a single letter at a time. Thus, we will work with a full probability vector \(x \in \Delta_{K-1}\).

**Proof of Theorem 5.** Let \(x \in \Delta_{K-1}\) be the vector we are quantizing, with \(i\)th element (out of \(K\), summing to 1) \(x_i\); since we are dealing with midpoint decoding, our (raw) decoded value of \(x_i\) is \(\bar{y}_{n,N(x_i)}\). For simplicity, let us denote it as \(\bar{y}_i\), and the normalized value as \(y_i = \bar{y}_i/(\sum_j \bar{y}_j)\).
Let $\delta_i = \bar{y}_i - x_i$ be the difference between the raw decoded value $\bar{y}_i$ and the original value $x_i$. Then:

$$D_{KL}(x \parallel y) = \sum_i x_i \log \frac{x_i}{y_i}$$

$$= \sum_i x_i \log \frac{x_i}{\bar{y}_i} + \log \left( \sum_i \bar{y}_i \right)$$

$$= \sum_i (\bar{y}_i - \delta_i) \log \frac{\bar{y}_i - \delta_i}{\bar{y}_i} + \log \left( 1 + \sum_i \delta_i \right) \tag{366}$$

Next we use that $\log(1 + z) \leq z$.

$$D_{KL}(x \parallel y) \leq \sum_i (\bar{y}_i - \delta_i) - \frac{\delta_i}{\bar{y}_i} + \sum_i \delta_i \tag{369}$$

$$= \sum_i -\delta_i + \sum_i \frac{\delta_i^2}{\bar{y}_i} + \sum_i \delta_i \tag{370}$$

$$= \sum_i \frac{(\bar{y}_i - x_i)^2}{\bar{y}_i} \tag{371}$$

(note that in (369) we used the inequality $\log(1 + z) \leq z$ on both appearances of the logarithm, as well as the fact that $\bar{y}_i - \delta_i = x_i \geq 0$).

We now consider each bin $I^{(n)}$ induced by $f$. For simplicity let the dividing points between the bins be denoted by

$$\beta_{(n)} = f^{-1}\left( \frac{n}{N} \right) = \bar{y}_{(n)} + r_{(n)}/2 \tag{372}$$

so that $I^{(n)} = (\beta_{(n-1)}, \beta_{(n)})$. Since all the companders we are discussing are strictly monotonic, there is no ambiguity. Then, the Mean Value Theorem (since the minimax compander, the approximate minimax compander, and the power compander are all continuous), for each $I^{(n)}$ there is some value $z_{(n)}$ such that

$$f'(z_{(n)}) = \frac{f(\beta_{(n)}) - f(\beta_{(n-1)})}{\beta_{(n)} - \beta_{(n-1)}} = N^{-1}r_{(n)}^{-1} \tag{373}$$

(since $f(\beta_{(n)}) - f(\beta_{(n-1)}) = n/N - (n-1)/N = 1/N$ and $\beta_{(n)} - \beta_{(n-1)} = r_{(n)}$ by definition).

Thus, we can re-write this as follows:

$$r_{(n)} = N^{-1} f'(z_{(n)})^{-1} \tag{374}$$

We will also denote the following for simplicity: $I_i = I^{(nN(x_i))}$; $r_i = r_{(nN(x_i))}$; and $z_i = z_{(nN(x_i))}$

(the bin, bin length, and bin mean value corresponding to $x_i$).
Trivially, since \( z_i \in I_i \), we know that \( \frac{\hat{y}_i}{2} \leq \hat{y}_i \). Thus, we can derive (since \( \hat{y}_i \) is the midpoint of \( I_i \) and \( x_i \in I_i \), we know that \( |\hat{y}_i - x_i| \leq r_i/2 \) that

\[
D_{KL}(x\|y) \leq \sum_i \frac{(\hat{y}_i - x_i)^2}{\hat{y}_i}
\]

(375)

\[
\leq \frac{1}{4} \sum_i \frac{r_i^2}{\hat{y}_i}
\]

(376)

\[
\leq \frac{1}{4} \sum_i \frac{N^2(z_i/2)(f'(z_i))^2}{\hat{y}_i}
\]

(377)

\[
= \frac{1}{2} N^{-2} \sum_i \frac{1}{z_i (f'(z_i))^2}.
\]

(378)

Note that while we are using midpoint decoding for our quantization, for the purposes of analysis, it is more convenient to express the all the terms in the KL divergence loss using the mean value.

We now examine the worst case performance of the three companders: the power compander, the minimax compander, and the approximate minimax compander.

**Power compander:** In this case, we have

\[
f(x) = x^s \quad \text{and} \quad f'(x) = sx^{s-1}
\]

(379)

for \( s = \frac{1}{\log K} \) (which is optimal for minimizing raw distortion against worst-case priors). This yields

\[
D_{KL}(x\|y) \leq \frac{1}{2} N^{-2} s^{-2} \sum_i \frac{1}{z_i^2 z_i^{2s-2}}
\]

(380)

\[
= \frac{1}{2} N^{-2} s^{-2} \sum_i z_i^{1-2s}
\]

(381)

So long as \( s < 1/2 \) (which occurs for \( K > 7 \)), the function \( z_i^{1-2s} \) is concave in \( z_i \). Thus, replacing all \( z_i \) by their average will increase the value. Furthermore, \( K^s = K^{\frac{1}{\log K}} = e \). Thus, we can derive:

\[
D_{KL}(x\|y) \leq \frac{1}{2} N^{-2} s^{-2} K \left( \frac{\sum_i z_i}{K} \right)^{1-2s}
\]

(382)

\[
= \frac{1}{2} N^{-2}(\log K)^2 e^2 \left( \sum_i z_i \right)^{1-2s}
\]

(383)

\[
\leq \frac{e^2}{2} N^{-2}(\log K)^2 \max \{1, \sum_i z_i\}
\]

(384)

Next, we need to bound \( \max \{1, \sum_i z_i\} \). Assume that \( \sum_i z_i > 1 \) (otherwise our bound is just 1). Then, we note the following: \( \sum_i x_i = 1 \) by definition; \( s^{-1} = \log K \); and

\[
r_i = N^{-1} f'(z_i)^{-1} = N^{-1} s^{-1} z_i^{1-s}.
\]

(385)
This allows us to make the following derivation:

\[ \sum_i |z_i - x_i| \leq \frac{1}{2} \sum_i r_i \quad (386) \]

\[ \implies \sum_i z_i \leq \sum_i x_i + \frac{1}{2} N^{-1} s^{-1} \sum_i z_i^{1-s} \quad (387) \]

\[ = 1 + \frac{1}{2} N^{-1} \log(K) \left( \frac{\sum_i z_i}{K} \right)^{1-s} \quad (388) \]

\[ = 1 + \frac{e}{2} N^{-1} \log(K) \left( \sum_i z_i \right)^{1-s} \quad (389) \]

\[ \leq 1 + \frac{e}{2} N^{-1} \log(K) \left( \sum_i z_i \right) \quad (390) \]

We get (388) by the same concavity trick: because \( z_i^{1-s} \) is concave in \( z_i \), replacing each individual \( z_i \) with their average can only increase the sum. We get (389) because \( K^s = K^{\frac{1}{\log K}} = e \).

We can combine terms with \( \sum_i z_i \).

\[ \left( 1 - \frac{e}{2} N^{-1} \log K \right) \sum_i z_i \leq 1 \quad (391) \]

This then implies that

\[ \sum_i z_i \leq \frac{1}{1 - \frac{e}{2} N^{-1} \log K} \]

\[ = \frac{N}{N - \frac{e}{2} \log K} = 1 + \frac{e}{2} \frac{\log K}{N - \frac{e}{2} \log K} \quad (392) \]

(393)

For the bound to hold we need that \( N > \frac{e}{2} \log K \). Furthermore, if \( N \geq e \log K \), we get that \( \sum_i z_i \leq 2 \). Assuming this and combining (384) with (393), we have

\[ D_{KL}(x\|y) \leq \frac{e^2}{2} N^{-2} (\log K)^2 \max \left\{ \frac{e^2}{2} \frac{\log K}{N - \frac{e}{2} \log K} \right\} \quad (394) \]

\[ = \frac{e^2}{2} N^{-2} (\log K)^2 \left( 1 + \frac{e}{2} \frac{\log K}{N - \frac{e}{2} \log K} \right) \quad (395) \]

When \( N \geq e \log K \), this becomes the pleasing

\[ D_{KL}(x\|y) \leq e^2 N^{-2} (\log K)^2 \quad (396) \]

**Minimax compander and approximate minimax compander:** Since they are very similar in form, it is convenient to do both at once. Let \( c \) be a constant which is either \( c_K \) if we
are considering the minimax compander, or \( \frac{1}{2} \) if we’re considering the approximate minimax compander; and let \( \gamma = cK \log K \). Then our compander and its derivative will have the form

\[
\begin{align*}
f(x) &= \frac{\text{ArcSinh}(\sqrt{\gamma}x)}{\text{ArcSinh}(\sqrt{\gamma})} \\
f'(x) &= \frac{1}{2\text{ArcSinh}(\sqrt{\gamma})} \frac{\sqrt{\gamma}}{\sqrt{x}\sqrt{1 - \gamma x}} \\
\implies f'(x)^{-1} &= 2\text{ArcSinh}(\sqrt{\gamma})\sqrt{\frac{x}{\gamma} + x^2}
\end{align*}
\]

This then yields that

\[
r_i = N^{-1}f'(z_i)^{-1} = 2N^{-1}\text{ArcSinh}(\sqrt{\gamma})\sqrt{\frac{z_i}{\gamma} + z_i^2}
\]

Then we can derive from (378) that

\[
D_{\text{KL}}(x\|y) \leq \frac{1}{2}N^{-2}(2\text{ArcSinh}(\sqrt{\gamma}))^2 \sum_i \frac{\frac{z_i}{\gamma} + z_i^2}{z_i}
\]

\[
= 2N^{-2}(\text{ArcSinh}(\sqrt{\gamma}))^2 \left( \frac{K}{\gamma} + \sum_i z_i \right)
\]

\[
\leq 2N^{-2}(\text{ArcSinh}(\sqrt{\gamma}))^2 \left( \frac{K}{\gamma} + \max \left\{ 1, \sum_i z_i \right\} \right)
\]

Assuming that \( \sum_i z_i > 1 \) (otherwise the bound is just 1),

\[
\sum_i |z_i - x_i| \leq \sum_i \frac{r_i}{2}
\]

\[
\implies \sum_i z_i \leq \sum_i x_i + N^{-1}\text{ArcSinh}(\sqrt{\gamma}) \sum_i \sqrt{\frac{z_i}{\gamma} + z_i^2}
\]

\[
= 1 + N^{-1}\text{ArcSinh}(\sqrt{\gamma}) \sum_i \sqrt{\frac{z_i}{\gamma} + z_i^2}
\]

To bound the sum in (407), using the fact that \( \sqrt{\cdot} \) is concave (so averaging the inputs of a sum
of square roots makes it bigger), we get

$$\sum_i \sqrt{\frac{z_i}{\gamma} + z_i^2} \leq \sum_i \sqrt{\frac{z_i}{\gamma} + z_i^2}$$

(408)

$$\leq K \left( \frac{\sum_i z_i}{K(cK \log K)} \right)^{1/2} + \sum_i z_i$$

(409)

$$\leq \left( \frac{\sum_i z_i}{c \log K} \right)^{1/2} + \sum_i z_i$$

(410)

$$\leq \frac{\sum_i z_i}{(c \log K)^{1/2}} + \sum_i z_i$$

(411)

$$= \left( \sum_i z_i \right) \left( 1 + \frac{1}{(c \log K)^{1/2}} \right)$$

(412)

$$\leq 2 \left( \sum_i z_i \right)$$

(413)

Then (407) becomes

$$\sum_i z_i \leq 1 + 2N^{-1} \text{ArcSinh}(\sqrt{\gamma}) \left( \sum_i z_i \right)$$

(414)

Since we have $\sum_i z_i$ on both sides of the equation, we can combine these terms like before.

$$(1 - 2N^{-1} \text{ArcSinh}(\sqrt{\gamma})) \sum_i z_i \leq 1$$

(415)

$$\implies \sum_i z_i \leq \frac{N}{N - 2\text{ArcSinh}(\sqrt{\gamma})}$$

(416)

Combining these and using the expression $\text{ArcSinh}(\sqrt{z}) = \log(\sqrt{z + 1} + \sqrt{z}) \leq \log(2\sqrt{z} + 1)$ we get from (404) that

$$D_{KL}(x\| y) \leq 2N^{-2} (\text{ArcSinh}(\sqrt{\gamma}))^2 \left( \frac{K}{\gamma} + \frac{N}{N - 2\text{ArcSinh}(\sqrt{\gamma})} \right)$$

(417)

$$= 2N^{-2} (\text{ArcSinh}(\sqrt{cK \log K}))^2 \left( \frac{K}{cK \log K} + \frac{N}{N - 2\text{ArcSinh}(\sqrt{cK \log K})} \right)$$

(418)

$$\leq 2N^{-2} (\log(2\sqrt{cK \log K} + 1))^2 \left( \frac{1}{c \log K} + \frac{N}{N - 2\log(2\sqrt{cK \log K} + 1)} \right)$$

(419)

This holds for all $N > 2 \log(2\sqrt{cK \log K} + 1)$; furthermore, if $N > 6 \log(2\sqrt{cK \log K} + 1)$, the second term in the parentheses is at most $3/2$ (and if $N$ is larger, this term goes to 1). Since $c$
is between $1/4$ and $3/4$ (as it is either $c_K$ or $1/2$), the first term is sufficiently small that we can
bound the entire parenthesis term by $2$. Then,

$$D_{KL}(x\|y) \leq 4N^{-2}(\log(2\sqrt{cK \log K} + 1))^2$$  \hspace{2cm} (420)

$$\leq 4N^{-2}(\log(3\sqrt{cK \log K}))^2$$ \hspace{2cm} (421)

$$= N^{-2}(\log(cK \log K) + 2 \log 3)^2$$ \hspace{2cm} (422)

$$= N^{-2}((\log K)^2 + O((\log K)(\log \log K))).$$ \hspace{2cm} (423)

Note that whether $c$ is $c_K$ or $1/2$, it is always between $1/4$ and $3/4$, and so it has no effect on
the order of growth. We also note that the above (stated more crudely) is an order of growth
within $O(N^{-2}(\log K)^2)$.  \hfill \Box