Memoryless channels: the benefits of feedback in the non-asymptotic regime

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Minimum energy per bit with and without feedback

Non-asymptotic problem statement

$$\begin{array}{c} Z \sim \mathcal{N}(0, \tfrac{1}{2}N_0) \\ \downarrow \\ X \longrightarrow \bigoplus \longrightarrow Y \end{array}$$

- (E, M, ϵ) code with feedback is given by a decoder $g : \mathbb{R}^\infty \to \{1, \ldots, M\}$ and a sequence of encoder functions $f_j : \{1, \ldots, M\} \times \mathbb{R}^{j-1} \to \mathbb{R}$, used to generate channel inputs:

$$X_j = f_j(W, Y^{j-1}),$$

and satisfying

$$\mathbb{E}\left[\sum_{j=1}^{\infty} X_j^2 \middle| W\right] \le E, \qquad \mathbb{P}[g(Y^\infty) \ne W] \le \epsilon.$$

- (E, M, ϵ) code (without feedback) is required to satisfy $f_j(W, Y^{j-1}) = f_j(W)$.
- Energy-information tradeoff:

$$\begin{aligned} M^*(E, \epsilon) &= \max\{M : \exists (M, E, \epsilon) \text{ code}\}, \\ M_f^*(E, \epsilon) &= \max\{M : \exists (M, E, \epsilon) \text{ code with feedback}\} \end{aligned}$$

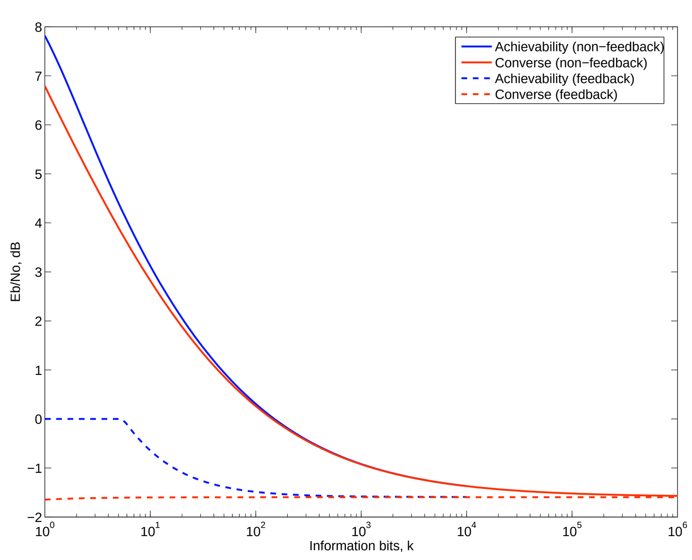
- Asymptotically we have [CS49]:

$$\lim_{E\to\infty} \frac{1}{E} \log M^*(E, \epsilon) = \lim_{E\to\infty} \frac{1}{E} \log M_f^*(E, \epsilon) = \frac{\log e}{N_0}, \quad 0 < \epsilon < 1$$

I.e. feedback does not improve minimum energy per bit.

- **Our work:** How do $M^*(E, \epsilon)$ and $M_f^*(E, \epsilon)$ compare for finite E?

Results



Highlights:

- Horizontal axis: $k = \log_2 M$
- Without feedback: convergence to Shannon limit of -1.59 dB is slow, $O(\frac{1}{\sqrt{k}})$.
- With feedback: convergence is very fast.
- *Surprisingly:* decision feedback is enough!
- Plot: block error probability $\epsilon = 10^{-3}$; see [PPV10a].

- Without feedback:

$$\log M^*(E, \epsilon) = \frac{E}{N_0} \log e - \sqrt{\frac{2E}{N_0}} Q^{-1}(\epsilon) \log e + \frac{1}{2} \log \frac{E}{N_0} + O(1), \quad \epsilon > 0$$

$$\log M^*(E, 0) = 0$$

- With feedback:

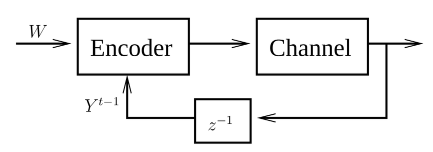
$$\log M_f^*(E, \epsilon) = \frac{E}{N_0} \frac{\log e}{1-\epsilon} + O\left(\log \frac{E}{N_0}\right), \quad \epsilon > 0$$

$$\log M_f^*(E, 0) \ge \left\lfloor \frac{E}{N_0} \right\rfloor$$

- Note: feedback makes zero-error communication possible.

Variable-length coding with feedback

Non-asymptotic problem statement



- (ℓ, M, ϵ) variable-length feedback (VLF) code: sequences of encoder f_n and decoder functions g_n and a stopping time τ of filtration $\sigma\{Y^n\}$ such that:

$$\mathbb{P}[g_\tau(Y^\tau) \ne W] \le \epsilon, \quad \mathbb{E}[\tau] \le \ell,$$

where distribution P_{WY^n} is given by

$$P_{WY^n}(w, y^n) = \frac{1}{M} \prod_{j=1}^{n} P_{Y|X}(y_j | f_j(w, y^{j-1})).$$

- Non-asymptotic fundamental limit:

$$M^*(\ell, \epsilon) = \max\{M : \exists (\ell, M, \epsilon)\text{-VLF code}\}.$$

- Burnashev [MB76] has shown:

$$\lim_{\ell\to\infty} \frac{1}{\ell} \log M^*(\ell, \exp\{-\ell E\}) = C\left(1 - \frac{E}{C_1}\right),$$

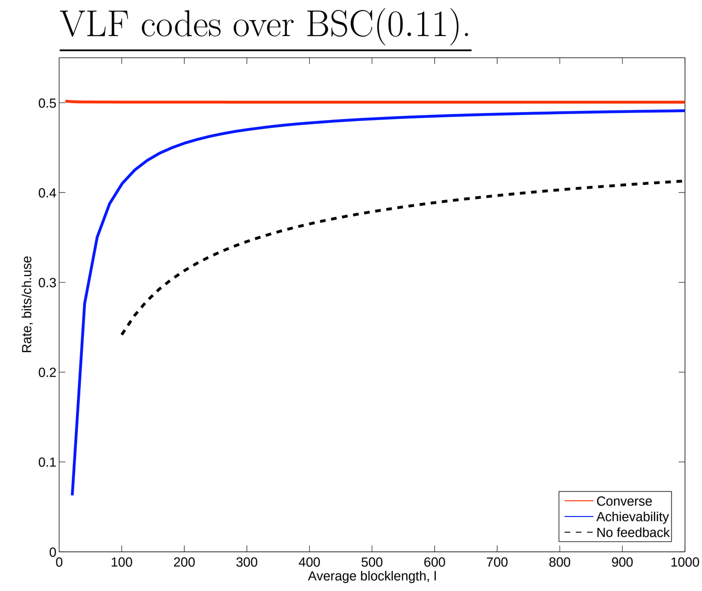
where C is the capacity and

$$C_1 = \max_{a_1, a_2 \in \mathcal{A}} D(P_{Y|X=a_1} || P_{Y|X=a_2}).$$

- **Our work:** What is the behavior of $M^*(\ell, \epsilon)$ for a fixed ϵ?

Results

VLF codes over BSC(0.11).



Highlights:

- Without feedback: convergence to capacity is slow, $O(\frac{1}{\sqrt{\ell}})$.
- VLF codes: *dispersion is zero*, i.e. convergence is much faster.
- Achievability (blue) bound: decision feedback only.
- Plot: probability of block error $\epsilon = 10^{-3}$.

- Main result [PPV10b]:

$$\boxed{\log M_f^*(\ell, \epsilon) = \frac{\ell C}{1-\epsilon} + O(\log \ell), \quad 0 < \epsilon < 1}$$

- Recall that without feedback [PPV10c]:

$$\log M^*(\ell, \epsilon) = \ell C - \sqrt{\ell V} Q^{-1}(\epsilon) + O(\log \ell),$$

where V is the channel dispersion.

Variable-length coding with termination

- ***Problem:*** Burnashev model assumes that control bits have the same reliability as information ones: in practice start/end is handled by upper layers.
- ***Modified model:*** "use-once" noiseless termination symbol.

$$010110T \to \boxed{\text{Channel}} \to 100110TTTTT\ldots$$

- ***VLFT code:*** a VLF code that employs $T \iff \tau$ is a stopping time of filtration $\sigma\{W, Y^n\}, n = 1, \ldots$

$$M_t^*(\ell, \epsilon) = \max\{M : \exists (\ell, M, \epsilon) \text{ VLFT code}\}.$$

- Examples of VLFT: ARQ, fountain codes.
- **Question:** Does $M_t^*(\ell, \epsilon)$ differ significantly from $M_f^*(\ell, \epsilon)$?

Results

- Without termination: for all channels with $C_1 < \infty$ (e.g. BSC) we have:

$$\text{VLF}: \quad \log M_f^*(\ell, 0) = 0$$

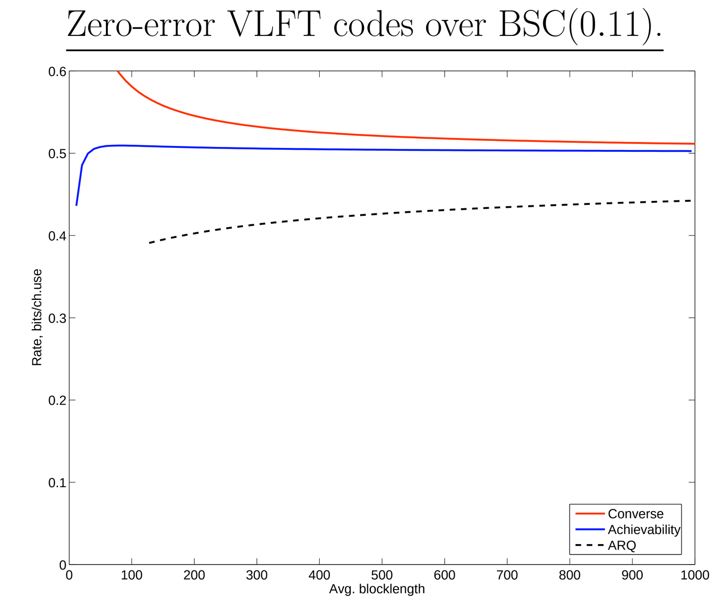
- With termination: ARQ paired with optimal fixed-length code achieves capacity [PPV10c]:

$$\text{ARQ}: \quad \log M_t^*(\ell, 0) \ge \ell C - \text{const}\sqrt{\ell \log \ell}$$

- But we can do much better [PPV10b]:

$$\text{VLFT}: \quad \log M_t^*(\ell, 0) \ge \ell C + O(1)$$

Zero-error VLFT codes over BSC(0.11).



Highlights:

– ARQ: very slow convergence even when the best possible block codes are used.

– New codes: capacity is achievable at *very short code-length* (and with zero error).

- In summary, the minimal blocklength to achieve $R = 0.9C$ for $BSC(0.11)$:

$$\begin{array}{rll} \text{fixed-length}: & \ell \approx 3100 & \text{penalty term: } O(\sqrt{\ell}) \\ \text{VLF}: & \ell \lesssim 200 & \text{penalty term: } O(\log \ell) \\ \text{VLF + termination}: & \ell \lesssim 20 & \text{penalty term: } O(1) \end{array}$$

References

CS49 C. E. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 37, pp. 10-21, Jan. 1949

MB76 M. V. Burnashev, "Data transmission over a discrete channel with feedback. Random transmission time," *Prob. Peredachi Inform.*, vol. 12, no. 4, pp. 10-30, 1976.

PPV10a Y. Polyanskiy, H. V. Poor and S. Verdú, "Minimum energy to send k bits with and without feedback," submitted to *IEEE Trans. Inform. Theory*, Feb 2010.

PPV10b Y. Polyanskiy, H. V. Poor and S. Verdú, "Variable-length coding with feedback in the non-asymptotic regime," submitted to *IEEE Trans. Inform. Theory*, Apr 2010.

PPV10c Y. Polyanskiy, H. V. Poor and S. Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inform. Theory*, vol. 56, no. 5, May 2010.