Comparison of channels: criteria for domination by a symmetric channel

Anuran Makur and Yury Polyanskiy

Abstract

This paper studies the basic question of whether a given channel \( V \) can be dominated (in the precise sense of being more noisy) by a \( q \)-ary symmetric channel. The concept of “less noisy” relation between channels originated in network information theory (broadcast channels) and is defined in terms of mutual information or Kullback-Leibler divergence. We give equivalent characterizations in terms of \( \chi^2 \) -divergence, Löwner (PSD) partial order, and spectral radius. Furthermore, we develop a simple criterion for domination by a symmetric channel in terms of the minimum entry of the stochastic matrix defining the channel \( V \). The criterion is strengthened for the special case of additive noise channels. Finally, it is shown that domination by a symmetric channel implies (via comparison of Dirichlet forms) a logarithmic Sobolev inequality for the original channel.

Index Terms

Less noisy, degradation, \( q \)-ary symmetric channel, additive noise channel, Dirichlet form, logarithmic Sobolev inequalities.

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* A. Makur and Y. Polyanskiy are with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139, USA (e-mail: a_makur@mit.edu; yp@mit.edu). This material is based upon work supported by the National Science Foundation CAREER award under grant agreement CCF-12-53205, and by the Center for Science of Information (CSoI), an NSF Science and Technology Center, under grant agreement CCF-09-39370.
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I. INTRODUCTION

For any Markov chain $W \rightarrow X \rightarrow Y$, it is well-known that $I(W;Y) \leq I(W;X)$. This result can be strengthened [1]:

$$I(W;Y) \leq \eta I(W;X)$$

where the coefficient $\eta \in [0,1]$ can be computed given the knowledge of $P_{Y|X}$ only. Frequently, one gets $\eta < 1$ and the resulting inequality is called a strong data processing inequality (SDPI). Such inequalities have been recently simultaneously rediscovered and applied in several disciplines; see [2, Sections 1-2] for a short survey. In [2, Section 6], it was noticed that the validity of (1) for all $P_{W,X}$ is equivalent to the statement that an erasure channel with erasure probability $1 - \eta$ is less noisy than the given channel $P_{Y|X}$. In this way, the entire field of SDPIs is equivalent to determining whether a given channel is dominated by an erasure channel.

This paper initiates the study of a natural extension of the concept of SDPI by replacing the distinguished role of the erasure channel by a $q$-ary symmetric channel. We give simple criteria for testing this type of domination and explain how the latter can be used to prove logarithmic Sobolev inequalities.

A. Preliminaries

We first briefly introduce some notation that will be pertinent to our ensuing discussion. For any $q, r \in \mathbb{N} \triangleq \{1,2,3,\ldots\}$, we let $\mathbb{R}^{q \times r}$ (respectively $\mathbb{C}^{q \times r}$) denote the set of all real (respectively complex) $q \times r$ matrices. Furthermore, we let $\mathbb{R}_{q \times q}^{\geq 0} \subseteq \mathbb{R}_{sym}^{q \times q}$ denote the sets of positive semidefinite and symmetric matrices, respectively. In fact, $\mathbb{R}_{q \times q}^{\geq 0}$ is a closed convex cone (with respect to the Frobenius norm). We also let $\succeq_{psd}$ denote the Löwner partial order over $\mathbb{R}_{sym}^{q \times q}$: for any two matrices $A, B \in \mathbb{R}_{sym}^{q \times q}$, we write $A \succeq_{psd} B$ (or equivalently, $A - B \succeq_{psd} 0$, where 0 is the zero matrix) if and only if $A - B \in \mathbb{R}_{q \times q}^{\geq 0}$. Finally, we let $\mathcal{P}_q \triangleq \{p = (p_1, \ldots, p_q) \in \mathbb{R}^q : p_1, \ldots, p_q \geq 0 \text{ and } p_1 + \cdots + p_q = 1\}$ be the probability simplex of row vectors in $\mathbb{R}^q$, and $\mathbb{R}_{sto}^{q \times r}$ be the convex set of stochastic matrices (which are matrices with rows in $\mathcal{P}_r$).

B. Channel preorders in information theory

Since we will study preorders over discrete channels that capture various notions of relative “noisiness” between channels, we provide an overview of some well-known channel preorders in the literature. Consider an input random variable $X \in \mathcal{X}$ and an output random variable $Y \in \mathcal{Y}$, where the alphabets are $\mathcal{X} = [q] \triangleq \{0,1,\ldots,q-1\}$ and $\mathcal{Y} = [r]$ for $q, r \in \mathbb{N}$ without loss of generality. We let $\mathcal{P}_q$ be the set of all probability mass functions (pmfs) of $\mathcal{X}$, where every pmf $P_X = (P_X(0),\ldots,P_X(q-1)) \in \mathcal{P}_q$ and is perceived as a row vector. Likewise, we let $\mathcal{P}_r$ be the set of all pmfs of $\mathcal{Y}$. A channel is the set of conditional distributions $W_{Y|X}$ that associates each $x \in \mathcal{X}$ with a conditional pmf $W_{Y|X}(\cdot|x) \in \mathcal{P}_r$. So, we represent each channel with a stochastic matrix $W \in \mathbb{R}^{q \times r}_{sto}$ that is defined entry-wise as:

$$\forall x \in \mathcal{X}, \forall y \in \mathcal{Y}, \quad [W]_{x+1,y+1} \triangleq W_{Y|X}(y|x)$$

where the $(x+1)$th row of $W$ corresponds to the conditional pmf $W_{Y|X}(\cdot|x) \in \mathcal{P}_r$, and each column of $W$ has at least one non-zero entry so that no output alphabet letters are redundant. Moreover, we think of such a channel as a linear map $W : \mathcal{P}_q \rightarrow \mathcal{P}_r$ that takes any row probability vector $P_X \in \mathcal{P}_q$ to the row probability vector $P_Y = P_X W \in \mathcal{P}_r$. 


One of the earliest preorders over channels was the notion of channel inclusion proposed by Shannon in [3]. Given two channels \( W \in \mathbb{R}^{q,x,r}_{sto} \) and \( V \in \mathbb{R}^{s,x,t}_{sto} \) for some \( q, r, s, t \in \mathbb{N} \), he stated that \( W \) includes \( V \), denoted \( W \succeq_{inc} V \), if there exist a pmf \( g \in \mathcal{P}_m \) for some \( m \in \mathbb{N} \), and two sets of channels \( \{A_k \in \mathbb{R}^{r,s,t}_{sto} : k = 1, \ldots, m\} \) and \( \{B_k \in \mathbb{R}^{r,s,t}_{sto} : k = 1, \ldots, m\} \), such that:

\[
V = \sum_{k=1}^{m} g_k B_k W A_k.
\]  

(3)

Shannon showed that channel inclusion is preserved under channel addition and multiplication (which are defined in [4]), and that the existence of a code for \( V \) implies the existence of as good a code for \( W \) in a probability of error sense. The channel inclusion preorder includes the input-output degradation preorder, which can be found in [5], as a special case. Indeed, \( V \) is an input-output degraded version of \( W \), denoted \( W \succeq_{inc} V \), if there exist channels \( A \in \mathbb{R}^{r,s,t}_{sto} \) and \( B \in \mathbb{R}^{s,x,q}_{sto} \) such that \( V = BWA \). We will study an even more specialized case of Shannon’s channel inclusion known as degradation [6].

**Definition 1 (Degradation Preorder).** A channel \( V \in \mathbb{R}^{q,x,s}_{sto} \) is said to be a degraded version of a channel \( W \in \mathbb{R}^{r,x,s}_{sto} \) with the same input alphabet, denoted \( W \succeq_{deg} V \), if \( V = WA \) for some channel \( A \in \mathbb{R}^{r,x,s}_{sto} \).

We note that when Definition 1 of degradation is applied to general matrices (rather than stochastic matrices), it is equivalent to Definition C.8 of matrix majorization in [7, Chapter 15]. Many other generalizations of the majorization preorder over vectors (briefly introduced in Appendix A) that apply to matrices are also presented in [7, Chapter 15].

Körner and Marton defined two other preorders over channels in [8] known as the more capable and less noisy preorders. While the original definitions of these preorders explicitly reflect their significance in channel coding, we will define them using equivalent mutual information characterizations proved in [8]. We say a channel \( W \in \mathbb{R}^{q,x,s}_{sto} \) is more capable than a channel \( V \in \mathbb{R}^{s,x,q}_{sto} \) with the same input alphabet, denoted \( W \succeq_{inc} V \), if \( I(P_X,W_{Y|X}) \geq I(P_X,V_{Y|X}) \) for every input pmf \( P_X \in \mathcal{P}_q \), where \( I(P_X,W_{Y|X}) \) denotes the mutual information of the joint pmf defined by \( P_X \) and \( W_{Y|X} \). The next definition presents the less noisy preorder, which will be a key player in our study.

**Definition 2 (Less Noisy Preorder).** Given two channels \( W \in \mathbb{R}^{q,x,r}_{sto} \) and \( V \in \mathbb{R}^{q,x,s}_{sto} \) with the same input alphabet, let \( Y_W \) and \( Y_V \) denote the output random variables of \( W \) and \( V \), respectively. Then, \( W \) is less noisy than \( V \), denoted \( W \preceq_{no} V \), if \( I(U;Y_W) \geq I(U;Y_V) \) for every joint distribution \( P_{U,X} \), where the random variable \( U \in \mathcal{U} \) has some arbitrary range, and \( U \to X \to Y_W \) and \( U \to X \to Y_V \) form Markov chains.

An analogous characterization of the less noisy preorder using Kullback-Leibler (KL) divergence or relative entropy is given in the next proposition.

**Proposition 1 (KL Divergence Characterization of Less Noisy [8]).** Given two channels \( W \in \mathbb{R}^{q,x,r}_{sto} \) and \( V \in \mathbb{R}^{q,x,s}_{sto} \) with the same input alphabet, \( W \preceq_{no} V \) if and only if \( D(P_XW||Q_XW) \geq D(P_XV||Q_XV) \) for every pair of input pmfs \( P_X, Q_X \in \mathcal{P}_q \), where \( D(\cdot||\cdot) \) denotes the KL divergence.

We will primarily use this KL divergence characterization in our discourse due to its simplicity. The more capable and less noisy preorders have both been used to study the capacity regions of broadcast channels. We refer readers to [9]–[11], and the references therein for further details. We also remark that the more capable and less noisy preorders tensorize, as shown in [12] and [2], [13], respectively. The next proposition illustrates certain well-known relationships between the various preorders discussed in this subsection.

**Proposition 2 (Relations between Channel Preorders).** Given two channels \( W \in \mathbb{R}^{q,x,r}_{sto} \) and \( V \in \mathbb{R}^{q,x,s}_{sto} \) with the same input alphabet, we have:

1) \( W \succeq_{deg} V \Rightarrow W \succeq_{inc} V \Rightarrow W \succeq_{nc} V \).

2) \( W \succeq_{deg} V \Rightarrow W \succeq_{nc} V \Rightarrow W \succeq_{no} V \).

These observations follow in a straightforward manner from the definitions of the various preorders. Perhaps the only nontrivial implication is \( W \succeq_{deg} V \Rightarrow W \succeq_{nc} V \), which can be proven using Proposition 1 and the data processing inequality.

**C. Erasure channels and contraction coefficients**

As we mentioned at the outset, our work is partly motivated by [2, Section 6], where the authors demonstrate an intriguing relation between less noisy domination by an erasure channel and the contraction coefficient of the SDPI for KL divergence. For a common input alphabet \( \mathcal{X} = \{q\} \), consider a channel \( V \in \mathbb{R}^{q,x,s}_{sto} \) with output alphabet \( \mathcal{Y} = \{s\} \), and the \( q \)-ary erasure channel \( E_\epsilon \in \mathbb{R}^{q,x,s}_{sto} \) with erasure probability \( \epsilon \in [0,1] \) and output alphabet \( \{\epsilon\} \cup \mathcal{X} \) (where
the symbol \( e \) stands for erasure. Recall that given an input \( x \in X \), the \( q \)-ary erasure channel erases \( x \) and outputs \( e \) with probability \( \epsilon \), and outputs \( x \) itself with probability \( 1 - \epsilon \). Then, [2, Proposition 14] states that \( E_\epsilon \geq_{\text{c}} n \) if and only if \( \eta_\epsilon(V) \leq 1 - \epsilon \), where \( \eta_\epsilon(V) \in [0, 1] \) is the contraction coefficient for KL divergence, which is defined as:

\[
\eta_\epsilon(V) \triangleq \sup_{P_X, Q_X \in P_q} \frac{D(P_X || Q_X)}{D(P_X || \epsilon X)},
\]

There are several simple upper bounds on \( \eta_\epsilon \). For example, if the \( \ell^2 \)-distances between the rows of \( V \) are bounded by \( 2(1 - \alpha) \), then \( \eta_\epsilon \leq 1 - \alpha \), cf. [14]. Another criterion follows from Doeblin minorization [15, Remark 3.2]: if for some pmf \( p \in P_q \) and some \( \alpha \in (0, 1) \), the channel \( V \) satisfies:

\[
V \geq \alpha 1_p \text{ entry-wise}
\]

where \( 1 \triangleq [1 \cdots 1]^T \), then \( E_\alpha \geq_{\text{c}} V \), and hence \( \eta_\epsilon(V) \leq 1 - \alpha \). One goal of this paper is to establish similar simple criteria for testing domination by a \( q \)-ary symmetric channel, instead of a \( q \)-ary erasure channel.

### D. Symmetric channels and their properties

Seeing that much of our discourse will be based on symmetric channels, we formally define symmetric channels in this subsection and convey some of their properties. To this end, we first introduce some properties of Abelian groups and define additive noise channels. Let us fix some \( q \in \mathbb{N} \) with \( q \geq 2 \) and consider an Abelian group \((X, \oplus)\) of order \( q \) equipped with a binary “addition” operation denoted by \( \oplus \). Without loss of generality, we let \( X = [q] \), and let \( 0 \) denote the identity element. This endows an ordering to the elements of \( X \). Each element \( x \in X \) permutes the entries of the row vector \((0, \ldots, q - 1) \) to \((\sigma_x(0), \ldots, \sigma_x(q - 1)) \) by (left) “addition” in the Cayley table of the group, where \( \sigma_x : [q] \to [q] \) denotes a permutation of \([q]\), and \( \sigma_x(y) = x \oplus y \) for every \( y \in X \). So, corresponding to each \( x \in X \), we can define a permutation matrix \( P_x \) such that:

\[
[v_0 \cdots v_{q-1}] P_x = [v_{\sigma_x(0)} \cdots v_{\sigma_x(q-1)}]
\]

for any \( v_0, \ldots, v_{q-1} \in \mathbb{R} \), where for each \( i \in [q] \), \( e_i \in \mathbb{R}^q \) is the \( i \)th standard basis column vector with unity in the \((i + 1)\)th position and zero elsewhere. The permutation matrices \( \{P_x \in \mathbb{R}^{q \times q} : x \in X\} \) (with the matrix multiplication operation) form a group that is isomorphic to \((X, \oplus)\) (see Cayley’s theorem, and permutation and regular representations of groups in [16, Sections 6.11, 7.1, 10.6]). In particular, these matrices commute as \((X, \oplus)\) is Abelian, and are jointly unitarily diagonalizable by a “Fourier” matrix (using [17, Theorem 2.5.5]). We now recall that given a row vector \( x = (x_0, \ldots, x_{q-1}) \in \mathbb{R}^q \), we may define a corresponding \( X \)-circular matrix, \( \text{cirk}(x) \in \mathbb{R}^{q \times q} \), that is defined entry-wise as [18, Chapter 3E, Section 4.1]:

\[
\forall a, b \in [q], \quad \text{cirk}(x)_{a+1,b+1} = x_{-a \oplus b}.
\]

where \(-a \in X\) denotes the inverse of \( a \in X \). Moreover, we can decompose this \( X \)-circular matrix as:

\[
\text{cirk}(x) = \sum_{i=0}^{q-1} x_i P_i^T
\]

since \( \sum_{i=0}^{q-1} x_i P_i^T \) is a generator cyclic permutation matrix as presented in [17, Section 0.9.6]:

\[
\forall a, b \in [q], \quad \left[ P_q \right]_{a+1,b+1} = \Delta_1(b-a \mod q)
\]
where $\Delta_{i,j}$ is the Kronecker delta function, which is unity if $i = j$ and zero otherwise. The matrix $P_q$ cyclically shifts any input row vector to the right once, i.e. $(x_1, x_2, \ldots, x_q) P_q = (x_q, x_1, \ldots, x_{q-1})$.

Let us now consider a channel with common input and output alphabet $\mathcal{X} = \mathcal{Y} = [q]$, where $(\mathcal{X}, \oplus)$ is an Abelian group. Such a channel operating on an Abelian group is called an additive noise channel when it is defined as:

$$Y = X \oplus Z$$  

where $X \in \mathcal{X}$ is the input random variable, $Y \in \mathcal{X}$ is the output random variable, and $Z \in \mathcal{X}$ is the additive noise random variable that is independent of $X$ with pmf $P_Z = (P_Z(0), \ldots, P_Z(q-1)) \in P_q$. The channel transition probability matrix corresponding to (11) is the $\mathcal{X}$-circulant stochastic matrix $\text{circ}_\mathcal{X}(P_Z) \in \mathbb{R}_{\text{sto}}^{q \times q}$, which is also doubly stochastic (i.e. both $\text{circ}_\mathcal{X}(P_Z), \text{circ}_\mathcal{X}(P_Z)^T \in \mathbb{R}_{\text{sto}}^{q \times q}$). Indeed, for an additive noise channel, it is well-known that the pmf of $Y$, $P_Y \in P_q$, can be obtained from the pmf of $X$, $P_X \in P_q$, by $\mathcal{X}$-circular convolution: $P_Y = P_X \text{circ}_\mathcal{X}(P_Z)$.

We remark that in the context of various channel symmetries in the literature (see [19, Section VI.B] for a discussion), additive noise channels correspond to “group-noise” channels, and are input symmetric, output symmetric, Dobrushin symmetric, and Gallager symmetric.

The $q$-ary symmetric channel is an additive noise channel on the Abelian group $(\mathcal{X}, \oplus)$ with noise pmf $P_Z = w_\delta \triangleq (1 - \delta, \delta/(q-1), \ldots, \delta/(q-1)) \in P_q$, where $\delta \in [0, 1]$. Its channel transition probability matrix is denoted $W_\delta \in \mathbb{R}_{\text{sto}}^{q \times q}$:

$$W_\delta \triangleq \text{circ}_\mathcal{X}(w_\delta) = \left[ w_\delta^{T} P_q^{T} w_\delta^{T} \cdots (P_q^{T})^{q-1} w_\delta^{T} \right]^{T}$$  

(12)

which has $1 - \delta$ in the principal diagonal entries and $\delta/(q-1)$ in all other entries regardless of the choice of group $(\mathcal{X}, \oplus)$. We may interpret $1 - \delta$ as the non-crossover probability of the symmetric channel. Indeed, when $q = 2$, $W_\delta$ represents the binary symmetric channel with crossover probability $\delta \in [0, 1]$. Although $W_\delta$ is only stochastic when $\delta \in [0, 1]$, we will refer to the parametrized convex set of matrices $\{W_\delta \in \mathbb{R}_{\text{sym}}^{q \times q} : \delta \in \mathbb{R}\}$ with parameter $\delta$ as the “symmetric channel matrices,” where each $W_\delta$ has the form (12) such that every row and column sums to unity. We conclude this section with a list of properties of symmetric channel matrices.

**Proposition 3** (Properties of Symmetric Channel Matrices). The symmetric channel matrices, $\{W_\delta \in \mathbb{R}_{\text{sym}}^{q \times q} : \delta \in \mathbb{R}\}$, satisfy the following properties:

1. $\forall \delta \in \mathbb{R}$, $W_\delta$ is a symmetric circulant matrix.
2. The discrete Fourier transform (DFT) matrix $F_q \in \mathbb{C}^{q \times q}$, defined entry-wise as $[F_q]_{j,k} = n^{-1/2} \omega^{(j-1)(k-1)}$ for $1 \leq j, k \leq q$ where $\omega = \exp(2\pi i/q)$ and $i = \sqrt{-1}$, jointly diagonalizes every $W_\delta$ for $\delta \in \mathbb{R}$. Moreover, the corresponding eigenvalues or Fourier coefficients, $\{\lambda_j(W_\delta) = [F_q^{H} W_\delta F_q]_{j,j} : j = 1, \ldots, q\}$ are real:

$$\lambda_j(W_\delta) = \begin{cases} 1 - \delta - \frac{\delta}{q-1} & , \ j = 1 \\ 1 - \delta & , \ j = 2, \ldots, q \end{cases}$$

where $F_q^{H}$ denotes the Hermitian transpose of $F_q$.
3. $\forall \delta \in [0, 1]$, $W_\delta$ is a doubly stochastic matrix that has the uniform distribution $u \triangleq \frac{1}{q} I_T$ as its stationary distribution: $u W_\delta = u$.
4. $\forall \delta \in \mathbb{R} \backslash \left\{ \frac{q-1}{q} \right\}$, $W^{-1}_\delta = W_T$ with $\tau = \frac{1 - \delta}{1 - \frac{\delta}{q-1}}$, and for $\delta = \frac{q-1}{q}$, $W_\delta = \frac{1}{q} I_T$ is unit rank and singular.
5. The set $\{W_\delta \in \mathbb{R}_{\text{sym}}^{q \times q} : \delta \in \mathbb{R} \backslash \left\{ \frac{q-1}{q} \right\}\}$ with the operation of matrix multiplication is an Abelian group.

**Proof.** See Appendix B.

II. MAIN RESULTS

In this section, we first delineate the guiding questions of our study, indicate the main results that address them, and then present these results in the ensuing subsections. We will delve into the following four leading questions:

1. **Can we test the less noisy preorder $\succeq_w$ without using KL divergence?**
   Yes, we can use $\chi^2$-divergence as shown in Theorem 1.
2. **Given a channel $V \in \mathbb{R}_{\text{sto}}^{T \times q}$, is there a simple sufficient condition for less noisy domination by a symmetric channel $W_\delta \succeq_w V$?**
   Yes, a condition using degradation (which implies less noisy domination) is presented in Theorem 2.
3. **Can we say anything stronger about less noisy domination by a symmetric channel when $V \in \mathbb{R}_{\text{sto}}^{T \times q}$ is an additive noise channel?**
   Yes, Theorem 3 outlines the structure of additive noise channels in this context (and Figure 1 depicts it).
4) Why do we care about less noisy domination by symmetric channels?

Because this permits us to compare Dirichlet forms as portrayed in Theorem 4.

We next elaborate on these aforementioned theorems.

A. Equivalent characterizations of the less noisy preorder

Our most general result illustrates that although less noisy domination is a preorder defined using KL divergence, one can equivalently define it using $\chi^2$-divergence, the Löwner partial order, or a spectral radius condition. Recall that for any two pmfs $P_X, Q_X \in \mathcal{P}_q$ on the alphabet $\mathcal{X} = [q]$, their $\chi^2$-divergence is given by:

$$\chi^2(P_X||Q_X) = \sum_{x \in \mathcal{X}} \frac{(P_X(x) - Q_X(x))^2}{Q_X(x)}$$

where we assume that $(0 - 0)^2/0 = 0$ and $(p - 0)^2/0 = +\infty$ for every $p > 0$ based on continuity arguments. The next theorem presents these characterizations of the less noisy preorder.

Theorem 1 (Equivalent Characterizations of $\succeq_n$). For any pair of channels $W \in \mathbb{R}^{q \times r}_{sto}$ and $V \in \mathbb{R}^{q \times s}_{sto}$ on the same input alphabet, the following are equivalent:

1) $W \succeq_n V$

2) For every $P_X, Q_X \in \mathcal{P}_q$:

$$\chi^2(P_X W||Q_X X W) \geq \chi^2(P_X V||Q_X V)$$

3) For every $P_X \in \mathcal{P}_q^o$:

$$W \text{diag}(P_X W)^{-1} W^T \succeq_{PSD} V \text{diag}(P_X V)^{-1} V^T$$

4) For every $P_X \in \mathcal{P}_q^o$:

$$\rho \left( \left( W \text{diag}(P_X W)^{-1} W^T \right)^\dagger V \text{diag}(P_X V)^{-1} V^T \right) = 1$$

where $\mathcal{P}_q^o$ denotes the relative interior of $\mathcal{P}_q$, for any $x = (x_1, \ldots, x_q) \in \mathbb{R}^q$, $\text{diag}(x)$ denotes a diagonal matrix with $[\text{diag}(x)]_{i,i} = x_i$ for each $i \in \{1, \ldots, q\}$, $X^\dagger$ is the Moore-Penrose pseudoinverse of any matrix $X$, and $\rho(X)$ is the spectral radius of any square matrix $X$ (which is the maximum of the absolute values of all complex eigenvalues of $X$).

We remark that the equivalent characterization of $\succeq_n$ using the Löwner partial order is the most convenient for proving some of our ensuing results.

B. Less noisy domination by symmetric channels

As we indicated in Section I, our remaining results are concerned with symmetric channels. This is partly due to our desire to complement the work mentioned in Subsection I-C, and compute symmetric channels that dominate a given channel in the less noisy and degraded senses. Suppose we are given a $q$-ary symmetric channel $W_\delta \in \mathbb{R}^{q \times q}_{sto}$ with $\delta \in [0, 1]$, and another channel $V \in \mathbb{R}^{q \times q}_{sto}$ with common input and output alphabets. Then, the next result provides a sufficient condition for when $W_\delta \succeq_{saq} V$.

Theorem 2 (Sufficient Condition for Degradation by Symmetric Channels). Given a channel $V \in \mathbb{R}^{q \times q}_{sto}$ with $q \geq 2$ and minimum probability $\nu = \min \{ |V|_{i,j} : 1 \leq i, j \leq q \}$, we have:

$$0 \leq \delta \leq \frac{\nu}{1 - (q - 1)\nu + \frac{\nu}{q-1}} \Rightarrow W_\delta \succeq_{saq} V$$

where $W_\delta \in \mathbb{R}^{q \times q}_{sto}$ is a symmetric channel.

This sufficient condition is tight as there exist channels $V$ that violate $W_\delta \succeq_{saq} V$ when $\delta > \nu/(1 - (q - 1)\nu + \frac{\nu}{q-1})$.

We also note that Theorem 2 provides a sufficient condition for $W_\delta \succeq_n V$ due to Proposition 2.
C. Structure of additive noise channels

Our next major result is concerned with understanding when symmetric channels operating on an Abelian group \((\mathcal{X}, \oplus)\) dominate other additive noise channels on \(\mathcal{X}\), which are defined in (11), in the less noisy and degraded senses. Given a symmetric channel \(W_\delta \in \mathbb{R}_{\text{sto}}^{q \times q}\) with \(\delta \in [0, 1]\), we define the additive less noisy domination region of \(W_\delta\) as:

\[
\mathcal{L}_{W_\delta}^{\text{add}} \triangleq \{ v \in \mathcal{P}_q : W_\delta = \text{circ}_\mathcal{X}(w_\delta) \succeq_{\text{in}} \text{circ}_\mathcal{X}(v) \}
\]

(14)

which is the set of all noise pmfs whose corresponding channel transition probability matrices are dominated by \(W_\delta\) in the less noisy sense. Likewise, we define the additive degradation region of \(W_\delta\) as:

\[
\mathcal{D}_{W_\delta}^{\text{add}} \triangleq \{ v \in \mathcal{P}_q : W_\delta = \text{circ}_\mathcal{X}(w_\delta) \succeq_{\text{deg}} \text{circ}_\mathcal{X}(v) \}
\]

(15)

which is the set of all noise pmfs whose corresponding channel transition probability matrices are degraded versions of \(W_\delta\). The next theorem exactly characterizes \(\mathcal{L}_{W_\delta}^{\text{add}}\), and “bounds” \(\mathcal{L}_{W_\delta}^{\text{add}}\) in a set theoretic sense.

**Theorem 3** (Additive Less Noisy Domination and Degradation Regions for Symmetric Channels). *Given a symmetric channel \(W_\delta \in \mathbb{R}_{\text{sto}}^{q \times q}\) with \(\delta \in [0, \frac{1}{q}]\) and \(q \geq 2\), we have:*

\[
\mathcal{D}_{W_\delta}^{\text{add}} = \text{conv} \left\{ \{ w_\delta^k P_q^k : k \in [q] \} \right\}
\]

\[
\subseteq \text{conv} \left\{ \{ w_\delta^k P_q^k : k \in [q] \} \cup \{ w_r^k P_q^k : k \in [q] \} \right\}
\]

\[
\subseteq \mathcal{L}_{W_\delta}^{\text{add}} \subseteq \{ v \in \mathcal{P}_q : \| v - u \|_2 \leq \| w_\delta - u \|_2 \}
\]

where the first set inclusion is strict for \(\delta \in (0, \frac{1}{q}]\) and \(q \geq 3\), \(\text{conv}(S)\) for a set of vectors \(S \subseteq \mathbb{R}^q\) is the convex hull of the vectors in \(S\), \(P_q\) denotes the generator cyclic permutation matrix as defined in (10), \(\| \cdot \|_2\) is the Euclidean \(\ell^2\)-norm, and:

\[
\gamma = \frac{1 - \delta}{1 - \delta + \frac{\delta}{(q-1)^2}}.
\]

Furthermore, \(\mathcal{L}_{W_\delta}^{\text{add}}\) is a closed and convex set that is symmetric with respect to the permutations \(\{ P_x \in \mathbb{R}^{q \times q} : x \in \mathcal{X} \}\) defined in (6) corresponding to the underlying Abelian group \((\mathcal{X}, \oplus)\) (i.e. \(v \in \mathcal{L}_{W_\delta}^{\text{add}} \Rightarrow v P_x \in \mathcal{L}_{W_\delta}^{\text{add}}\) for every \(x \in \mathcal{X}\)).

We remark that according to numerical evidence, the second and third set inclusions in Theorem 3 appear to be strict, and \(\mathcal{L}_{W_\delta}^{\text{add}}\) seems to be a strictly convex set. The content of Theorem 3 and these observations are illustrated in Figure 1, which portrays the probability simplex of noise pmfs for \(q = 3\) and the pertinent regions which capture less noisy domination and degradation by a \(q\)-ary symmetric channel.

D. Comparison of Dirichlet forms

Besides complementing Subsection I-C, another reason we study symmetric channels and prove Theorems 2 and 3 is because less noisy domination implies useful bounds between Dirichlet forms. Recall that the symmetric channel \(W_\delta \in \mathbb{R}_{\text{ sto}}^{q \times q}\) with \(\delta \in [0, 1]\) has uniform stationary distribution \(u \in \mathcal{P}_q\) (see part 3 of Proposition 3). For any channel \(V \in \mathbb{R}_{\text{sto}}^{q \times q}\) that is doubly stochastic and has uniform stationary distribution (like the symmetric channel), we may define a corresponding Dirichlet form:

\[
\forall f \in \mathbb{R}^q, \ E_V (f, f) = \frac{1}{q} f^T (I_q - V) f
\]

(16)

where \(f = [f_1 \cdots f_q]^T \in \mathbb{R}^q\) are column vectors, and \(I_q\) denotes the identity matrix in \(\mathbb{R}^{q \times q}\) (as shown in [20] or [21]). Our final theorem portrays that \(W_\delta \succeq_{\text{sto}} V\) implies that the Dirichlet form corresponding to \(V\) dominates the Dirichlet form corresponding to \(W_\delta\). The Dirichlet form corresponding to \(W_\delta\) is in fact a scaled version of the so called standard Dirichlet form:

\[
\forall f \in \mathbb{R}^q, \ E_{\text{std}} (f, f) \triangleq \forall A \in \mathbb{R}_+ (f) = \frac{1}{q} \sum_{k=1}^q f_k^2 - \left( \frac{1}{q} \sum_{k=1}^q f_k \right)^2
\]

(17)

which is the Dirichlet form corresponding to the symmetric channel with all uniform conditional pmfs \(W_{\delta/q} = \frac{1}{q} I_q\). Indeed, we have:

\[
\forall f \in \mathbb{R}^q, \ E_{W_\delta} (f, f) = \frac{q \delta}{q - 1} E_{\text{std}} (f, f).
\]

(18)

The standard Dirichlet form is the usual choice for Dirichlet form comparison because its logarithmic Sobolev constant has been precisely computed in [20, Appendix, Theorem A.1]. So, we present Theorem 4 using \(E_{\text{std}} (\cdot, \cdot)\) rather than \(E_{W_\delta} (\cdot, \cdot)\).
Given the doubly stochastic channels $W_\delta \in \mathbb{R}^{q \times q}_{\text{std}}$ with $\delta \in \left[0, \frac{q-1}{q}\right]$ and $V \in \mathbb{R}^{q \times q}_{\text{std}}$, if $W_\delta \succeq_n V$, then:

$$\forall f \in \mathbb{R}^q, \quad \mathcal{E}_V (f, f) \geq \frac{q\delta}{q-1} \mathcal{E}_{\text{std}} (f, f).$$

The domination of Dirichlet forms shown in Theorem 4 has several useful consequences. A major consequence is that we can immediately establish Poincaré (spectral gap) inequalities and logarithmic Sobolev inequalities for the channel $V$ using the corresponding inequalities for symmetric channels. For example, the logarithmic Sobolev inequality for the symmetric channel $W_\delta \in \mathbb{R}^{q \times q}_{\text{std}}$ with $q > 2$ is:

$$D \left( f^2 \mid u \mid u \right) \leq \frac{(q-1) \log(q-1)}{(q-2)\delta} \mathcal{E}_{W_\delta} (f, f)$$

(19)

for every $f \in \mathbb{R}^q$ such that $\sum_{k=1}^q f_k^2 = q$, where we use (51) and the logarithmic Sobolev constant computed in part 1 of Proposition 14. As shown in Appendix F, (19) is easily established using the known logarithmic Sobolev constant corresponding to the standard Dirichlet form. Using the logarithmic Sobolev inequality for the channel $V$ that follows from (19) and Theorem 4, we immediately obtain guarantees on the convergence rate and hypercontractivity properties of the associated Markov semigroup $\{\exp(-t(I_q - V)) : t \geq 0\}$. We refer readers to [20] and [21] for comprehensive accounts of such topics.

### E. Outline

We briefly outline the content of the ensuing sections. In Section III, we study the structure of less noisy domination and degradation regions of channels. In Section IV, we prove the equivalent characterizations of $\succeq_n$ presented in Theorem 1. We then derive several necessary and sufficient conditions for less noisy domination among additive noise channels in Section V, which together with the results of Section III, culminates in a proof of Theorem 3. Section VI
provides a proof of Theorem 2, and Section VII introduces logarithmic Sobolev inequalities and proves an extension of Theorem 4. Finally, we conclude our discussion in Section VIII.

III. LESS NOISY DOMINATION AND DEGRADATION REGIONS

In this section, we focus on understanding the geometric aspects of less noisy domination and degradation by channels. We begin by considering the less noisy domination and degradation regions (which will be defined soon), and derive some simple characteristics of the sets of channels that are dominated by some fixed channel in the less noisy and degraded senses. We then specialize our results for additive noise channels, and this culminates in the complete characterization of $D^{\text{add}}_{W_{\lambda}}$ and derivations of certain properties of $L^{\text{add}}_{W_{\lambda}}$ presented in Theorem 3.

Let $W \in \mathbb{R}^{q \times r}_{\text{sto}}$ be a fixed channel with $q,r \in \mathbb{N}$, and define its less noisy domination region:

$$L_W \triangleq \{ V \in \mathbb{R}^{q \times r}_{\text{sto}} : W \succeq_h V \}$$

(20)

to be the set of all channels on the same input and output alphabets that are dominated by $W$ in the less noisy sense. Moreover, we define the degradation region of $W$ as:

$$D_W \triangleq \{ V \in \mathbb{R}^{q \times r}_{\text{sto}} : W \succeq_{\text{deg}} V \}$$

(21)

which is the set of all channels on the same input and output alphabets that are degraded versions of $W$. Then, $L_W$ and $D_W$ satisfy the properties delineated below.

**Proposition 4** (Less Noisy Domination and Degradation Regions). Given the channel $W \in \mathbb{R}^{q \times r}_{\text{sto}}$, its less noisy domination region $L_W$ and its degradation region $D_W$ are non-empty, closed, convex, and output alphabet permutation symmetric (i.e. $V \in L_W \Rightarrow VP \in L_W$ and $V \in D_W \Rightarrow VP \in D_W$ for every permutation matrix $P \in \mathbb{R}^{r \times r}$).

**Proof.**

**Non-Emptyness of $L_W$ and $D_W$:** Fix any two pmfs $P_X, Q_X \in \mathcal{P}_q$, and consider a sequence of channels $V_k \in L_W$ such that $V_k \rightarrow V \in \mathbb{R}^{q \times r}_{\text{sto}}$ (with respect to any equivalent Schatten $p$-norm$^1$). Then, we also have $P_X V_k \rightarrow P_X V$ and $Q_X V_k \rightarrow Q_X V$ (with respect to any equivalent $p$-norm), which means that the sequences of probability measures $P_X V_k$ and $Q_X V_k$ converge in total variation distance to $P_X V$ and $Q_X V$, respectively. Hence, we get:

$$D( P_X V || Q_X V) \leq \liminf_{k \rightarrow \infty} D( P_X V_k || Q_X V_k) \leq D( P_X W || Q_X W)$$

where the first inequality follows from the lower semicontinuity of KL divergence (see [22, Theorem 1], or [23, Section 3.5] for a broader exposition of this property), and the second inequality holds because $V_k \in L_W$. This implies that for any two pmfs $P_X, Q_X \in \mathcal{P}_q$, $S( P_X, Q_X) = \{ V \in \mathbb{R}^{q \times r}_{\text{sto}} : D( P_X W || Q_X W) \geq D( P_X V || Q_X V) \}$ is a closed set. Using Proposition 1, we have that:

$$L_W = \bigcap_{P_X, Q_X \in \mathcal{P}_q} S( P_X, Q_X).$$

So, $L_W$ is a closed set as it is an intersection of closed sets [24]. Note that this proof assumes that if $D( P_X W || Q_X W) = \infty$ and $D( P_X V || Q_X V) = \infty$, then $D( P_X W || Q_X W) \geq D( P_X V || Q_X V)$ is not violated.

**Closure of $D_W$:** Consider a sequence of channels $V_k \in D_W$ such that $V_k \rightarrow V \in \mathbb{R}^{q \times r}_{\text{sto}}$. Since $V_k \in D_W$, we have that $V_k = WA_k$ for some sequence of channels $A_k \in \mathbb{R}^{r \times r}_{\text{sto}}$ belonging to the compact set $\mathbb{R}^{r \times r}_{\text{sto}}$. Using the Bolzano-Weierstrass theorem, this means that there exists a subsequence $A_{k_m} \in \mathbb{R}^{r \times r}_{\text{sto}}$ that converges [24]: $A_{k_m} \rightarrow A \in \mathbb{R}^{r \times r}_{\text{sto}}$. Hence, $V \in D_W$ since $V_{k_m} = WA_{k_m} \rightarrow WA = V$, and $D_W$ is a closed set.

**Convexity of $L_W$:** Suppose $V_1, V_2 \in L_W$ and let $\lambda \in [0,1]$. Then, for every $P_X, Q_X \in \mathcal{P}_q$, we have:

$$D( P_X W || Q_X W) \geq \lambda D( P_X V_1 || Q_X V_1) + (1-\lambda) D( P_X V_2 || Q_X V_2) \geq D( P_X (\lambda V_1 + (1-\lambda) V_2) || Q_X (\lambda V_1 + (1-\lambda) V_2))$$

by the convexity of KL divergence. This implies that $\lambda V_1 + (1-\lambda) V_2 \in L_W$. Hence, $L_W$ is a convex set.

**Convexity of $D_W$:** If $V_1, V_2 \in D_W$ so that $V_1 = WA_1$ and $V_2 = WA_2$ for some $A_1, A_2 \in \mathbb{R}^{r \times r}_{\text{sto}}$, then $\lambda V_1 + (1-\lambda) V_2 = \left( \lambda A_1 + (1-\lambda) A_2 \right) W$ is a valid channel.

$^1$For any $1 \leq p \leq \infty$, the Schatten $p$-norm of a matrix is defined as the $p$-norm of its singular values. In particular, the Schatten $p$-norm is the trace class or nuclear norm when $p = 1$, the Frobenius norm when $p = 2$, and the operator or spectral norm when $p = \infty$. 

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W (λA_1 + (1 − λ)A_2) ∈ D_W for every λ ∈ [0, 1]. Hence, D_W is also a convex set.

**Symmetry of L_W:** Given V ∈ L_W and any permutation matrix P ∈ R^{r × r}, we have for every P_X, Q_X ∈ P_q:

\[
D(P_X W || Q_X W) \geq D(P_X V || Q_X V)
\]

\[
\geq D(P_X V P || Q_X V P)
\]

\[
\geq D(P_X V P P^{-1} || Q_X V P P^{-1})
\]

\[
= D(P_X V || Q_X V)
\]

where the second and third inequalities follow from the data processing inequality. Thus, VP ∈ L_W for every permutation matrix P ∈ R^{r × r}.

**Symmetry of D_W:** Given V ∈ D_W so that V = WA for some A ∈ R^{r × r}, we have that VP = WAP ∈ D_W for every permutation matrix P ∈ R^{r × r}. This completes the proof.

We remark that while the channels in L_W and D_W all have the same input and output alphabets as W, as defined in (20) and (21), we may extend the output alphabet of W by adding zero probability letters. So, separate convex less noisy domination and degradation regions can be defined for each output alphabet size that is at least as large as the original output alphabet size of W. On a separate note, given a channel W ∈ R^{r × r}, it is straightforward to observe that, L_W = L_W P and D_W = D_W P for every permutation matrix P ∈ R^{r × r}. Indeed, it can be distilled from the proofs of output alphabet symmetry in Proposition 5 that:

\[
W \geq_{in} WP \geq_{in} W
\]

\[
W \geq_{deg} WP \geq_{deg} W
\]

for every permutation matrix P ∈ R^{r × r}.

**A. Less noisy domination and degradation regions for additive noise channels**

Often in information theory, we are concerned with additive noise channels on an Abelian group (X = [q], +) with q ∈ N, as defined in (11). Such channels are completely defined by a noise pmf P_Z ∈ P_q with corresponding channel transition probability matrix circ_X(P_Z) ∈ R_q^{q × q}. Suppose W = circ_X(w) ∈ R_q^{q × q} is an additive noise channel with noise pmf w ∈ P_q. Then, we are often only interested in the set of additive noise channels that are dominated by W. We define the *additive less noisy domination region* of W as:

\[
L_W^{\text{add}} \triangleq \{ v \in P_q : W \geq_{n} \text{circ}_X(v) \}
\]

which is the set of all noise pmfs whose corresponding channel transition probability matrices are dominated by W in the less noisy sense. This definition generalizes (14) and can hold for any (non-additive noise) channel W. Likewise, we define the *additive degradation region* of W:

\[
D_W^{\text{add}} \triangleq \{ v \in P_q : W \geq_{\text{deg}} \text{circ}_X(v) \}
\]

to be the set of all noise pmfs whose corresponding channel transition probability matrices are degraded versions of W. This definition generalizes (15) and can also hold for any (non-additive noise) channel W. The next proposition illustrates certain properties of L_W^{\text{add}} (analogous to Proposition 4) and explicitly characterizes D_W^{\text{add}}.

**Proposition 5** (Additive Less Noisy Domination and Degradation Regions). Given the additive noise channel W = circ_X(w) ∈ R_q^{q × q} with noise pmf w ∈ P_q, we have:

1) The additive less noisy domination region L_W^{\text{add}} and the additive degradation region D_W^{\text{add}} are non-empty, closed, convex, and symmetric with respect to the permutation matrices \{ P_x ∈ R_q^{q × q} : x ∈ X \} defined in (6) (i.e. \( v ∈ L_W^{\text{add}} \Rightarrow v P_x ∈ L_W^{\text{add}} \) and \( v ∈ D_W^{\text{add}} \Rightarrow v P_x ∈ D_W^{\text{add}} \) for every \( x ∈ X \)).

2) Furthermore, the additive degradation region of W is given by:

\[
D_W^{\text{add}} = \text{conv}(\{ w P_x : x ∈ X \}) = \{ v ∈ P_q : w \geq_X v \}
\]

where \( \geq_X \) denotes the group majorization preorder as defined in Appendix A.

To prove Proposition 5, we will need the following lemma.

**Lemma 1** (Additive Noise Channel Degradation). Given two additive noise channels W = circ_X(w) ∈ R_q^{q × q} and V = circ_X(v) ∈ R_q^{q × q} with noise pmfs w, v ∈ P_q, W \geq_{\text{deg}} V if and only if V = W circ_X(z) = circ_X(z) W for some \( z ∈ P_q \) (i.e. for additive noise channels W \geq_{\text{deg}} V, the channel that degrades W to produce V is also an additive noise channel without loss of generality).
Proof. Since $X$-circulant matrices commute, we must have $W\circ \text{circ}_X(z) = \text{circ}_X(z)W$ for every $z \in \mathcal{P}_q$. Furthermore, $V = W\circ \text{circ}_X(z)$ for some $z \in \mathcal{P}_q$ implies that $W \geq_{\text{add}} V$ by Definition 1. So, it suffices to prove that $W \geq_{\text{add}} V$ implies $V = W\circ \text{circ}_X(z)$ for some $z \in \mathcal{P}_q$. By Definition 1, $W \geq_{\text{add}} V$ implies that $V = WR$ for some doubly stochastic channel $R \in \mathbb{R}_{\text{sto}}^{q \times q}$ (as $V$ and $W$ are doubly stochastic). Let $r$ with $r^T \in \mathcal{P}_q$ be the first column of $R$, and $s = Wr$ with $s^T \in \mathcal{P}_q$ be the first column of $V$. Then, it is straightforward to verify using (9) that:

$$V = \begin{bmatrix} s & P_1 s & P_2 s & \cdots & P_{q-1}s \\ Wr & P_1 Wr & P_2Wr & \cdots & P_{q-1}Wr \\ \end{bmatrix}$$

where the third equality holds because $\{P_x : x \in X\}$ are $X$-circulant matrices which commute with $W$. Hence, $V$ is the product of $W$ and an $X$-circulant stochastic matrix, i.e. $V = W\circ \text{circ}_X(z)$ for some $z \in \mathcal{P}_q$. This concludes the proof.

We emphasize that in Lemma 1, the channel that degrades an additive noise channel $W$ to produce another additive noise channel $V$ is only an additive noise channel without loss of generality. We can certainly have $V = WR$ with a non-additive noise channel $R$. Consider for instance, $V = W = 11^T/q$, where every doubly stochastic matrix $R$ satisfies $V = WR$. However, when we consider $V = WR$ with an additive noise channel $R$, $V$ corresponds to the channel $W$ with an additional independent additive noise term associated with $R$. We now prove Proposition 5.

Proof of Proposition 5.

Part 1: Non-emptiness, closure, and convexity of $L_{W}^{\text{add}}$ and $D_{W}^{\text{add}}$ can be proved in exactly the same way as in Proposition 4, with the additional observation that the set of $X$-circulant matrices is closed (with respect to any equivalent Schatten $p^*$-norm) and convex. Moreover, in analogy with (22) and (23), observe that for every $x \in X$:

$$W \geq_{\text{add}} WP_x = \text{circ}_X(wp_x) \geq_{\text{add}} W$$

(26)

$$W \geq_{\text{add}} WP_x = \text{circ}_X(wp_x) \geq_{\text{add}} W$$

(27)

where the equalities follow from (9). (This implies that $L_{W}^{\text{add}} = L_{WP_x}^{\text{add}}$ and $D_{W}^{\text{add}} = D_{WP_x}^{\text{add}}$ for every $x \in X$.) Using (26), (27), and the transitive property of the preorders $\geq_{\text{add}}$ and $\geq_{\text{add}}$, we obtain the symmetry of $L_{W}^{\text{add}}$ and $D_{W}^{\text{add}}$ with respect to the permutation matrices $\{P_x \in \mathbb{R}^{q \times q} : x \in X\}$.

Part 2: Lemma 1 is equivalent to the fact that $v \in D_{W}^{\text{add}}$ if and only if $\text{circ}_X(v) = \text{circ}_X(w)\circ \text{circ}_X(z)$ for some $z \in \mathcal{P}_q$. This implies that $v \in D_{W}^{\text{add}}$ if and only if $v = w \circ \text{circ}_X(z)$ for some $z \in \mathcal{P}_q$ (due to (9) and the fact that $X$-circulant matrices commute). Using Proposition 16 in Appendix A, we have:

$$D_{W}^{\text{add}} = \text{conv} \left( \{wp_x : x \in X\} \right) = \{v \in \mathcal{P}_q : w \geq_X v\}$$

which completes the proof.

We remark that part 1 of Proposition 5 does not require $W$ to be an additive noise channel. The proofs of closure, convexity, and symmetry with respect to $\{P_x \in \mathbb{R}_\text{sto}^{q \times q} : x \in X\}$ hold for general $W \in \mathbb{R}_\text{sto}^{q \times q}$. Since $W \neq \text{circ}_X(w)$ for some $w \in \mathcal{P}_q$ in general, we cannot claim that $w \in L_{W}^{\text{add}}$ and $w \in D_{W}^{\text{add}}$ to prove non-emptiness. Instead, we note that every channel $V \in \mathbb{R}_\text{sto}^{q \times q}$ satisfies $V \geq_{\text{add}} \text{circ}_X(u)$ and $V \geq_{\text{add}} \text{circ}_X(u)$ for the additive noise channel $\text{circ}_X(u) = 1u \in \mathbb{R}_\text{sto}^{q \times q}$. Hence, $L_{W}^{\text{add}}$ and $D_{W}^{\text{add}}$ are non-empty since $u \in L_{W}^{\text{add}}$ and $u \in D_{W}^{\text{add}}$.

B. Less noisy domination and degradation regions for symmetric channels

Since $q$-ary symmetric channels for $q \in \mathbb{N}$ are additive noise channels, the results in Subsection III-A, particularly Proposition 5, hold for $q$-ary symmetric channels. In this subsection, we deduce some simple results that are unique to symmetric channels. The first of these is a specialization of part 2 of Proposition 5 which states that the additive degradation region of a symmetric channel can be characterized by traditional majorization instead of group majorization.

Corollary 1 (Degradation Region of Symmetric Channel). The symmetric channel $W_\delta = \text{circ}_X(w_\delta) \in \mathbb{R}_\text{sto}^{q \times q}$ for $\delta \in [0, 1]$ has additive degradation region:

$$D_{W_\delta}^{\text{add}} = \{v \in \mathcal{P}_q : w_\delta \geq_{\text{maj}} v\} = \text{conv} \left( \{wp_k : k \in [q]\} \right)$$

where $\geq_{\text{maj}}$ denotes the majorization preorder defined in Appendix A, and $P_q \in \mathbb{R}^{q \times q}$ is the generator cyclic permutation matrix defined in (10).
Proof. From part 2 of Proposition 5, we have that:
\[
P_{W^\text{add}} = \text{conv}\left(\{w_q P_x : x \in X\}\right)
= \text{conv}\left(\{w_q P_k^q : k \in [q]\}\right)
= \text{conv}\left(\{w_q P : P \in \mathbb{R}^{q \times q} \text{ is a permutation matrix}\}\right)
= \{v \in P_q : w \succeq_{\text{major}} v\}
\]
where the second and third equalities hold regardless of the choice of group \((\mathcal{X}, \oplus)\), because the sets of all cyclic or regular permutations of \(w_q = (1 - \delta, \delta/(q-1), \ldots, \delta/(q-1))\) equal \(\{w_q P_x : x \in X\}\). The final equality follows from the definition of majorization in Appendix A. ■

With this geometric characterization of the additive degradation region, it is straightforward to find the extremal symmetric channel \(W_\gamma\) that is a degraded version of \(W_\tau\) for some fixed \(\delta \in [0, 1]\). We compute \(\tau\) by realizing that the noise pmf \(w_\gamma \in P_q\) lives in \(\text{conv}\left(\{w_q P^k_q : k = 1, \ldots, q - 1\}\right)\), or equivalently:
\[
w_\gamma = \sum_{i=1}^{q-1} \lambda_i w_q^i P^i_q
\]
for some \(\lambda_1, \ldots, \lambda_{q-1} \in [0, 1]\) such that \(\lambda_1 + \cdots + \lambda_{q-1} = 1\). Solving these equations immediately reveals that:
\[
\tau = 1 - \frac{\delta}{q - 1}
\]
and \((\lambda_i - \lambda_j)(1 - \delta - (\delta/(q-1))) = 0\) for every \(1 \leq i, j \leq q - 1\), which means that \(\lambda_1 = \cdots = \lambda_{q-1} = \frac{1}{q - 1}\). Hence, we have:
\[
w_\gamma = \frac{1}{q - 1} \sum_{i=1}^{q-1} w_q^i P^i_q
\]
which is illustrated in Figure 1 for the case where \(\delta \in (0, \frac{q-1}{q})\) and \(\tau > \frac{q-1}{q} > \delta\). For \(\delta \in (0, \frac{q-1}{q})\), the symmetric channels that are degraded versions of \(W_\delta\) are \(\{W_\gamma : \gamma \in [0, 1)\}\). In particular, for such \(\gamma \in [\delta, 1]\), \(W_\gamma = W_\delta W_\beta\), where \(\beta = (\gamma - \delta)/(1 - \delta - \frac{\delta}{q - 1})\) using the proof of part 5 of Proposition 3 in Appendix B.

In the spirit of comparing symmetric and erasure channels as done in [11] for the binary input case, our next result shows that a q-ary erasure channel can never be less noisy than a q-ary erasure channel.

**Proposition 6** (Symmetric Channel \(\preceq_n\) Erasure Channel). For \(q \in \mathbb{N} \setminus \{1\}\), given a q-ary erasure channel \(E_\epsilon \in \mathbb{R}^{q \times (q + 1)}\) with erasure probability \(\epsilon \in (0, 1)\), there does not exist \(\delta \in (0, 1)\) such that the corresponding q-ary symmetric channel \(W_\delta \in \mathbb{R}^{q \times q}\) on the same input alphabet satisfies \(W_\delta \succeq_n E_\epsilon\).

**Proof.** For a q-ary erasure channel \(E_\epsilon\) with \(\epsilon \in (0, 1)\), we always have \(D(uE_\epsilon || \Delta_0 E_\epsilon) = +\infty\) for \(u, \Delta_0 = (1, 0, \ldots, 0) \in P_q\). On the other hand, for any q-ary symmetric channel \(W_\delta\) with \(\delta \in (0, 1)\), we have \(D(P_X W_\delta || Q_X W_\delta) < +\infty\) for every \(P_X, Q_X \in P_q\). Thus, \(W_\delta \not\succeq_n E_\epsilon\) for any \(\delta \in (0, 1)\). ■

In fact, the argument for Proposition 6 conveys that a symmetric channel \(W_\delta \in \mathbb{R}^{q \times q}\) with \(\delta \in (0, 1)\) satisfies \(W_\delta \succeq_n V\) for some channel \(V \in \mathbb{R}^{q \times r}\) only if \(D(P_X W_\delta || Q_X V) < +\infty\) for every \(P_X, Q_X \in P_q\). Typically, we are only interested in studying q-ary symmetric channels with \(q \geq 2\) and \(\delta \in (0, \frac{q-1}{q})\). For example, the binary symmetric channel with flip-over probability \(p\) is usually studied for \(p \in (0, \frac{1}{2})\). Indeed, the less noisy domination characteristics of the extremal symmetric channels with \(\delta = 0\) or \(\delta = \frac{q-1}{q}\) are quite elementary. Given \(q \geq 2\), \(W_0 = I_q \in \mathbb{R}^{q \times q}\) satisfies \(W_0 \succeq_n V\), and \(W_{(q-1)/q} = 1u \in \mathbb{R}^{q \times q}\) satisfies \(V \succeq_n W_{(q-1)/q}\), for every channel \(V \in \mathbb{R}^{q \times r}\) on a common input alphabet. For the sake of completeness, we also note that for \(q \geq 2\), the extremal erasure channels \(E_0 \in \mathbb{R}^{q \times (q+1)}\) and \(E_1 \in \mathbb{R}^{q \times (q+1)}\), with \(\epsilon = 0\) and \(\epsilon = 1\) respectively, satisfy \(E_0 \succeq_n V\) and \(V \succeq_n E_1\) for every channel \(V \in \mathbb{R}^{q \times r}\) on a common input alphabet.
The result that the symmetric channel with uniform noise pmf $W_{(q-1)/q}$ is more noisy than every channel on the same input alphabet has an analogue concerning additive white Gaussian noise (AWGN) channels. Consider all additive noise channels of the form:

\[ Y = X + Z \]

where $X, Y \in \mathbb{R}$, the input $X$ is uncorrelated with the additive noise $Z$: $\mathbb{E}[XZ] = 0$, and the noise $Z$ has power constraint $\mathbb{E}[Z^2] \leq \sigma_Z^2$ for some fixed $\sigma_Z > 0$. Let $X = X_g \sim \mathcal{N}(0, \sigma_X^2)$ for some $\sigma_X > 0$. Then, we have:

\[ I(X_g; X_g + Z) \geq I(X_g; X_g + Z_g) \]

where $Z_g \sim \mathcal{N}(0, \sigma_Z^2)$. This states that Gaussian noise is the “worst case additive noise” for a Gaussian source. Hence, the AWGN channel is not more capable than any other additive noise channel with the same constraints. As a result, the AWGN channel is not less noisy than any other additive noise channel with the same constraints (using Proposition 2).

IV. EQUIVALENT CHARACTERIZATIONS OF LESS NOISY PREORDER

Having studied the structure of less noisy domination and degradation regions of channels, we now address the problem of verifying whether a channel $W$ is less noisy than another channel $V$. Directly using Definition 2 or Proposition 1 is difficult, and we typically resort to checking whether $V$ is a degraded version of $W$. In this section, we derive some equivalent characterizations of the less noisy preorder.

A. Characterization using $\chi^2$-divergence

It is well-known that KL divergence is locally approximated by the $\chi^2$-divergence, e.g. [23, Section 4.2]. While this approximation sometimes fails globally, cf. [25], the following notable result was first shown by Ahlswede and Gács in [1]:

\[ \eta_{\chi^2}(W) = \eta_{\chi^2}(W) \triangleq \sup_{P_X, Q_X \in \mathcal{P}_q} \frac{\chi^2(P_X W || Q_X W)}{\chi^2(P_X || Q_X)} \]

for any channel $W \in \mathbb{R}^{q \times q}$, where $\eta_{\chi^2}(W)$ is defined in (4), and the second equality defines the contraction coefficient for $\chi^2$-divergence $\eta_{\chi^2}(W)$. The next proposition generalizes this result to less noisy domination by an arbitrary channel (recall from Subsection I-C that $\eta_{\chi^2}$ characterizes less noisy domination with respect to an erasure channel).

**Proposition 7** ($\chi^2$-Divergence Characterization of $\succeq_n$). For any pair of channels $W \in \mathbb{R}^{q \times q}$ and $V \in \mathbb{R}^{q \times q}$ on the same input alphabet $[q]$, we have $W \succeq_n V$ if and only if:

\[ \forall P_X, Q_X \in \mathcal{P}_q, \quad \chi^2(P_X W || Q_X W) \geq \chi^2(P_X V || Q_X V). \]

**Proof.** In order to prove the forward direction, we recall the local approximation of KL divergence using $\chi^2$-divergence from [23, Proposition 4.2] which states that for any $P_X \in \mathcal{P}_q$ and $Q_X \in \mathcal{P}^\circ_q$:

\[ \lim_{\lambda \to 0^+} \frac{2}{\lambda^2} D(\lambda P_X + \lambda Q_X || Q_X) = \chi^2(P_X || Q_X) \]

for any $\lambda \in [0,1]$, and the KL divergence is defined using natural logarithms. Let us fix some $P_X \in \mathcal{P}_q$ and $Q_X \in \mathcal{P}^\circ_q$. Since $W \succeq_n V$, we have from Proposition 1:

\[ D(\lambda P_X W + \lambda Q_X W || Q_X W) \geq D(\lambda P_X V + \lambda Q_X V || Q_X V) \]

for any $\lambda \in [0,1]$. Taking appropriate limits as shown in (34) produces:

\[ \chi^2(P_X W || Q_X W) \geq \chi^2(P_X V || Q_X V) \]

for any $P_X \in \mathcal{P}_q$ and any $Q_X \in \mathcal{P}^\circ_q$. Furthermore, for any $Q_X \in \mathcal{P}_q \setminus \mathcal{P}^\circ_q$ and $P_X \in \mathcal{P}_q$, we can consider a sequence of pmfs $\{Q^k_X \in \mathcal{P}^\circ_q : k \in \mathbb{N}\}$ such that $Q^k_X \to Q_X$ as $k \to \infty$. Then, $\chi^2(P_X W || Q^k_X W) \geq \chi^2(P_X V || Q^k_X V)$ for every $k \in \mathbb{N}$, and letting $k \to \infty$ gives us $\chi^2(P_X W || Q_X W) \geq \chi^2(P_X V || Q_X V)$ using the continuity of $\chi^2$-divergence in its second argument with fixed first argument. (Note that we let $+\infty \geq +\infty$ be true in this inequality.) Thus, $\chi^2(P_X W || Q_X W) \geq \chi^2(P_X V || Q_X V)$ for all $P_X, Q_X \in \mathcal{P}_q$, which completes the forward direction.

To prove the converse direction, we assume that:

\[ \forall P_X, Q_X \in \mathcal{P}_q, \quad \chi^2(P_X W || Q_X W) \geq \chi^2(P_X V || Q_X V). \]

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We recall an integral representation of KL divergence using χ²-divergence presented in [2, Appendix A.2]:

\[ D(P_X||Q_X) = \int_0^\infty \chi^2(P_X||Q_X^t) \, dt \]  

for any \( P_X, Q_X \in \mathcal{P}_q \), where \( Q_X^t = \frac{1}{1+t}P_X + \frac{t}{1+t}Q_X \) for \( t \in [0, \infty) \). Hence, for every \( P_X, Q_X \in \mathcal{P}_q \), we have:

\[ \int_0^\infty \chi^2(P_X W||Q_X^t W) \, dt \geq \int_0^\infty \chi^2(P_X V||Q_X^t V) \, dt \]

\[ D(P_X W||Q_X W) \geq D(P_X V||Q_X V) \]

which means that \( W \succeq_n V \) using Proposition 1. This completes the proof. ■

We remark that Proposition 7 can be extended to hold for more general Markov kernels since both (34) and (35) hold more generally.

B. Characterization via the Löwner partial order

The χ²-divergence characterization in Proposition 7 can be used to derive a characterization of \( \succeq_n \) using the Löwner partial order. This is presented next.

**Proposition 8** (Löwner Characterization of \( \succeq_n \)). For any pair of channels \( W \in \mathbb{R}_s^{d \times r} \) and \( V \in \mathbb{R}_s^{d \times s} \) on the same input alphabet \([q]\), we have:

\[ \forall P_X, Q_X \in \mathcal{P}_q, \; \chi^2(P_X W||Q_X W) \geq \chi^2(P_X V||Q_X V) \]

if and only if:

\[ \forall P_X \in \mathcal{P}_q^o, \; W \text{diag}(P_X W)^{-1} W^T \succeq_{\text{PSD}} V \text{diag}(P_X V)^{-1} V^T. \]

**Proof.** First observe that for every \( P_X \in \mathcal{P}_q \) and every \( Q_X \in \mathcal{P}_q^o \), we have:

\[ \chi^2(P_X||Q_X) = (P_X - Q_X) \text{diag}(Q_X)^{-1} (P_X - Q_X)^T. \]  

(36)

To prove the converse direction, note that for every \( P_X \in \mathcal{P}_q \) and every \( Q_X \in \mathcal{P}_q^o \), we can let \( J_X = P_X - Q_X \) and get:

\[ J_X W \text{diag}(Q_X W)^{-1} W^T J_X^T \geq J_X V \text{diag}(Q_X V)^{-1} V^T J_X^T. \]

Using (36), this is equivalent to:

\[ \chi^2(P_X W||Q_X W) \geq \chi^2(P_X V||Q_X V) \]

for every \( P_X \in \mathcal{P}_q \) and every \( Q_X \in \mathcal{P}_q^o \), where we may extend this inequality to all \( Q_X \in \mathcal{P}_q \) using the continuity of χ²-divergence in its second argument with fixed first argument (as shown in the proof of Proposition 7).

We prove the forward direction by contraposition. Let:

\[ M(Q_X) \triangleq W \text{diag}(Q_X W)^{-1} W^T - V \text{diag}(Q_X V)^{-1} V^T \]

for \( Q_X \in \mathcal{P}_q^o \), and suppose \( \exists Q_X \in \mathcal{P}_q^o \) such that \( M(Q_X) \not\succeq_{\text{PSD}} 0 \). Observe that \( \text{diag}(\sqrt{Q_X}) M(Q_X) \text{diag}(\sqrt{Q_X}) \not\succeq_{\text{PSD}} 0 \) if and only if \( M(Q_X) \not\succeq_{\text{PSD}} 0 \), because the former is non-singular *-congruent to the latter (where \( \sqrt{Q_X} \) denotes the entry-wise square root of \( Q_X \)). Let \( \lambda < 0 \) be a strictly negative eigenvalue of \( \text{diag}(\sqrt{Q_X}) M(Q_X) \text{diag}(\sqrt{Q_X}) \) with corresponding left (or row) eigenvector \( x \in \mathbb{R}^q \). Since \( Q_X M(Q_X) = 0 \) (zero vector), which means that \( Q_X \) is in the nullspace of \( M(Q_X) \), \( \sqrt{Q_X} \) is in the nullspace of \( \text{diag}(\sqrt{Q_X}) M(Q_X) \text{diag}(\sqrt{Q_X}) \). Therefore, \( \sqrt{Q_X} \) and \( x \) are orthogonal, and we can construct a pmf \( P_X = Q_X + x \text{diag}(\sqrt{Q_X}) \in \mathcal{P}_q \) (where \( x \) is rescaled appropriately). Then, we have:

\[ (P_X - Q_X)^T M(Q_X) (P_X - Q_X) = \lambda \|x\|^2 < 0 \]

which implies that:

\[ \chi^2(P_X W||Q_X W) < \chi^2(P_X V||Q_X V) \]

using (36). This proves the contrapositive. ■

The Löwner condition of Proposition 8 is particularly alluring if one wishes to deduce whether \( W \succeq_n V \). Indeed, the positive semidefiniteness of the matrices \( W \text{diag}(P_X W)^{-1} W^T - V \text{diag}(P_X V)^{-1} V^T \) (for different \( P_X \in \mathcal{P}_q^o \)) can be tested using standard sufficient conditions for positive semidefiniteness such as diagonal dominance. The ensuing subsection presents yet another equivalent characterization of the less noisy preorder which follows from Proposition 8.
C. Spectral characterization

We will now derive a spectral characterization of the less noisy preorder. In order to prove this characterization, we will require a slight generalization of a rather useful result from [17, Theorem 7.7.3 (a)] that provides a direct connection between the L"owner characterization in Proposition 8 and a spectral radius condition.

**Lemma 2** (Löwner Domination and Spectral Radius). Given matrices $A, B \in \mathbb{R}_{\geq 0}^{q \times q}$ where $A$ is non-zero, we have:

$$A \succeq_{\text{PSD}} B \iff \rho(A^\dagger B) \leq 1.$$  

*Proof.* See Appendix C.  

We remark that when $A$ is invertible, $\rho(A^{-1}B)$ in Lemma 2 is the largest generalized eigenvalue of the matrix pencil $(B, A)$. The next result presents the spectral characterization of the less noisy preorder.

**Proposition 9** (Spectral Characterization of $\succeq_{\mu}$). For any pair of channels $W \in \mathbb{R}_{\text{sto}}^{q \times r}$ and $V \in \mathbb{R}_{\text{sto}}^{q \times s}$ on the same input alphabet $[q]$, we have $W \succeq_{\mu} V$ if and only if for every $P_X \in \mathcal{P}_q^0$, the spectral radius below is unity:

$$\rho \left( \left( W \text{diag}(P_X W)^{-1} W^T \right)^\dagger V \text{diag}(P_X V)^{-1} V^T \right) = 1.$$  

Furthermore, if $q \geq r$ and $W$ has full column rank, then $W \succeq_{\mu} V$ if and only if for every $P_X \in \mathcal{P}_q^0$:

$$\rho \left( (W^\dagger)^T \text{diag}(P_X W) W \text{diag}(P_X V)^{-1} V^T \right) = 1.$$  

*Proof.* Using Propositions 7 and 8, $W \succeq_{\mu} V$ if and only if for every $P_X \in \mathcal{P}_q^0$:

$$W \text{diag}(P_X W)^{-1} W^T \succeq_{\text{PSD}} V \text{diag}(P_X V)^{-1} V^T.$$  

Since $W \text{diag}(P_X W)^{-1} W^T, V \text{diag}(P_X V)^{-1} V^T \in \mathbb{R}_{\geq 0}^{q \times q}$, Lemma 2 gives us the equivalent condition:

$$\rho \left( \left( W \text{diag}(P_X W)^{-1} W^T \right)^\dagger V \text{diag}(P_X V)^{-1} V^T \right) \leq 1$$

for every $P_X \in \mathcal{P}_q^0$. Observe that $P_X W \text{diag}(P_X W)^{-1} W^T = 1^T$ and $P_X V \text{diag}(P_X V)^{-1} V^T = 1^T$, which means that:

$$1^T \left( W \text{diag}(P_X W)^{-1} W^T \right)^\dagger V \text{diag}(P_X V)^{-1} V^T = 1^T.$$  

Indeed, when $A, B \in \mathbb{R}_{\geq 0}^{q \times q}$ and $A \succeq_{\text{PSD}} B$, if $xA = xB = y$ for some row vectors $x, y \in \mathbb{R}^q$, then $yA^\dagger B = xPB = xB = y$, where $P = AA^\dagger$ is the orthogonal projection matrix onto the range of $A$, and $PB = B$ is shown in Appendix C. Thus, $W \succeq_{\mu} V$ if and only if for every $P_X \in \mathcal{P}_q^0$:

$$\rho \left( \left( W \text{diag}(P_X W)^{-1} W^T \right)^\dagger V \text{diag}(P_X V)^{-1} V^T \right) = 1$$

where we use the fact that the largest eigenvalue of the matrix $(W \text{diag}(P_X W)^{-1} W^T)^\dagger V \text{diag}(P_X V)^{-1} V^T$ equals its spectral radius as shown in Appendix C. When $W$ has full column rank, we can simplify the Moore-Penrose pseudoinverse to:

$$\left( W \text{diag}(P_X W)^{-1} W^T \right)^\dagger = (W^\dagger)^T \text{diag}(P_X W) W^\dagger.$$  

This completes the proof.  

The spectral radius condition in Proposition 9 is even more useful than the Löwner condition in Proposition 8 when one wishes to computationally deduce whether $W \succeq_{\mu} V$. In closing this subsection, we emphasize that Theorem 1 is an immediate consequence of Propositions 7, 8, and 9.
D. Relation to a proof of Ahlswede and Gács

Curiously, Ahlswede and Gács’ technique to prove (33) in [1] also proves the Löwner characterization of $\succeq_n$ presented in Proposition 8. We next illustrate this alternative proof.

**Proposition 10** (Alternative Löwner Characterization of $\succeq_n$). For any pair of channels $W \in \mathbb{R}^{q \times r}_{\text{sfo}}$ and $V \in \mathbb{R}^{q \times s}_{\text{sfo}}$ on the same input alphabet $[q]$, we have $W \succeq_n V$ if and only if:

$$\forall P_X \in \mathcal{P}^o_q, \quad W \text{diag}(P_X W)^{-1} W^T \succeq_{\text{psd}} V \text{diag}(P_X V)^{-1} V^T.$$  

**Proof.** For fixed channels $W$ and $V$, and any pmf $Q_X \in \mathcal{P}^o_q$, consider the function $F : \mathcal{P}_q \rightarrow \mathbb{R}$ that is defined by:

$$\forall P_X \in \mathcal{P}_q, \quad F(P_X) = D(P_X W||Q_X W) - D(P_X V||Q_X V)$$

in analogy with the proof of (33) in [1]. This function is non-negative for every $Q_X \in \mathcal{P}^o_q$ if and only if $W \succeq_n V$ (modulo some analytical details shown later). We will show that regardless of the choice of $Q_X$, the function $F$ has a stationary point with value zero, and is therefore non-negative when it is convex.

We first compute the Hessian matrix of $F$. The function $F$ can be written as:

$$\forall P_X \in \mathcal{P}_q, \quad F(P_X) = \sum_{y=0}^{r-1} P_X W(y) \log \left( \frac{P_X W(y)}{Q_X W(y)} \right) - \sum_{y=0}^{s-1} P_X V(y) \log \left( \frac{P_X V(y)}{Q_X V(y)} \right)$$

where all logarithms are natural, and the output alphabets of $W$ and $V$ are $[r]$ and $[s]$, respectively. The first derivatives of $F$ are:

$$\forall P_X \in \mathcal{P}_q, \quad \frac{\partial F}{\partial P_X(x)} (P_X) = \sum_{y=0}^{r-1} [W]_{x,y} \log \left( \frac{P_X W(y)}{Q_X W(y)} \right) - \sum_{y=0}^{s-1} [V]_{x,y} \log \left( \frac{P_X V(y)}{Q_X V(y)} \right)$$

for every $x \in [q]$, and the second derivatives of $F$ are:

$$\forall P_X \in \mathcal{P}_q, \quad \frac{\partial^2 F}{\partial P_X(x) \partial P_X(x')} (P_X) = \sum_{y=0}^{r-1} [W]_{x,y} [W]_{x',y} \log \left( \frac{P_X W(y)}{Q_X W(y)} \right) - \sum_{y=0}^{s-1} [V]_{x,y} [V]_{x',y} \log \left( \frac{P_X V(y)}{Q_X V(y)} \right)$$

for every $x, x' \in [q]$ (where we index the matrices $W$ and $V$ starting at 0 rather than 1). The Hessian of $F$, $\nabla^2 F : \mathcal{P}_q \rightarrow \mathbb{R}^{q \times q}$, is defined entry-wise for every $x, x' \in [q]$ as:

$$[\nabla^2 F (P_X)]_{x,x'} = \frac{\partial^2 F}{\partial P_X(x) \partial P_X(x')} (P_X)$$

and we may write it in matrix form as:

$$\nabla^2 F (P_X) = W \text{diag}(P_X W)^{-1} W^T - V \text{diag}(P_X V)^{-1} V^T$$

for every $P_X \in \mathcal{P}_q$ (which ensures that the matrix inverses are well-defined). This Hessian matrix is useful because it does not depend on $Q_X$, and any result derived from it holds for every $Q_X \in \mathcal{P}_q$.

We next prove the converse direction. Observe that $F|_{\mathcal{P}_q} : \mathcal{P}_q \rightarrow \mathbb{R}$ is convex if and only if $\nabla^2 F (P_X) \succeq_{\text{psd}} 0$ for every $P_X \in \mathcal{P}_q$ since $\mathcal{P}_q$ is a convex set [26, Section 3.1.4]. Moreover, as the function $F$ is continuous on its domain $\mathcal{P}_q$, $F : \mathcal{P}_q \rightarrow \mathbb{R}$ is convex if and only if $\nabla^2 F (P_X) \succeq_{\text{psd}} 0$ for every $P_X \in \mathcal{P}_q$. Suppose $\nabla^2 F (P_X) \succeq_{\text{psd}} 0$ for every $P_X \in \mathcal{P}_q$ so that $F$ is convex, or equivalently:

$$\forall P_X \in \mathcal{P}_q, \quad W \text{diag}(P_X W)^{-1} W^T \succeq_{\text{psd}} V \text{diag}(P_X V)^{-1} V^T.$$  

The function $F$ has a stationary point at $P_X = Q_X \in \mathcal{P}_q$:

$$\forall x \in [q], \quad \frac{\partial F}{\partial P_X(x)} (Q_X) = 0$$

with value $F(Q_X) = 0$. Since $F$ is convex, this is a global minimum. Hence, for every $P_X \in \mathcal{P}_q$ and every $Q_X \in \mathcal{P}_q$, we have $F(P_X) \geq 0$:

$$D(P_X W||Q_X W) \geq D(P_X V||Q_X V).$$

Furthermore, for any $Q_X \in \mathcal{P}_q \setminus \mathcal{P}_q$ and $P_X \in \mathcal{P}_q$, we can consider a sequence of pmfs $\{Q_X^k \in \mathcal{P}_q : k \in \mathbb{N}\}$ such that $Q_X^k \rightarrow Q_X$ as $k \rightarrow \infty$. Then, $D(P_X W||Q_X^k W) \geq D(P_X V||Q_X^k V)$ for every $k \in \mathbb{N}$, and letting $k \rightarrow \infty$ gives us $D(P_X W||Q_X W) \geq D(P_X V||Q_X V)$ using the continuity of KL divergence in its second argument with fixed first
argument. Thus, $D(P_X W || Q_X W) \geq D(P_X V || Q_X V)$ for all $P_X, Q_X \in \mathcal{P}_q$, which means $W \succeq_v V$ (using Proposition 1).

Finally, we prove the forward direction by contraposition. Suppose $\exists P_X \in \mathcal{P}^o_q$ such that $\nabla^2 F(P_X) \not\succeq_{\mathbb{PSD}} 0$. Choosing $Q_X = P_X$, we have using Taylor’s theorem:\(^2\)

$$F(P_X + J_X) = \frac{1}{2} J_X \nabla^2 F(P_X) J_X + O\left(\|J_X\|^2\right)$$

for any perturbation (row) vector $J_X \in \mathbb{R}^q$ that satisfies $J_X 1 = 0$ so that $P_X + J_X \in \mathcal{P}^o_q$, where we assume that $J_X$ is in a fixed neighborhood around $P_X$ so that $\|J_X\|_2$ is small. This Taylor approximation uses the facts that $F$ is three times continuously differentiable, $F'(P_X) = 0$, and $\nabla^2 F(P_X) = 0$. Following the proof of the forward direction of Proposition 8, recall that $\text{diag}(\nabla^2 F(P_X)) \succeq_{\mathbb{PSD}} 0$ if and only if $\nabla^2 F(P_X) \not\succeq_{\mathbb{PSD}} 0$.\(^3\) We let $J_X = x \text{diag}(\nabla^2 F(P_X))$ be the valid perturbation (because $J_X 1 = 0$) in the earlier Taylor approximation, where $x \in \mathbb{R}^q$ is an appropriately scaled left (or row) eigenvector of $\text{diag}(\nabla^2 F(P_X)) \text{diag}(\nabla^2 F(P_X))$ corresponding to a strictly negative eigenvalue $\lambda < 0$. Then, $F(P_X + J_X) < 0$ for sufficiently small $\|J_X\|_2$ as $x \text{diag}(\nabla^2 F(P_X)) \text{diag}(\nabla^2 F(P_X)) x^T = \lambda \|x\|^2_2 < 0$. So, we have proved the contrapositive that if $\exists P_X \in \mathcal{P}^o_q$ such that $\nabla^2 F(P_X) \not\succeq_{\mathbb{PSD}} 0$, then $W \not\succeq_v V$ because:


This completes the proof. \(\square\)

We note that by defining $F$ using $\chi^2$-divergences instead of KL divergences and following the proof of Proposition 10, we could have also directly proved the equivalence between the $\chi^2$-divergence and Löwner characterizations of $\succeq_v$ given in Proposition 8. It is worth mentioning that the proofs of the forward directions of Propositions 7 and 10 both employ (in essence) the local approximation of KL divergence using $\chi^2$-divergence. On the other hand, the proof of the converse direction of Proposition 10 exploits the fact that the KL divergence map $P_X \mapsto D(P_X W || Q_X W)$ has a Hessian $P_X \mapsto W \text{diag}(P_X W)^{-1} W^T$ that does not depend on $Q_X$. Although the quadratic form $(Q_X - P_X) W \text{diag}(P_X W)^{-1} W^T (Q_X - P_X)^T$ defined by this Hessian resembles $\chi^2(Q_X W || P_X W)$ (using (36)), the converse proof of Proposition 10 does not utilize the local approximation of KL divergence using $\chi^2$-divergence. Furthermore, the converse proof of Proposition 7, which uses an integral representation of KL divergence, also differs from the converse proof of Proposition 10.

V. CONDITIONS FOR LESS NOISY DOMINATION OVER ADDITIVE NOISE CHANNELS

We now turn our attention to deriving several conditions for determining when symmetric channels are less noisy than other channels. Our interest in symmetric channels arises from the analytical tractability of many problems involving symmetric channels; Proposition 3 from Subsection I-D, Proposition 14 from Section VII, and [28, Theorem 4.5.2] (which conveys that symmetric channels with uniform capacity achieving input distributions) serve as illustrations of this tractability. We focus on additive noise channels in this section, and on general channels in the next section.

A. Necessary conditions

We first present some straightforward necessary conditions for when an additive noise channel $W \in \mathbb{R}^{q \times q}_{\text{sto}}$ with $q \in \mathbb{N}$ is less noisy than another additive noise channel $V \in \mathbb{R}^{q \times q}_{\text{sto}}$ on an Abelian group $(X, \oplus)$. These conditions can obviously be specialized for less noisy domination by symmetric channels.

**Proposition 11** (Necessary Conditions for $\succeq_v$ Domination over Additive Noise Channels). Suppose $W = \text{circ}_X(w)$ and $V = \text{circ}_X(v)$ are additive noise channels with noise pmfs $w, v \in \mathcal{P}_q$ such that $W \succeq_v V$. Then, the following are true:

1. (Circle Condition) $\|w - u\|_2 \geq \|v - u\|_2$.
2. (Contraction Condition) $\eta_{\text{ent}}(W) \geq \eta_{\text{ent}}(V)$.
3. (Entropy Condition) $H(u) \geq H(v)$, where $H: \mathcal{P}_q \to \mathbb{R}^+$ is the Shannon entropy function.

**Proof.**

**Part 1:** Using part 2 of Theorem 1, we have:

$$\forall P_X, Q_X \in \mathcal{P}_q, \quad \chi^2(P_X W || Q_X W) \geq \chi^2(P_X V || Q_X V).$$

\(^2\)We use the big-O notation here, which is defined as: $f(v) = O(g(v)) \iff \limsup_{v \to +v} |f(v)/g(v)| < +\infty$.

\(^3\)Let the divergence transition matrix (DTM) corresponding to the channel $W$ be $B_W = \text{diag}(\nabla^2 F(P_X))^{-1} W^T \text{diag}(\nabla^2 F(P_X))$, and the DTM corresponding to the channel $V$ be $B_V = \text{diag}(\nabla^2 F(P_X))^{-1} V^T \text{diag}(\nabla^2 F(P_X))$ (see [27] and the references therein for uses of such matrices). Then, $\text{diag}(\nabla^2 F(P_X)) \text{diag}(\nabla^2 F(P_X)) = B_W^T B_W - B_V^T B_V$. Hence, $\nabla^2 F(P_X) \succeq_{\mathbb{PSD}} 0$ if and only if $B_W^T B_W \succeq_{\mathbb{PSD}} B_V^T B_V$.\(\square\)
Letting $P_X = \Delta_0 \triangleq (1, 0, \ldots, 0)$ and $Q_X = u$ produces:

$$q \|w - u\|_2^2 = \chi^2 (w\|u) \geq \chi^2 (v\|u) = q \|v - u\|_2^2$$

since $uW = uV = u$ (i.e., $W$ and $V$ are doubly stochastic), and $P_X W = w$ and $P_X V = v$ using (9). Hence, we have $\|w - u\|_2 \geq \|v - u\|_2$. An alternative proof of this result using Fourier analysis is given in Appendix D.

**Part 2:** This easily follows from Proposition 1 and (4).

**Part 3:** Since $W \succeq_n V$, we have:

$$D (\Delta_0 W\|uW) \geq D (\Delta_0 V\|uV)$$

$$D (w\|u) \geq D (v\|u)$$

$$\log (q) - H (w) \geq \log (q) - H (v)$$

$$H (v) \geq H (w)$$

using Proposition 1.  

We remark that the aforementioned necessary conditions have many generalizations. Firstly, if $W, V \in \mathbb{R}^{q \times q}_{\text{stof}}$ are doubly stochastic matrices, then the generalized circle condition holds:

$$\left\| W - W_{\frac{q-1}{q}} \right\|_{\text{Fro}} \geq \left\| V - W_{\frac{q-1}{q}} \right\|_{\text{Fro}}$$

(37)

where $W_{(q-1)/q}$ is the symmetric channel whose conditional pmfs are all uniform, and $\|\cdot\|_{\text{Fro}}$ denotes the Frobenius norm. Indeed, letting $P_X = \Delta_x \triangleq (0, \ldots, 1, \ldots, 0)$ for $x \in [q]$, which has unity in the $(x + 1)$th position, in the proof of part 1 and then adding the inequalities corresponding to every $x \in [q]$ produces (37). Secondly, the contraction condition in Proposition 11 actually holds for any pair of general channels $W \in \mathbb{R}^{q \times r}_{\text{stof}}$ and $V \in \mathbb{R}^{q \times s}_{\text{stof}}$ on a common input alphabet (not necessarily additive noise channels). Moreover, we can start with Proposition 7 and take the suprema of the ratios in $\chi^2 (P_X W\|Q_X W) / \chi^2 (P_X)\|Q_X)$ over all $P_X \neq Q_X$ to get:

$$\rho_{\text{max}} (Q_X, W) \geq \rho_{\text{max}} (Q_X, V)$$

(38)

for any $Q_X \in P_q$, where $\rho_{\text{max}} (\cdot)$ denotes maximal correlation which is defined later in part 3 of Proposition 14 (also see [29]), and we use [27, Theorem 3] (or the results of [30]). A similar result also holds for the contraction coefficient for KL divergence with fixed input pmf (which is defined in [27, Definition 1]).

**B. Sufficient conditions**

We next portray a sufficient condition for when an additive noise channel $V \in \mathbb{R}^{q \times q}_{\text{stof}}$ is a degraded version of a symmetric channel $W_\delta \in \mathbb{R}^{q \times q}_{\text{stof}}$. By Proposition 2, this is also a sufficient condition for $W_\delta \succeq_n V$.

**Proposition 12** (Degradation by Symmetric Channels). *Given an additive noise channel $V = \text{circ}_X (v)$ with noise pmf $v \in P_q$ and minimum probability $\tau = \min \{ [V]_{i,j} : 1 \leq i, j \leq q \}$, we have:

$$0 \leq \delta \leq (q - 1) \tau \Rightarrow W_\delta \succeq_{\text{deg}} V$$

where $W_\delta \in \mathbb{R}^{q \times q}_{\text{stof}}$ is a symmetric channel.*/

**Proof.** Using Corollary 1, we observe that it suffices to prove that $w_{(q-1)\tau} \succeq_{\text{maj}} v$. Since $0 \leq \tau \leq \frac{1}{q}$, we must have $0 \leq (q - 1) \tau \leq \frac{q-1}{q}$. Let $\tau = w_{(1)} = \cdots = w_{(q-1)} \leq w_{(q)} = 1 - (q - 1)\tau$ denote the entries of the noise pmf $w_{(q-1)\tau}$ in ascending order, and $v_{(1)} \leq \cdots \leq v_{(q)}$ denote the entries of the noise pmf $v$ in ascending order. As $v \geq \tau$ (entry-wise), the conditions of part 3 in Proposition 15 in Appendix A hold:

$$k\tau = \sum_{i=1}^{k} w_{(i)} \leq \sum_{i=1}^{q} v_{(i)} \text{, for } k = 1, \ldots, q - 1,$$

and

$$\sum_{i=1}^{q} w_{(i)} = \sum_{i=1}^{q} v_{(i)} = 1.$$

Hence, $w_{(q-1)\tau} \succeq_{\text{maj}} v$, which completes the proof.  

It is compelling to find a sufficient condition for $W_\delta \succeq_n V$ that does not simply ensure $W_\delta \succeq_{\text{deg}} V$ (such as Proposition 12 and Theorem 2). The ensuing proposition elucidates such a sufficient condition for additive noise channels. The general strategy for finding such a condition for additive noise channels is to identify a noise pmf that
belongs to $\mathcal{L}_{W_{\delta}}^{\text{add}}, D_{W_{\delta}}^{\text{add}}$. One can then use Proposition 5 to explicitly construct a set of noise pmfs that is a subset of $\mathcal{L}_{W_{\delta}}^{\text{add}}$ but strictly includes $D_{W_{\delta}}^{\text{add}}$. The proof of the Proposition 13 finds such a noise pmf (that corresponds to a symmetric channel).

**Proposition 13 (Less Noisy Domination by Symmetric Channels).** Given an additive noise channel $V = \text{circ}_X(v)$ with noise pmf $v \in \mathcal{P}_q$ and $q \geq 2$, if for $\delta \in \left[0, \frac{q-1}{q}\right]$ we have:

$$v \in \text{conv}\left(\left\{w_{\gamma}P^k_x : k \in [q]\right\} \cup \left\{w_{\gamma}P^k_q : k \in [q]\right\}\right)$$

then $W_{\delta} \succeq_m V$, where $P_q$ denotes the generator cyclic permutation matrix as defined in (10), and:

$$\gamma = \frac{1 - \delta}{1 - \delta + \frac{\delta}{(q-1)^2}} \in \left[0, \frac{q-1}{q}\right].$$

**Proof.** Due to Proposition 5 and $\left\{w_{\gamma}P^k_x : x \in \mathcal{X}\right\} = \left\{w_{\gamma}P^k_q : k \in [q]\right\}$, it suffices to prove that $W_{\delta} \succeq_m W_{\gamma}$. Since $\delta = 0 \Rightarrow \gamma = 1$ and $\delta = \frac{q-1}{q} \Rightarrow \gamma = \frac{q-1}{q}$, $W_{\delta} \succeq_m W_{\gamma}$ is certainly true for $\delta \in \left\{0, \frac{q-1}{q}\right\}$. So, we assume that $\delta \in \left(0, \frac{q-1}{q}\right)$, which implies that:

$$\gamma = 1 - \frac{\delta}{1 - \delta + \frac{\delta}{(q-1)^2}} \in \left(0, \frac{q-1}{q}\right).$$

Since our goal is to show that $W_{\delta} \succeq_m W_{\gamma}$, we prove the equivalent condition in Theorem 1 that for every $P_X \in \mathcal{P}_q^o$:

$$W_{\delta} \text{ diag}(P_X W_{\delta})^{-1} W_{\delta}^T \succeq_{\text{Psd}} W_{\gamma} \text{ diag}(P_X W_{\gamma})^{-1} W_{\gamma}^T$$

$$\Leftrightarrow W_{\gamma}^{-1} \text{ diag}(P_X W_{\gamma}) W_{\delta}^{-1} \succeq_{\text{Psd}} \text{ diag}(P_X W_{\delta}) W_{\gamma}^{-1}$$

$$\Leftrightarrow \text{ diag}(P_X W_{\gamma}) \succeq_{\text{Psd}} W_{\delta} W_{\gamma}^{-1} \text{ diag}(P_X W_{\delta})$$

$$\Leftrightarrow I_q \succeq_{\text{Psd}} \text{ diag}(P_X W_{\gamma})^{-\frac{1}{2}} W_{\delta} \text{ diag}(P_X W_{\delta}) W_{\gamma}^{-\frac{1}{2}}$$

$$\Leftrightarrow 1 \geq \left\|\text{ diag}(P_X W_{\gamma})^{-\frac{1}{2}} W_{\delta} \text{ diag}(P_X W_{\delta}) W_{\gamma}^{-\frac{1}{2}}\right\|_{\text{op}}$$

where the second equivalence holds because $W_{\delta}$ and $W_{\gamma}$ are symmetric and invertible (see part 4 of Proposition 3 and [17, Corollary 7.7.4]), the third and fourth equalities are non-singular *-congruences, and:

$$\tau = \frac{\gamma - \delta}{1 - \delta - \frac{\delta}{q-1}} > 0$$

since $W_{\tau} = W_{\delta}^{-1} W_{\gamma} = W_{\gamma} W_{\delta}^{-1}$ (see the proof of Proposition 17 in Appendix E).\(^4\)

It is instructive to note that if $W_{\tau} \in \mathbb{R}_{\text{atol}}^{q \times q}$, then the matrix diag($P_X W_{\gamma})^{-\frac{1}{2}} W_{\gamma} \text{ diag}(P_X W_{\delta})^{-\frac{1}{2}}$ in the final inequality above is known as the *divergence transition matrix*, and this matrix has right singular vector $\sqrt{P_X W_{\gamma}^T}$ and left singular vector $\sqrt{P_X W_{\gamma}^T}$ corresponding to its maximum singular value of unity (see [27] and the references therein).

So, $W_{\tau} \in \mathbb{R}_{\text{atol}}^{q \times q}$ is a sufficient condition for $W_{\delta} \succeq_m W_{\gamma}$.

Observe that $W_{\tau} \in \mathbb{R}_{\text{atol}}^{q \times q}$ if and only if:

$$0 \leq \tau \leq 1$$

$$0 \leq \gamma - \delta \leq 1 - \delta - \frac{\delta}{q-1}$$

$$\Leftrightarrow \delta \leq \gamma \leq 1 - \frac{\delta}{q-1}.$$  

Hence, if $\delta \leq \gamma \leq 1 - \frac{\delta}{q-1}$, then $W_{\delta} \succeq_m W_{\gamma}$. However, we recall from (29) in Subsection III-B that $W_{\delta} \succeq_{\text{seq}} W_{\gamma}$ for $\delta \leq \gamma \leq 1 - \frac{\delta}{q-1}$, while we seek some $1 - \frac{\delta}{q-1} < \gamma \leq 1$ for which $W_{\delta} \succeq_m W_{\gamma}$. When $q = 2$, we only have:

$$\gamma = \frac{1 - \delta}{1 - \delta + \frac{\delta}{(q-1)^2}} = 1 - \frac{\delta}{q-1} = 1 - \delta$$

\(^4\)Note that we cannot use the strict Löwner partial order $\succ_{\text{Psd}}$ (which is defined for any matrices $A, B \in \mathbb{R}^{2 \times 2}$ as $A \succ_{\text{Psd}} B$ if and only if $A - B$ is positive definite) for these equivalences as $1^T \text{ diag}(P_X W_{\gamma}) 1 = 1^T W_{\gamma} W_{\delta}^{-1} \text{ diag}(P_X W_{\delta}) W_{\delta}^{-1} W_{\gamma} 1$. 

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which implies that $W_\delta \succeq_{\nu} W_\gamma$ is true for $q = 2$. When $q \geq 3$, we can verify that $\gamma > 1 - \frac{\delta}{q-1}$ from the following sequence of equivalences:

$$\gamma = \frac{1 - \delta}{1 - \delta + \frac{\delta}{(q-1)^2}} > 1 - \frac{\delta}{q-1}$$

$$1 - \delta > \left(1 - \frac{\delta}{q-1}\right)\left(1 - \delta + \frac{\delta}{(q-1)^2}\right)$$

$$0 > \frac{\delta}{(q-1)^2} + \frac{\delta^2 - \delta}{q-1} - \frac{\delta^2}{(q-1)^3}$$

$$1 - \frac{1}{q-1} > \delta \left(1 - \frac{1}{(q-1)^2}\right)$$

$$\frac{q-1}{q} > \delta$$

where the last inequality is assumed to be true. Hence, $\gamma \in (1 - \frac{\delta}{q-1}, 1)$ when $\delta \in (0, \frac{q-1}{q})$ and $q \geq 3$.

From the preceding discussion, it suffices to prove for $q \geq 3$ that for every $P_X \in \mathcal{P}_q$:

$$\left\| \operatorname{diag}(P_X W_\gamma)^{-\frac{1}{2}} W_r \operatorname{diag}(P_X W_\delta) W_r \operatorname{diag}(P_X W_\gamma)^{-\frac{1}{2}} \right\|_{\text{op}} \leq 1.$$

Since $\tau > 0$, and $0 \leq \tau \leq 1$ does not produce $\gamma > 1 - \frac{\delta}{q-1}$, we require that $\tau > 1$ ($\Leftrightarrow \gamma > 1 - \frac{\delta}{q-1}$) so that $W_r$ has strictly negative entries along the diagonal. Notice that:

$$\forall x \in [q], \quad \operatorname{diag}(\Delta_x W_\gamma) \preceq_{\text{psd}} W_\gamma W_\delta^{-1} \operatorname{diag}(\Delta_x W_\delta) W_\delta^{-1} W_\gamma$$

where $\Delta_x = (0, \ldots, 1, \ldots, 0) \in \mathcal{P}_q$ denotes the Kronecker delta pmf with unity at the $(x+1)$th position, implies that:

$$\operatorname{diag}(P_X W_\gamma) \preceq_{\text{psd}} W_\gamma W_\delta^{-1} \operatorname{diag}(P_X W_\delta) W_\delta^{-1} W_\gamma$$

for every $P_X \in \mathcal{P}_q$, because convex combinations preserve the Löwner relation. So, it suffices to prove that for every $x \in [q]$:}

$$\left\| \operatorname{diag}(w_r P_\gamma^x)^{-\frac{1}{2}} W_r \operatorname{diag}(w_x P_\gamma^x) W_r \operatorname{diag}(w_r P_\gamma^x)^{-\frac{1}{2}} \right\|_{\text{op}} \leq 1$$

as $\Delta_x M$ is the $(x+1)$th row of $M \in \mathbb{R}^{q \times q}$, where $P_q$ is the generator cyclic permutation matrix defined in (10). Let us define $A_x \triangleq \operatorname{diag}(w_x P_\gamma^x)^{-\frac{1}{2}} W_r \operatorname{diag}(w_x P_\gamma^x) W_r \operatorname{diag}(w_x P_\gamma^x)^{-\frac{1}{2}}$ for each $x \in [q]$. Observe that for every $x \in [q]$, the symmetric matrix $A_x$ is orthogonally diagonalizable by the real spectral theorem [31, Theorem 7.13], and has a strictly positive eigenvector $\sqrt{w_r P_\gamma^x}$ corresponding to the eigenvalue of unity:

$$\forall x \in [q], \quad \sqrt{w_r P_\gamma^x} A_x = \sqrt{w_r P_\gamma^x}$$

so that all other eigenvectors of $A_x$ have some strictly negative entries since they are orthogonal to $\sqrt{w_r P_\gamma^x}$. Suppose $A_x$ is entry-wise non-negative for every $x \in [q]$. Then, the largest eigenvalue (known as the Perron-Frobenius eigenvalue) and the spectral radius of each $A_x$ is unity by the Perron-Frobenius theorem [17, Theorem 8.3.4], which proves that $\|A_x\|_{\text{op}} \leq 1$ for every $x \in [q]$. Therefore, it is sufficient to prove that $A_x$ is entry-wise non-negative for every $x \in [q]$.

Equivalently, we can prove that $W_r \operatorname{diag}(w_x P_r^x) W_r$ is entry-wise non-negative for every $x \in [q]$, since $\operatorname{diag}(w_x P_r^x)^{-\frac{1}{2}}$ scales the rows or columns of the matrix it is pre- or post-multiplied with using strictly positive scalars.

We now show the equivalent condition that the minimum possible entry of $W_r \operatorname{diag}(w_x P_r^x) W_r$ is non-negative:

$$0 \leq \min_{x \in [q]} \sum_{1 \leq i,j \leq q} \left[ \underbrace{W_{r,1,r}}_{= W_r \operatorname{diag}(w_x P_r^x) W_r} \right]_{i,j} [W_\gamma x_{1,r}] [W_\gamma x_{1,r}] [W_\gamma x_{1,r}] = \frac{\tau(1 - \delta)(1 - \tau)}{q-1} + \frac{\delta \tau (1 - \tau)}{(q-1)^2} + \frac{\delta^2 (1 - \tau)}{(q-1)^3}$$

\[^5\text{It is interesting to note that forcing the divergence transition matrix } \operatorname{diag}(P_X W_\gamma)^{-\frac{1}{2}} W_r \operatorname{diag}(P_X W_\delta)^\frac{1}{2} \text{ (or equivalently, } W_r) \text{ to be entry-wise non-negative for every } P_X \in \mathcal{P}_q \text{ is not enough to find a symmetric channel that is more noisy than } W_\gamma \text{ but not a degraded version of it. However, forcing the dual Gramian of the divergence transition matrix } \operatorname{diag}(P_X W_\gamma)^{-\frac{1}{2}} W_r \operatorname{diag}(P_X W_\delta) W_r \operatorname{diag}(P_X W_\gamma)^{-\frac{1}{2}} \text{ to be entry-wise negative for } P_X = \Delta_x \text{ for every } x \in [q] \text{ produces such a symmetric channel.}\]
where the equality holds because for \( i \neq j \):
\[
\frac{\delta}{q-1} \sum_{r=1}^{q} [W_{\tau}]_{i,r} [W_{\tau}]_{r,i} \geq \frac{\delta}{q-1} \sum_{r=1}^{q} [W_{\tau}]_{i,r} [W_{\tau}]_{r,j}
\]
is clearly true (for example, one can use the rearrangement inequality in [32, Section 10.2]). Furthermore, adding \((1 - \delta - \frac{\delta}{q-1}) [W_{\tau}]_{i,k} \geq 0\) (regardless of the value of \(1 \leq k \leq q\)) to the left summation increases its value, while adding \((1 - \delta - \frac{\delta}{q-1}) [W_{\tau}]_{i,p} [W_{\tau}]_{p,j} < 0\) (which exists for an appropriate value \(1 \leq p \leq q\) as \(\tau > 1\)) to the right summation decreases its value. As a result, the minimum possible entry of \(W_{\tau} \text{diag}(w_{\delta} P_{\tau}^q) W_{\tau}\) can be achieved with \(x + 1 = i \neq j\) or \(i \neq j \neq x + 1\). Now observe that:
\[
0 \leq \frac{\tau(1 - \delta)(1 - \tau)}{q-1} + \frac{\delta \tau(1 - \tau)}{(q-1)^2} + (q-2) \frac{\delta^2}{(q-1)^3} \leq \frac{(\gamma - \delta)(1 - \delta - \frac{\delta}{q-1})}{(q-1)^2} \left(1 - \delta + \frac{\delta}{q-1}\right) + \frac{(q-2)\delta^2}{(q-1)^3} \left(1 - \delta - \frac{\delta}{q-1}\right)
\]
where the equality follows from the substitution \(\tau = (\gamma - \delta)/(1 - \delta - \frac{\delta}{q-1})\), and we can further simplify this to:
\[
0 \leq (\gamma - \delta) \left(1 - \frac{\delta}{q-1} - \frac{\delta}{1 - \delta + \frac{\delta}{q-1}}\right) + \frac{(q-2)\delta(\gamma - \delta)}{(q-1)^2}.
\]
The right hand side of this inequality is quadratic in \(\gamma\) with roots \(\gamma = \delta\) and:
\[
\gamma = \frac{(q-2)\delta^2}{(q-1)^2} - \frac{(1 - \frac{\delta}{q-1})(1 - \delta + \frac{\delta}{q-1})}{(q-1)^2} = \frac{1 - \delta}{(1 - \frac{\delta}{q-1})^2}.
\]
Since the coefficient of \(\gamma^2\) in this quadratic is strictly negative:
\[
\frac{(q-2)\delta}{(q-1)^2} - \frac{(1 - \delta + \frac{\delta}{q-1})}{(q-1)^2} < 0 \Leftrightarrow 1 - \delta + \frac{\delta}{(q-1)^2} > 0
\]
the minimum possible entry of \(W_{\tau} \text{diag}(w_{\delta} P_{\tau}^q) W_{\tau}\) is non-negative if and only if:
\[
\delta \leq \gamma \leq \frac{1 - \delta}{1 - \frac{\delta}{(q-1)^2}}
\]
where we use the fact that \(\frac{1 - \delta}{1 - \delta + \frac{\delta}{(q-1)^2}} \geq 1 - \frac{\delta}{q-1} \geq \delta\). Therefore, \(\gamma = \frac{1 - \delta}{1 - \delta + \frac{\delta}{(q-1)^2}}\) produces \(W_{\delta} \lesssim W_{\tau}\), which completes the proof. 

Heretofore we have derived results concerning less noisy domination and degradation regions in Section III, and proven several necessary and sufficient conditions for less noisy domination of additive noise channels by symmetric channels in this section. We finally have all the pieces in place to establish Theorem 3 from Section II. In closing this section, we indicate the pertinent results that coalesce to justify it.

**Proof of Theorem 3.** The first equality follows from Corollary 1. The first set inclusion is obvious, and its strictness follows from the proof of Proposition 13. The second set inclusion follows from Proposition 13. The third set inclusion follows from the circle condition (part 1) in Proposition 11. Lastly, the properties of \(L_{W_{\delta}}^{add}\) are derived in Proposition 5. 

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VI. SUFFICIENT CONDITIONS FOR DEGRADATION OVER GENERAL CHANNELS

While Propositions 12 and 13 present sufficient conditions for a symmetric channel $W_δ \in \mathbb{R}^{q \times q}$ to be less noisy than an additive noise channel, our more comprehensive objective is to find the maximum $δ \in [0, \frac{q-1}{q}]$ such that $W_δ \succeq_n V$ for any given general channel $V \in \mathbb{R}^{q \times r}$ on a common input alphabet. We may formally define this maximum $δ$ (that characterizes the extremal symmetric channel that is less noisy than $V$) as:

$$δ^* (V) \triangleq \sup \left\{ δ \in \left[0, \frac{q-1}{q}\right] : W_δ \succeq_n V \right\}$$

(39)

and for every $0 \leq δ < δ^* (V)$, $W_δ \succeq_n V$ (see the Remark in Appendix E). Alternatively, we can define a non-negative (less noisy) domination factor function, $μ_V : (0, \frac{q-1}{q}) \rightarrow \mathbb{R}^+$, for any channel $V \in \mathbb{R}^{q \times r}$:

$$μ_V (δ) \triangleq \sup_{P_X, Q_X \in P_q} \frac{D (P_X V || Q_X V)}{D (P_X W_δ || Q_X W_δ)}$$

(40)

where we assume that the codomain of the function does not include $\infty$. This function is analogous to the contraction coefficient for KL divergence since $μ_V (0) \triangleq \eta_\mathcal{C} (V)$ (although we exclude zero from the domain because the domination factor is only interesting for $δ \in (0, \frac{q-1}{q})$, and this exclusion also affords us some analytical simplicity). Indeed, we may perceive $P_X W_δ$ and $Q_X W_δ$ in the denominator of (40) as pmfs inside the “shrunk” simplex $\text{conv} \{ w_k P_k : k \in [q] \}$, and (40) represents a contraction coefficient of $V$ where the supremum is taken over this “shrunk” simplex.\footnote{Pictorially, the “shrunk” simplex is the magenta triangle in Figure 1 while the simplex itself is the larger gray triangle.} One can verify that $μ_V : (0, \frac{q-1}{q}) \rightarrow \mathbb{R}^+$ is a continuous, convex, and strictly increasing function that has a vertical asymptote at $δ = \frac{q-1}{q}$ (see Proposition 17 in Appendix E). Since for every $P_X, Q_X \in P_q$:

$$μ_V (δ) D (P_X W_δ || Q_X W_δ) \geq D (P_X V || Q_X V)$$

we have $μ_V (δ) \leq 1$ if and only if $W_δ \succeq_n V$. Hence, using the strictly increasing property of $μ_V : (0, \frac{q-1}{q}) \rightarrow \mathbb{R}^+$, we can also write $δ^* (V)$ as:

$$δ^* (V) = μ_V^{-1} (1)$$

(42)

where $μ_V^{-1}$ denotes the inverse function of $μ_V$ (which exists as $μ_V$ is injective), and we assume that unity is in the range of $μ_V$.

We briefly delineate how one might computationally approximate $δ^* (V)$ for a given general channel $V \in \mathbb{R}^{q \times r}$. From Theorem 1, $W_δ \in \mathbb{R}^{q \times q}$ is less noisy than $V$ if and only if:

$$W_δ \text{diag} (P_X W_δ)^{-1} W_δ^T - V \text{diag} (P_X V)^{-1} V^T \succeq_{\text{PSD}} 0$$

(43)

for every $P_X \in P_q^o$. For any fixed $P_X \in P_q^o$, verifying (43) corresponds to verifying a collection of rational (ratio of polynomials) inequalities in $δ$, because the positive semidefiniteness of the matrix in (43) is equivalent to the non-negativity of all its principal minors by Sylvester’s criterion\footnote{Sylvester’s criterion states that a matrix is positive semidefinite if and only if all its principal minors are non-negative.} [17, Theorem 7.2.5]. Therefore, one can computationally estimate the maximum value of $δ \in [0, \frac{q-1}{q}]$ that satisfies these rational inequalities for any fixed $P_X \in P_q^o$, and then repeat this process for various other choices of $P_X \in P_q^o$. The minimum among these maximum $δ$ values (one for each choice of $P_X \in P_q^o$) is an estimate for $δ^* (V)$. Unfortunately, this procedure appears to be rather cumbersome.

Since computing $δ^* (V)$ is also analytically intractable (as one would presume), we now prove Theorem 2′, which is a restatement of Theorem 2 in Section II. Theorem 2′ provides a sufficient condition for $W_δ \succeq_{\text{deg}} V$ (which implies $W_δ \succeq_n V$ using Proposition 2) by restricting its attention to the case where $V \in \mathbb{R}^{q \times q}_{\text{sto}}$ with $q \geq 2$. Moreover, it can be construed as a lower bound on $δ^* (V)$:

$$δ^* (V) \geq \frac{ν}{1 - (q-1)ν + \frac{ν}{q-1}}$$

(44)

where $ν = \min \{(V)_{i,j} : 1 \leq i,j \leq q\}$ is the minimum conditional probability in $V$.

**Theorem 2′** (Sufficient Condition for Degradation by Symmetric Channels). Given a channel $V \in \mathbb{R}^{q \times q}_{\text{sto}}$ with $q \geq 2$ and minimum probability $ν = \min \{(V)_{i,j} : 1 \leq i,j \leq q\}$, we have:

$$0 \leq δ \leq \frac{ν}{1 - (q-1)ν + \frac{ν}{q-1}} \Rightarrow W_δ \succeq_{\text{deg}} V$$

where $W_δ \in \mathbb{R}^{q \times q}_{\text{sto}}$ is a symmetric channel.
Suppose certain have input-output degradation: \( \nu \) when the colons separate the rows of the matrix. If \( W \), i.e. \( W \), we can write this as:

\[
\forall i \in \{1, \ldots, q\}, \quad v_i = \sum_{j=1}^{q} p_{i,j} w_{(q-1)\nu} P_q^{j-1}
\]

where \( P_q \) is the generator cyclic permutation matrix defined in (10), and \( \{p_{i,j} \geq 0 : 1 \leq i, j \leq q\} \) are the convex weights such that:

\[
\forall i \in \{1, \ldots, q\}, \quad \sum_{j=1}^{q} p_{i,j} = 1.
\]

Defining \( P \in \mathbb{R}^{q \times q} \) entry-wise as \([P]_{i,j} = p_{i,j} \) for every \( 1 \leq i, j \leq q \), we can also write this equation as \( V = PW_{(q-1)\nu} \).

\[ P = \sum_{1 \leq j_1, \ldots, j_q \leq q} \left( \prod_{i=1}^{q} p_{i,j_i} \right) E_{j_1,\ldots,j_q} \]

where \( \{\prod_{i=1}^{q} p_{i,j_i} : 1 \leq j_1, \ldots, j_q \leq q\} \) form a product pmf of convex weights, and for every \( 1 \leq j_1, \ldots, j_q \leq q \):

\[ E_{j_1,\ldots,j_q} \triangleq [e_{j_1} \ e_{j_2} \ \cdots \ e_{j_q}]^T \]

where \( e_i \in \mathbb{R}^q \) is the \( i \)th standard basis (column) vector that has unity at the \( i \)th entry and zero elsewhere. Hence, we get:

\[
V = \sum_{1 \leq j_1, \ldots, j_q \leq q} \left( \prod_{i=1}^{q} p_{i,j_i} \right) E_{j_1,\ldots,j_q} W_{(q-1)\nu}.
\]

Suppose \( \exists \delta \in \left[0, \frac{q-1}{q}\right] \) such that \( \forall j_1, \ldots, j_q \in \{1, \ldots, q\} \):

\[
\exists M_{j_1,\ldots,j_q} \in \mathbb{R}^{q \times q}_{\text{sDo}}, \quad E_{j_1,\ldots,j_q} W_{(q-1)\nu} = W_\delta M_{j_1,\ldots,j_q}
\]

i.e. \( W_\delta \preceq_{\text{sDo}} E_{j_1,\ldots,j_q} W_{(q-1)\nu} \). Then, we would have:

\[
V = W_\delta \sum_{1 \leq j_1, \ldots, j_q \leq q} \left( \prod_{i=1}^{q} p_{i,j_i} \right) M_{j_1,\ldots,j_q}
\]

which implies that \( W_\delta \preceq_{\text{sDo}} V \).

We will prove that \( \forall j_1, \ldots, j_q \in \{1, \ldots, q\} \), \( \exists M_{j_1,\ldots,j_q} \in \mathbb{R}^{q \times q}_{\text{sDo}} \) such that \( E_{j_1,\ldots,j_q} W_{(q-1)\nu} = W_\delta M_{j_1,\ldots,j_q} \) when:

\[
0 \leq \delta \leq \frac{\nu}{1 - (q-1)\nu + \frac{\nu}{q-1}}.
\]

Since \( 0 \leq \nu \leq \frac{1}{q} \), the above inequality implies that \( 0 \leq \delta \leq \frac{q-1}{q} \), where \( \delta = \frac{q-1}{q} \) is possible if and only if \( \nu = \frac{1}{q} \).

When \( \nu = \frac{1}{q} \), \( V = W_{(q-1)/q} \) is the channel with all uniform conditional pmfs, and \( W_{(q-1)/q} \preceq_{\text{sDo}} V \) clearly holds.

\[7\text{Matrices of the form } V = PW_{(q-1)\nu} \text{ with } P \in \mathbb{R}^{q \times q} \text{ are not necessarily degraded versions of } W_{(q-1)\nu}; \text{ when } W_{(q-1)\nu} \preceq_{\text{sDo}} V \text{ (although we certainly have input-output degradation: } W_{(q-1)\nu} \preceq_{\text{sDo}} V). \text{ As a counterexample, consider } W_{1/2} \text{ for } q = 3, \text{ and } P = [1\ 0\ 0; 1\ 0\ 0; 0\ 1\ 0], \text{ where the colons separate the rows of the matrix. If } W_{1/2} \preceq_{\text{sDo}} PW_{1/2}, \text{ then } \exists \delta \in \mathbb{R}^{3 \times 3} \text{ such that } PW_{1/2} = W_{1/2} \delta. \text{ However, } A = W_{1/2}^{-1} PW_{1/2} = (1/4) [301; 301; -141] \text{ has a strictly negative entry, which leads to a contradiction.}
Hence, we assume that $0 \leq \nu < \frac{1}{q}$ so that $0 \leq \delta < \frac{q-1}{q}$. Note that for every $j_1, \ldots, j_q \in \{1, \ldots, q\}$, there exists $M_{j_1, \ldots, j_q} \in \mathbb{R}_{sto}^{q \times q}$ such that $E_{j_1, \ldots, j_q} W_{(q-1)\nu} = W_{\delta} M_{j_1, \ldots, j_q}$ if and only if for every $j_1, \ldots, j_q \in \{1, \ldots, q\}$:

$$M_{j_1, \ldots, j_q} = W_{\delta}^{-1} E_{j_1, \ldots, j_q} W_{(q-1)\nu} = W_{\tau}^{-1} F_{j_1, \ldots, j_q} W_{(q-1)\nu}$$

is a valid stochastic matrix, where $\tau = \frac{1 - \delta}{1 - \delta - \frac{\delta}{q-1}}$ using part 4 of Proposition 3. Clearly, all the rows of each $M_{j_1, \ldots, j_q}$ sum to unity. So, it remains to verify that each $M_{j_1, \ldots, j_q}$ has non-negative entries. For any $j_1, \ldots, j_q \in \{1, \ldots, q\}$ and any $i, j \in \{1, \ldots, q\}$:

$$[M_{j_1, \ldots, j_q}]_{i,j} \geq \nu (1 - \tau) + \tau (1 - (q - 1) \nu)$$

where the right hand side is the minimum possible entry of any $M_{j_1, \ldots, j_q}$ (with equality when $j_1 > 1$ and $j_2 = j_3 = \cdots = j_q = 1$ for example) as $\tau < 0$ and $1 - (q - 1) \nu > 0$. To ensure each $M_{j_1, \ldots, j_q}$ is entry-wise non-negative, the minimum possible entry must satisfy:

$$\nu (1 - \tau) + \tau (1 - (q - 1) \nu) \geq 0$$

This final inequality completes the proof.

We remark that if $V = E_{2,1,\ldots,1} W_{(q-1)\nu} \in \mathbb{R}_{sto}^{q \times q}$, then this proof illustrates that $W_{\delta} \succeq_{\text{stoch}} V$ if and only if $0 \leq \delta \leq \nu / (1 - (q - 1) \nu + \frac{\nu}{q-1})$. Hence, the condition in Theorem 2’ is tight when no further information about $V$ is known. It is worth juxtaposing Theorem 2’ and Proposition 12. The upper bounds on $\delta$ from these results satisfy:

$$\frac{\nu}{1 - (q - 1) \nu + \frac{\nu}{q-1}} \leq \frac{(q - 1) \nu}{(q - 1) \nu + \frac{\nu}{q-1}}$$

where we have equality if and only if $\nu = \frac{1}{q}$, and it is straightforward to verify that (45) is equivalent to $\nu \leq \frac{1}{q}$. Moreover, assuming that $q$ is large and $\nu = o(1/q)$, the upper bound in Theorem 2’ is $\nu / (1 + o(1) + o(1/q^2)) = \Theta (\nu)$, while the upper bound in Proposition 12 is $\Theta (q \nu)$.8 (Note that both bounds are $\Theta (1)$ if $\nu = \frac{1}{q}$.) Therefore, when $V \in \mathbb{R}_{\text{stoch}}^{q \times q}$ is an additive noise channel, $\delta = O(q \nu)$ is enough for $W_{\delta} \succeq_{\text{stoch}} V$, but a general channel $V \in \mathbb{R}_{\text{stoch}}^{q \times q}$ requires $\delta = O(\nu)$ for such degradation. So, in order to account for $q$ different conditional pmfs in the general case (as opposed to a single conditional pmf which characterizes the channel in the additive noise case), we loosen a factor of $q$ in the upper bound on $\delta$. Furthermore, we can check using simulations that $W_{\delta} \in \mathbb{R}_{\text{stoch}}^{q \times q}$ is not in general less noisy than $V \in \mathbb{R}_{\text{stoch}}^{q \times q}$ for $\delta = (q - 1) \nu$. Indeed, counterexamples can be easily obtained by letting $V = E_{j_1, \ldots, j_q} W_{\delta}$ for specific values of $1 \leq j_1, \ldots, j_q \leq q$, and computationally verifying that $W_{\delta} \not\succeq_{\text{stoch}} V + J \in \mathbb{R}_{\text{stoch}}^{q \times q}$ for appropriate choices of perturbation matrices $J \in \mathbb{R}_{\text{stoch}}^{q \times q}$ with sufficiently small Frobenius norm.

We have now proved Theorems 1, 2, and 3 from Section II. The next section relates our results regarding less noisy and degradation preorders to logarithmic Sobolev inequalities. It can be considered as partly the motivation for this work. We will prove Theorem 4 in the next section, which will complete our exposition of the main results in Section II.

8We use the Bachmann-Landau asymptotic notation here. The little-o notation is defined as: $f(q) = o(g(q)) \iff \lim_{q \to \infty} f(q)/g(q) = 0$. The big-O notation is defined as: $f(q) = O(g(q)) \iff \limsup_{q \to \infty} |f(q)/g(q)| < +\infty$. Finally, the big-Θ notation is defined as: $f(q) = \Theta (g(q)) \iff 0 < \liminf_{q \to \infty} |f(q)/g(q)| \leq \limsup_{q \to \infty} |f(q)/g(q)| < +\infty$. 
VII. LESS NOISY DOMINATION AND LOGARITHMIC SOBOLEV INEQUALITIES

Logarithmic Sobolev inequalities are a class of functional inequalities associated with Markov processes that shed light on several important phenomena such as concentration of measure, hypercontractivity, and ergodicity of Markov semigroups. We refer readers to [33] for a general treatment of such inequalities for Markov semigroups, and more pertinently to [20] and [21], which present logarithmic Sobolev inequalities in the context of finite state-space Markov chains. In this section, we illustrate that proving a channel $W \in \mathbb{R}^{q \times q}_{\text{sto}}$ is less noisy than a channel $V \in \mathbb{R}^{q \times q}_{\text{sto}}$ allows us to translate the logarithmic Sobolev inequality for $W$ to the logarithmic Sobolev inequality for $V$. Thus, important information about $V$ can be deduced (from its logarithmic Sobolev inequality) by proving \(W \succeq_n V\) for an appropriate channel $W$ (perhaps a symmetric channel) that has known logarithmic Sobolev inequality.

We commence by introducing some appropriate notation and terminology associated with logarithmic Sobolev inequalities. For fixed input and output alphabet $\mathcal{X} = \mathcal{Y} = [q]$ with $q \in \mathbb{N}$, we think of a channel $W \in \mathbb{R}^{q \times q}_{\text{sto}}$ as a Markov kernel on $\mathcal{X}$. We assume that the “time homogeneous” discrete-time Markov chain defined by $W$ is irreducible, and has unique stationary distribution (or invariant measure) $\pi \in \mathcal{P}_q$ such that $\pi W = \pi$. Furthermore, we define the Hilbert space $L^2(\mathcal{X}, \pi)$ of all real functions with domain $\mathcal{X}$ endowed with the inner product:

$$\forall f, g \in L^2(\mathcal{X}, \pi), \quad \langle f, g \rangle_\pi = \sum_{x \in \mathcal{X}} \pi(x) f(x) g(x)$$

and induced norm $\| \cdot \|_\pi$. We construe $W : L^2(\mathcal{X}, \pi) \to L^2(\mathcal{X}, \pi)$ as a conditional expectation operator that takes a function $f \in L^2(\mathcal{X}, \pi)$, which is a column vector $f = [f(0) \cdots f(q-1)]^T \in \mathbb{R}^q$, to another function $Wf \in L^2(\mathcal{X}, \pi)$, which is also a column vector $Wf \in \mathbb{R}^q$. Corresponding to the discrete-time Markov chain $W$, we may also define a continuous-time Markov semigroup:

$$\forall t \geq 0, \quad H_t \triangleq \exp(-t(I_q - W)) \in \mathbb{R}^{q \times q}_{\text{sto}}$$

where the “discrete-time derivative” $W-I_q$ is the Laplacian operator that forms the generator of the Markov semigroup. The unique stationary distribution of this Markov semigroup is also $\pi$, and we may interpret $H_t : L^2(\mathcal{X}, \pi) \to L^2(\mathcal{X}, \pi)$ as a conditional expectation operator for each $t \geq 0$ as well.

In order to present logarithmic Sobolev inequalities, we define the Dirichlet form $\mathcal{E}_W : L^2(\mathcal{X}, \pi) \times L^2(\mathcal{X}, \pi) \to \mathbb{R}$:

$$\forall f, g \in L^2(\mathcal{X}, \pi), \quad \mathcal{E}_W(f, g) \triangleq \langle (I_q - W) f, g \rangle_\pi$$

which is used to study properties of the discrete-time Markov chain $W$ and its associated continuous-time Markov semigroup $\{H_t \in \mathbb{R}^{q \times q}_{\text{sto}} : t \geq 0\}$. $\mathcal{E}_W$ is technically only a Dirichlet form when $W$ is a reversible Markov chain (which means $W$ is a self-adjoint operator on $L^2(\mathcal{X}, \pi)$, or equivalently, $W$ and $\pi$ satisfy the detailed balance condition) as observed in [20, Section 2.3, page 705]. Moreover, the quadratic form defined by this Dirichlet form represents the energy of its input function, and satisfies:

$$\forall f \in L^2(\mathcal{X}, \pi), \quad \mathcal{E}_W(f, f) = \langle \left( I_q - \frac{W + W^*}{2} \right) f, f \rangle_{\pi}$$

where $W^* : L^2(\mathcal{X}, \pi) \to L^2(\mathcal{X}, \pi)$ is the adjoint operator of $W$. Finally, we introduce a particularly important Dirichlet form corresponding to the channel $W_{(q-1)/q} = I_1^T/q$, which has all uniform conditional pmfs and uniform stationary distribution $\pi = u$, known as the standard Dirichlet form:

$$\mathcal{E}_{\text{std}}(f, g) \triangleq \mathcal{E}_{1_{1/q}^T}(f, g) = \text{COV}_u(f, g)$$

$$= \sum_{x \in \mathcal{X}} \frac{f(x) g(x)}{q} - \left( \sum_{x \in \mathcal{X}} \frac{f(x)}{q} \right) \left( \sum_{x \in \mathcal{X}} \frac{g(x)}{q} \right)$$

for any $f, g \in L^2(\mathcal{X}, u)$. The quadratic form defined by the standard Dirichlet form is presented in (17) in Subsection II-D.

We now present the logarithmic Sobolev inequalities associated with the Markov chain $W$ and the Markov semigroup $\{H_t \in \mathbb{R}^{q \times q}_{\text{sto}} : t \geq 0\}$ it defines. The logarithmic Sobolev inequality for the Markov semigroup with constant $\alpha \in \mathbb{R}$ states that for every $f \in L^2(\mathcal{X}, \pi)$ such that $\|f\|_{\pi} = 1$, we have:

$$D (f^2 \pi \parallel \pi) = \sum_{x \in \mathcal{X}} \pi(x) f^2(x) \log(f^2(x)) \leq \frac{1}{\alpha} \mathcal{E}_W(f, f)$$

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where the logarithms are natural, we construe $\mu = f^2 \pi \in P_q$ as a pmf such that $\forall x \in X, \mu(x) = f^2(x)\pi(x)$, and $f^2$ behaves like the Radon-Nikodym derivative (or density) of $\mu$ with respect to $\pi$. The largest constant $\alpha$ such that (51) holds:

$$\alpha(W) \triangleq \inf_{f \in L^2(X,\pi)} \frac{\mathcal{E}_W(f,f)}{D(f^2\pi||\pi)}$$

is called the *logarithmic Sobolev constant* of the Markov chain $W$ (or the Markov chain $(W + W^*)/2$). Likewise, the *logarithmic Sobolev inequality* for the discrete-time Markov chain with constant $\alpha \in \mathbb{R}$ states that for every $f \in L^2(X,\pi)$ such that $\|f\|_2 = 1$, we have:

$$D(f^2\pi||\pi) \leq 1 \alpha \mathcal{E}_{WW^*}(f,f)$$

where $\mathcal{E}_{WW^*} : L^2(X,\pi) \times L^2(X,\pi) \to \mathbb{R}$ is the "discrete" Dirichlet form. The largest constant $\alpha$ such that (53) holds is the logarithmic Sobolev constant of the Markov chain $WW^*$, $\alpha(WW^*)$, and we refer to it as the *discrete logarithmic Sobolev constant* of the Markov chain $W$. As we mentioned earlier, there are many useful consequences of such logarithmic Sobolev inequalities. For example, if (51) holds with constant (52), then for every pmf $\mu \in P_q$:

$$\forall t \geq 0, \quad D(\mu^t H || \pi) \leq e^{-2\alpha(W)t} D(\mu || \pi)$$

where the exponent $-2\alpha(W)t$ can be improved to $-4\alpha(W)t$ when $W$ is reversible, as shown in [20, Theorem 3.6]. This is a measure of ergodicity of the Markov semigroup $\{H_t \in \mathbb{R}^{q \times q} : t \geq 0\}$. Likewise, if (53) holds with constant $\alpha(WW^*)$, then for every pmf $\mu \in P_q$:

$$\forall n \in \mathbb{N}, \quad D(\mu^n || \pi) \leq (1 - \alpha(WW^*))^n D(\mu || \pi)$$

as mentioned in [20, Remark, page 725] and proved in [34]. This is also a measure of ergodicity of the Markov chain $W$.

Although logarithmic Sobolev inequalities have many useful consequences, logarithmic Sobolev constants are difficult to compute analytically. Fortunately, logarithmic Sobolev constants corresponding to the standard Dirichlet form (or the channel $11^T/q$) have been computed in [20, Appendix, Theorem A.1]. Recall from (18) that:

$$\forall f \in L^2(X,\mu), \quad \mathcal{E}_{W^*}(f,f) = \frac{q\delta}{q-1} \mathcal{E}_{\text{std}}(f,f)$$

where we use $I_q - W^*_\delta = \frac{q\delta}{q-1} (I_q - \frac{1}{q} 11^T)$. Therefore, we can compute logarithmic Sobolev constants for symmetric channels as well. The next proposition collects the logarithmic Sobolev constants for symmetric channels (which are irreducible for $\delta \in (0,1]$) as well as some other related quantities.

**Proposition 14** (Constants of Symmetric Channels). The $q$-ary symmetric channel $W_\delta \in \mathbb{R}^{q \times q}_{\text{std}}$ with $q \geq 2$ has:

1. *logarithmic Sobolev constant*:

$$\alpha(W_\delta) = \begin{cases} \frac{(q-2)\delta}{(q-1)\log(q-1)}, & q > 2 \\ \delta, & q = 2 \end{cases}$$

for $\delta \in (0,1]$.

2. *discrete logarithmic Sobolev constant*:

$$\alpha(W_\delta W^*_\delta) = \alpha(W_{\delta'}) = \begin{cases} \frac{(q-2)(2q-2-q\delta)\delta}{(q-1)^2 \log(q-1)}, & q > 2 \\ 2\delta(1-\delta), & q = 2 \end{cases}$$

for $\delta \in (0,1]$, where $\delta' = 2 - \frac{q\delta}{q-1}$.

3. *(Hirschfeld-Gebelein-Rényi) maximal correlation* corresponding to the uniform stationary distribution $\mu \in P_q$:

$$\rho_{\text{max}}(u, W_\delta) = \left| 1 - \delta - \frac{\delta}{q-1} \right|$$

for $\delta \in [0,1]$, where for any channel $W \in \mathbb{R}^{2q \times q}$ and any source pmf $P_X \in P_q$, we define the maximal correlation between the input random variable $X \in [q]$ and the output random variable $Y \in [r]$ (with joint pmf $P_{X,Y}(x,y) = P_X(x)W_{Y|x}(y|x)$) as:

$$\rho_{\text{max}}(P_X, W) \triangleq \sup_{f: [q] \to [r], g: [r] \to \mathbb{R}} \frac{\mathbb{E}[f(X)g(Y)]}{\mathbb{E}[f(X)]\mathbb{E}[g(Y)]} = 0 \quad \mathbb{E}[f^2(X)]\mathbb{E}[g^2(Y)] = 1$$

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4) contraction coefficient for KL divergence bounded by:

\[
\left(1 - \frac{\delta}{q-1}\right)^2 \leq \eta_{\text{ac}}(W) \leq \left|1 - \frac{\delta}{q-1}\right|
\]

for \(\delta \in [0, 1]\).

**Proof.** See Appendix F. \(\blacksquare\)

In view of Proposition 14 and the intractability of computing logarithmic Sobolev constants for general Markov chains, we often “compare” a given irreducible channel \(V \in \mathbb{R}_{\text{sto}}^{q \times q}\) with a symmetric channel \(W_\delta \in \mathbb{R}_{\text{sto}}^{q \times q}\) to try and establish a logarithmic Sobolev inequality for it. We assume for the sake of simplicity that \(V\) is doubly stochastic and has the uniform pmf as its stationary distribution (just like the symmetric channel). Usually, such a comparison between \(W_\delta\) and \(V\) requires us to prove domination of Dirichlet forms, such as:

\[
\forall f \in \mathcal{L}^2(\mathcal{X}, \mu), \quad \mathcal{E}_V(f, f) \geq \mathcal{E}_{W_\delta}(f, f) = \frac{q\delta}{q-1} \mathcal{E}_{\text{std}}(f, f).
\] (57)

Such domination results immediately produce logarithmic Sobolev inequalities (51) and (53) for \(V\). They also lower bound its logarithmic Sobolev constants; for example:

\[\alpha(V) \geq \alpha(W_\delta).\] (58)

This turn begets other results such as (54) and (55) for the channel \(V\) (albeit with worse constants in the exponents, i.e. the logarithmic Sobolev constants of \(W_\delta\) are used instead of those for \(V\)). More general versions of Dirichlet form domination between Markov chains on different state spaces with different stationary distributions, and the resulting bounds on their logarithmic Sobolev constants are presented in [20, Lemmata 3.3 and 3.4]. We next illustrate that the information theoretic notion of less noisy domination is a sufficient condition for various kinds of Dirichlet form domination.

**Theorem 4'** (Domination of Dirichlet Forms). Let \(W, V \in \mathbb{R}_{\text{sto}}^{q \times q}\) be doubly stochastic channels, and \(\pi = \mu\) be the uniform stationary distribution. Then, the following are true:

1) If \(W \succeq_{\mu} V\), then:

\[
\forall f \in \mathcal{L}^2(\mathcal{X}, \mu), \quad \mathcal{E}_{V}\cdot(f, f) \geq \mathcal{E}_{W}\cdot(f, f).
\]

2) If \(W \in \mathbb{R}_{\geq 0}^{q \times q}\) is positive semidefinite, \(V\) is normal (i.e. \(V^TV = VV^T\)), and \(W \succeq_{\mu} V\), then:

\[
\forall f \in \mathcal{L}^2(\mathcal{X}, \mu), \quad \mathcal{E}_{V}(f, f) \geq \mathcal{E}_{W}(f, f).
\]

3) If \(W = W_\delta \in \mathbb{R}_{\text{sto}}^{q \times q}\) is any \(q\)-ary symmetric channel with \(\delta \in [0, \frac{q-1}{q}]\) and \(W_\delta \succeq_{\mu} V\), then:

\[
\forall f \in \mathcal{L}^2(\mathcal{X}, \mu), \quad \mathcal{E}_{V}(f, f) \geq \frac{q\delta}{q-1} \mathcal{E}_{\text{std}}(f, f).
\]

**Proof.**

**Part 1:** First observe that:

\[
\forall f \in \mathcal{L}^2(\mathcal{X}, \mu), \quad \mathcal{E}_{W}\cdot(f, f) = \frac{1}{q} f^T (I_q - WW^T) f
\]

\[
\forall f \in \mathcal{L}^2(\mathcal{X}, \mu), \quad \mathcal{E}_{V}\cdot(f, f) = \frac{1}{q} f^T (I_q - VV^T) f
\]

where we use the facts that \(WW^T = W^a\) and \(VV^T = V^a\) because the stationary distribution is uniform. This implies that \(\mathcal{E}_{V}\cdot(f, f) \geq \mathcal{E}_{W}\cdot(f, f)\) for every \(f \in \mathcal{L}^2(\mathcal{X}, \mu)\) if and only if \(I_q - VV^T \succeq_{\text{psd}} I_q - WW^T\), which is true if and only if \(WW^T \succeq_{\text{psd}} VV^T\). Since \(W \succeq_{\mu} V\), we get \(WW^T \succeq_{\text{psd}} VV^T\) from part 3 of Theorem 1 after letting \(P_X = \mu = P_XW = P_XV\).

**Part 2:** Once again, we first observe using (49) that:

\[
\forall f \in \mathcal{L}^2(\mathcal{X}, \mu), \quad \mathcal{E}_{W}(f, f) = \frac{1}{q} f^T \left(I_q - \frac{W + WW^T}{2}\right) f,
\]

\[
\forall f \in \mathcal{L}^2(\mathcal{X}, \mu), \quad \mathcal{E}_{V}(f, f) = \frac{1}{q} f^T \left(I_q - \frac{V + VV^T}{2}\right) f.
\]
So, \( E_V(f, f) \geq E_W(f, f) \) for every \( f \in L^2(\mathcal{X}, \mathbf{u}) \) if and only if \( I_q - (V + V^T)/2 \geq \text{psd} I_q - (W + W^T)/2 \), which is true if and only if \((W + W^T)/2 \geq \text{psd} (V + V^T)/2 \). From the proof of part 1, we know that \( WW^T \geq \text{psd} VV^T \). Thus, it is sufficient to prove that:

\[
WW^T \geq \text{psd} VV^T \Leftrightarrow \frac{W + W^T}{2} \geq \text{psd} \frac{V + V^T}{2}
\]

where \( WW^T = W^2 \) and \((W + W^T)/2 = W\) as \( W \) is symmetric. Since \( W \in \mathbb{R}^q_{\geq 0} \) and \( V \) is a normal matrix, Lemma 3 in Appendix G proves the claim in (59).

**Part 3:** We note that when \( V \) is a normal matrix, this result follows from part 2 because \( W_\Delta \in \mathbb{R}^q_{\geq 0} \) for \( \delta \in [0, \frac{q-1}{q}] \), as can be seen from part 2 of Proposition 3. For a general doubly stochastic channel \( V \), we need to prove that \( E_V(f, f) \geq E_{W_\Delta}(f, f) = \frac{q^2}{T} E_{\text{std}}(f, f) \) for every \( f \in L^2(\mathcal{X}, \mathbf{u}) \) (where we use (56)). Following the proof of part 2, it is sufficient to prove (59) with \( W = W_\Delta \):\(^9\)

\[
W_\Delta^2 \geq \text{psd} VV^T \Rightarrow W_\Delta \geq \text{psd} \frac{V + V^T}{2}
\]

where \( W_\Delta^2 = W_\Delta W_\Delta^T \) and \( W_\Delta = (W_\Delta + W_\Delta^T)/2 \). Recall the Löwner-Heinz theorem (originally proved in [35] and [36]) which states that for \( A, B \in \mathbb{R}^{T \times q} \) and \( 0 \leq p \leq 1 \):

\[
A \geq \text{psd} B \Rightarrow A^p \geq \text{psd} B^p
\]

or equivalently, \( f : [0, \infty) \to \mathbb{R}, f(x) = x^p \) is an operator monotone function for \( p \in [0, 1] \); also see [37, Section 6.6, Problem 17], or [38] for a short operator theoretic proof. Using (60) with \( p = \frac{1}{2} \) (cf. [17, Corollary 7.7.4 (b)]), we have:

\[
W_\Delta^2 \geq \text{psd} VV^T \Rightarrow W_\Delta \geq \text{psd} (VV^T)^{\frac{1}{2}}
\]

because the Gramian matrix \( VV^T \in \mathbb{R}^{q \times q} \). Let \( VV^T = Q\Lambda Q^T \) and \((V + V^T)/2 = U\Sigma U^T \) be the spectral decompositions of \( VV^T \) and \((V + V^T)/2 \), where \( Q \) and \( U \) are orthogonal matrices with eigenvectors as columns, and \( \Lambda \) and \( \Sigma \) are diagonal matrices with eigenvalues on the principal diagonal. Since \( VV^T \) and \((V + V^T)/2 \) are both doubly stochastic, they both have the eigenvector \( 1/\sqrt{q} \) corresponding to the maximum eigenvalue of unity. In fact, we have:

\[
(VV^T)^{\frac{1}{2}} \frac{1}{\sqrt{q}} = 1 \cdot \frac{1}{\sqrt{q}}
\]

\[
\left( \frac{V + V^T}{2} \right)^{\frac{1}{2}} \frac{1}{\sqrt{q}} = 1 \cdot \frac{1}{\sqrt{q}}
\]

where we use the fact that \( (VV^T)^{\frac{1}{2}} = Q\Lambda^{\frac{1}{2}}Q^T \) is the spectral decomposition of \((VV^T)^{\frac{1}{2}} \). For any matrix \( X \in \mathbb{R}^{q \times q}_{\text{sym}} \), let \( \lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_q(X) \) denote the eigenvalues of \( X \) in descending order. Without loss of generality, we assume that \([\Lambda]_{1,j} = \lambda_j(VV^T) \) and \([\Sigma]_{1,j} = \lambda_j((V + V^T)/2) \) for every \( 1 \leq j \leq q \). So, \( \lambda_1((VV^T)^{\frac{1}{2}}) = \lambda_1((V + V^T)/2) = 1 \), and the first columns of both \( Q \) and \( U \) are equal to \( 1/\sqrt{q} \). From part 2 of Proposition 3, we have:

\[
W_\Delta = F_qDF_q^H = QDQ^T = UD^T
\]

where \( F_q \) is the DFT matrix, and \( D \) is the diagonal matrix of eigenvalues such that \([D]_{1,1} = \lambda_1(W_\Delta) = 1 \) and \([D]_{j,1} = \lambda_j(W_\Delta) = 1 - \delta - \frac{\delta}{q}\) for \( 2 \leq j \leq q \). Note that we may use any of the eigenbases \( F_q, Q, \) or \( U, \) because they all have first column \( 1/\sqrt{q} \), which is the eigenvector of \( W_\Delta \) corresponding to \( \lambda_1(W_\Delta) = 1 \) since \( W_\Delta \) is doubly stochastic, and the remaining eigenvalue columns are permitted to be any orthonormal basis of \( \text{span}(1/\sqrt{q}) \) as \( \lambda_j(W_\Delta) = 1 - \delta - \frac{\delta}{q} \) for \( 2 \leq j \leq q \). Hence, we have:

\[
W_\Delta \geq \text{psd} (VV^T)^{\frac{1}{2}} \Leftrightarrow QDQ^T \geq \text{psd} Q\Lambda^{\frac{1}{2}}Q^T \Leftrightarrow D \geq \text{psd} \Lambda^{\frac{1}{2}},
\]

\[
W_\Delta \geq \text{psd} \frac{V + V^T}{2} \Leftrightarrow UD^T \geq \text{psd} U\Sigma U^T \Leftrightarrow D \geq \text{psd} \Sigma.
\]

\(^9\)Note that (59) trivially holds for \( W = W_\Delta \) with \( \delta = (q - 1)/q \), because \( W_{(q - 1)/q} = W_\Delta^2 \) implies that \( V = W_{(q - 1)/q} \).
In order to show that $D \succeq_{\text{PSD}} \Lambda^\frac{1}{2} \Rightarrow D \succeq_{\text{PSD}} \Sigma$, it suffices to prove that $\Lambda^\frac{1}{2} \succeq_{\text{PSD}} \Sigma$. Recall from [37, Corollary 3.1.5] that for any matrix $A \in \mathbb{R}^{q \times q}$, we have:\footnote{This states that for any matrix $A \in \mathbb{R}^{q \times q}$, the $i$th largest eigenvalue of the symmetric part of $A$ is less than or equal to the $i$th largest singular value of $A$ (which is the $i$th largest eigenvalue of the unique positive semidefinite part $(AA^T)^{1/2}$ in the polar decomposition of $A$) for every $1 \leq i \leq q$.}

$$\forall i \in \{1, \ldots, q\}, \quad \lambda_i \left( (A A^T)^{\frac{1}{2}} \right) \geq \lambda_i \left( \frac{A + A^T}{2} \right).$$

(61)

Hence, $\Lambda^\frac{1}{2} \succeq_{\text{PSD}} \Sigma$ is true, cf. [39, Lemma 2.5]. This completes the proof.

Theorem 4' includes Theorem 4 from Section II as part 3, and also provides two other useful Dirichlet form domination results. Part 1 of Theorem 4' states that less noisy domination implies discrete Dirichlet form domination. In particular, if we have $W_\delta \succeq_n V$ for some irreducible symmetric channel $W_\delta \in \mathbb{R}^{q \times q}$ and irreducible doubly stochastic channel $V \in \mathbb{R}^{q \times q}_{\text{std}}$, then part 1 implies that:

$$\forall \mu \in \mathcal{P}_q, \quad D(\mu V^n || u) \leq (1 - \alpha(W_\delta W_\delta^*))^n D(\mu || u)$$

(62)

for all pmfs $\mu \in \mathcal{P}_q$, where $\alpha(W_\delta W_\delta^*)$ is computed in part 2 of Proposition 14. However, it is worth mentioning that (55) for $W_\delta$ and Proposition 1 directly produce (62). So, such ergodicity results for the discrete-time Markov chain $V$ do not require the full power of the Dirichlet form domination in part 1. Regardless, Dirichlet form domination results, such as in parts 2 and 3, yield several functional inequalities (like Poincaré and logarithmic Sobolev inequalities) which have many other potent consequences as well.

Parts 2 and 3 of Theorem 4' convey that less noisy domination also implies vanilla Dirichlet form domination under regularity conditions. We note that in part 2, the channel $W$ is more general than that in part 3, but the channel $V$ is restricted to be normal (which includes the case where $V$ is an additive noise channel). The proofs of these parts essentially consist of two segments. The first segment uses part 1, and the second segment illustrates that domination of discrete Dirichlet forms implies domination of Dirichlet forms (as shown in (57)). This latter segment is encapsulated in Lemma 3 of Appendix G for part 2, and requires a slightly more sophisticated proof pertaining to symmetric channels in part 3.

VIII. Conclusion

In closing, we briefly reiterate our main results by delineating a possible program for proving logarithmic Sobolev inequalities for certain Markov chains. Given an arbitrary irreducible doubly stochastic channel $V \in \mathbb{R}^{q \times q}_{\text{std}}$ with minimum entry $\nu = \min\{V_{i,j} : 1 \leq i, j \leq q\} > 0$ and $q \geq 2$, we can first use Theorem 2 to generate a $q$-ary symmetric channel $W_\delta \in \mathbb{R}^{q \times q}_{\text{std}}$ with $\delta = \nu/((q - 1)\nu + \frac{\nu}{q - 1})$ such that $W_\delta \succeq_n V$. This also means that $W_\delta \succeq V$, using Proposition 2. Moreover, the $\delta$ parameter can be improved using Theorem 3 (or Propositions 12 and 13) if $V$ is an additive noise channel. We can then use Theorem 4' to deduce a domination of Dirichlet forms. Since $W_\delta$ satisfies the logarithmic Sobolev inequalities (51) and (53) with corresponding logarithmic Sobolev constants given in Proposition 14, Theorem 4' establishes the following logarithmic Sobolev inequalities for $V$:

$$D \left( f^2 || u \right) \leq \frac{1}{\alpha(W_\delta)} \mathcal{E}_V (f, f)$$

(63)

$$D \left( f^2 || u \right) \leq \frac{1}{\alpha(W_\delta W_\delta^*)} \mathcal{E}_{VV^*} (f, f)$$

(64)

for every $f \in L^2 (\lambda; \mu)$ such that $\|f\|_u = 1$. These inequalities can be used to derive a myriad of important facts about $V$. We note that the equivalent characterizations of the less noisy preorder in Theorem 1 are exceptionally useful for proving some of these results. Finally, we accentuate that Theorems 2 and 3 also provide analogs of the relationship between less noisy domination by erasure channels and contraction coefficients in the context of symmetric channels; this addresses our motivation in Subsection I-C.

APPENDIX A

Basics of majorization theory

Since we use some majorization arguments in our analysis, we briefly introduce the notion of group majorization over row vectors in $\mathbb{R}^q$ (with $q \in \mathbb{N}$) in this appendix. Given a group $G \subseteq \mathbb{R}^{q \times q}$ of matrices (with the operation of matrix multiplication), we may define a preorder called $G$-majorization over row vectors in $\mathbb{R}^q$. For two row vectors $x, y \in \mathbb{R}^q$, we say that $x$ $G$-majorizes $y$ if $y \in \text{conv} (\{ xG : G \in G \})$, where $\text{conv} (\cdot)$ is the convex hull and $\{ xG : G \in G \}$ is the...
orbit of $x$ under the group $G$. Group majorization intuitively captures a notion of “spread” of vectors. So, $x$ $G$-majorizes $y$ when $x$ is more spread out than $y$ with respect to $G$. We refer readers to [7, Chapter 14, Section C] and the references therein for a thorough treatment of group majorization. If we let $G$ be the symmetric group of all permutation matrices in $\mathbb{R}^{q \times q}$, then $G$-majorization corresponds to traditional majorization of vectors in $\mathbb{R}^q$ as introduced in [32]. The next proposition collects some results about traditional majorization.

**Proposition 15** (Majorization [7], [32]). Given two row vectors $x = (x_1, \ldots, x_q), y = (y_1, \ldots, y_q) \in \mathbb{R}^q$, let $x(1) \leq \cdots \leq x(q)$ and $y(1) \leq \cdots \leq y(q)$ denote the re-orderings of $x$ and $y$ in ascending order. Then, the following are equivalent:

1) $x$ majorizes $y$, or equivalently, $y$ resides in the convex hull of all permutations of $x$.
2) $y = xD$ for some doubly stochastic matrix $D \in \mathbb{R}^{q \times q}_\text{sto}$.
3) The entries of $x$ and $y$ satisfy:

$$\sum_{i=1}^{k} x(i) \leq \sum_{i=1}^{k} y(i), \text{ for } k = 1, \ldots, q-1,$$

and $$\sum_{i=1}^{q} x(i) = \sum_{i=1}^{q} y(i).$$

When these conditions are true, we will write $x \succeq^G y$.

In the context of Subsection I-D, given an Abelian group $(\mathcal{X}, \oplus)$ of order $q$, another useful notion of $G$-majorization can be obtained by letting $G = \{P_z \in \mathbb{R}^{q \times q} : x \in \mathcal{X}\}$ be the group of permutation matrices defined in (6) that is isomorphic to $(\mathcal{X}, \oplus)$. For such choice of $G$, we write $x \succeq_x y$ when $x$ $G$-majorizes (or $\mathcal{X}$-majorizes) $y$ for any two row vectors $x, y \in \mathbb{R}^q$. We will only require one fact about such group majorization, which we present in the next proposition.

**Proposition 16** (Group Majorization). Given two row vectors $x, y \in \mathbb{R}^q$, $x \succeq_x y$ if and only if there exists $\lambda \in \mathcal{P}_q$ such that $y = x \circ \mathcal{C}_x(\lambda)$.

**Proof.** Observe that:

$$x \succeq_x y \iff y \in \text{conv} (\{xP_z : z \in \mathcal{X}\})$$

$$\iff y = \lambda \circ \mathcal{C}_\mathcal{X}(x) \text{ for some } \lambda \in \mathcal{P}_q$$

$$\iff y = x \circ \mathcal{C}_\mathcal{X}(\lambda) \text{ for some } \lambda \in \mathcal{P}_q$$

where the first step follows by definition, the second step follows from (9), and the final step follows from the commutativity of $\mathcal{X}$-circular convolution. This completes the proof. \qed

Proposition 16 parallels the equivalence between parts 1 and 2 of Proposition 15, because $\circ \mathcal{C}_\mathcal{X}(\lambda)$ is a doubly stochastic matrix for every pmf $\lambda \in \mathcal{P}_q$. In closing this appendix, we mention a well-known special case of such group majorization. When $(\mathcal{X}, \oplus)$ is the cyclic Abelian group $\mathbb{Z}/q\mathbb{Z}$ of integers with addition modulo $q$, $G = \{I_q, P_q, P_q^2, \ldots, P_q^{q-1}\}$ is the group of all cyclic permutation matrices in $\mathbb{R}^{q \times q}$, where $P_q \in \mathbb{R}^{q \times q}$ is the generator cyclic permutation matrix defined in (10). The corresponding notion of $G$-majorization is known as cyclic majorization. We refer readers to [40] for basic results on cyclic majorization.

**APPENDIX B**

**PROOF OF PROPOSITION 3**

**Proof.**

**Part 1:** This is obvious from (12).

**Part 2:** It is well-known that the DFT matrix jointly diagonalizes all circulant matrices. In fact, this is of tremendous importance in discrete-time signal processing [41, Chapters 8-10]. Hence, the DFT matrix diagonalizes every $W_\delta$ for $\delta \in \mathbb{R}$ from part 1, and the corresponding eigenvalues are all real because $W_\delta$ is symmetric. To find the explicit form of the eigenvalues, we refer to [17, Problem 2.2.P10]. Observe that the circulant matrix corresponding to a row vector $x = (x_0, \ldots, x_{q-1}) \in \mathbb{R}^q$ satisfies:

$$\circ \mathcal{C}_{\mathbb{Z}/q\mathbb{Z}}(x) = \sum_{k=0}^{q-1} x_k P_q^k = F_q \left( \sum_{k=0}^{q-1} x_k D_q^k \right) F_q^H$$

$$= F_q \text{ diag}(\sqrt{q} x F_q) F_q^H$$
where the first equality uses (8) for the group $\mathbb{Z}/q\mathbb{Z}$ of integers with addition modulo $q$ [17, Section 0.9.6], $P_q \in \mathbb{R}^{q \times q}$ denotes the generator cyclic permutation matrix defined in (10) which satisfies $P_q = F_q D_q F_q^H$ with $D_q = \text{diag}((1, \omega, \omega^2, \ldots, \omega^{q-1}))$, and $\text{diag}(\cdot)$ constructs a diagonal matrix from a vector. Hence, we have:

$$\lambda_j(W_\delta) = \sum_{k=1}^{q} (w_\delta)_k \omega^{(j-1)(k-1)}$$

$$= 1 + \frac{\delta}{q - 1} \left( \sum_{k=1}^{q} \omega^{(j-1)(k-1)} - 1 \right)$$

$$= 1 + \frac{\delta (q \Delta_{j,k} - 1)}{q - 1}$$

$$= \begin{cases} 1 & , \ j = 1 \\ 1 - \delta - \frac{\delta}{q - 1} & , \ j = 2, \ldots, q \end{cases}$$

where $\Delta_{j,k}$ is the Kronecker delta function, which is unity if $j = k$ and zero otherwise.

**Part 3:** This is straightforward to see. Indeed, recall that a square stochastic matrix is doubly stochastic if and only if its stationary distribution is the uniform distribution [17, Section 8.7].

**Part 4:** For $\delta \neq \frac{q-1}{q}$, we can verify that $W_\tau W_\delta = I_q$ when $\tau = \frac{-\delta}{1 - \delta - \frac{\delta}{q - 1}}$ by direct computation:

$$[W_\tau W_\delta]_{j,j} = (1 - \tau) (1 - \delta) + (q - 1) \left( \frac{\tau}{q - 1} \right) \left( \frac{\delta}{q - 1} \right)$$

$$= (1 - \delta) \left( \frac{1}{q - 1} - \frac{\delta}{q - 1} \right) - \frac{\delta^2}{q - 1}$$

$$= \frac{1 - \delta^2 q - 1}{(q - 1)^2}$$

$$= 1$$, for $j = 1, \ldots, q$,

$$[W_\tau W_\delta]_{j,k} = \frac{\delta (1 - \tau)}{q - 1} + \frac{\tau (1 - \delta) + (q - 2)}{(q - 1)^2} \frac{\delta}{q - 1}$$

$$= \frac{\delta}{q - 1} - \frac{\delta^2}{(q - 1)^2} \frac{\delta}{q - 1} + \frac{\delta^2}{q - 1} + \frac{(q - 2) \delta^2}{(q - 1)^2}$$

$$= 0$$, for $j \neq k$ and $1 \leq j, k \leq q$.

The $\delta = \frac{q-1}{q}$ case follows from (12).

**Part 5:** The set $\{W_\delta : \delta \in \mathbb{R} \setminus \{\frac{q-1}{q}\}\}$ is closed over matrix multiplication. Indeed, for $\epsilon, \delta \in \mathbb{R} \setminus \{\frac{q-1}{q}\}$, we can verify with a straightforward calculation that $W_\epsilon W_\delta = W_\epsilon$ with $\tau = 1 - (1 - \epsilon) (1 - \delta) - \frac{\delta}{q - 1} = \epsilon + \delta - \epsilon \delta - \frac{\epsilon \delta}{q - 1}$. Moreover, $\tau \neq \frac{q-1}{q}$ because $W_\epsilon$ is invertible (since $W_\epsilon$ and $W_\delta$ are invertible using part 4). The set also includes the identity matrix as $W_0 = I_q$, and multiplicative inverses (using part 4). Finally, the associativity of matrix multiplication and the commutativity of circulant matrices (since they are jointly diagonalizable by the DFT matrix) proves that $\{W_\delta : \delta \in \mathbb{R} \setminus \{\frac{q-1}{q}\}\}$ is an Abelian group.

### APPENDIX C

**PROOF OF LEMMA 2**

**Proof.** We provide a proof based on that of [17, Theorem 7.7.3 (a)], which handles the $A$ positive definite case. In this proof, we let $X^{\frac{1}{2}} \in \mathbb{R}^{q \times q}_{\geq 0}$ denote the unique positive semidefinite square root of a positive semidefinite matrix $X \in \mathbb{R}^{q \times q}_{\geq 0}$, and $0 \in \mathbb{R}^q$ denote the zero (column) vector. We first observe that:

$$A \succeq_{\text{PSD}} B \Leftrightarrow \left( A^{\frac{1}{2}} \right)^\dagger A \left( A^{\frac{1}{2}} \right)^\dagger \succeq_{\text{PSD}} \left( A^{\frac{1}{2}} \right)^\dagger B \left( A^{\frac{1}{2}} \right)^\dagger$$

$$\Leftrightarrow P \succeq_{\text{PSD}} \left( A^{\frac{1}{2}} \right)^\dagger B \left( A^{\frac{1}{2}} \right)^\dagger$$

$$\Leftrightarrow P A P^T \succeq_{\text{PSD}} P B P^T$$

$$\Leftrightarrow A \succeq_{\text{PSD}} B$$

where $P \triangleq A^{\frac{1}{2}} (A^{\frac{1}{2}})^\dagger = AA^\dagger = P^T$ is the orthogonal projection matrix onto the range of $A^{\frac{1}{2}}$ or $A$ [42, Section 5.5.4], the second equivalence follows from the facts that $A = A^{\frac{1}{2}} A^{\frac{1}{2}}$ and $P^2 = P$ (idempotency), the third implication
easily follows from the (right hand side of the) first implication, and the final implication holds because $PAP^T = A$ and $PBPT = B$. Note that $PBPT = B$ because $\text{nullspace}(A) \subseteq \text{nullspace}(B)$ (indeed, $Ax = 0 \Rightarrow 0 = x^T Ax \geq x^T Bx \geq 0 \Rightarrow Bx = 0$ for every $x \in \text{nullspace}(A)$). Hence, we have:

$$A \succeq_{\text{psd}} B \iff P \succeq_{\text{psd}} (A^\frac{1}{2})^T B (A^\frac{1}{2})^T$$

$$\iff \lambda_1 \left( (A^\frac{1}{2})^T B (A^\frac{1}{2})^T \right) \leq \lambda_1 (P) = 1$$

$$\iff \lambda_1 (A^1 BPT) = \lambda_1 (A^1 B) \leq 1$$

$$\iff \rho (A^1 B) \leq 1$$

where $\lambda_1 (\cdot)$ denotes the largest eigenvalue and $\|\cdot\|_{\text{op}}$ denotes the operator norm or largest singular value of a matrix. The forward direction of the second equivalence follows from the Courant-Fischer min-max theorem in [17, Theorem 4.2.6], and the converse direction holds because $(A^\frac{1}{2})^T B (A^\frac{1}{2})^T \in \mathbb{R}^{q \times q}$ and $\lambda_1 ((A^\frac{1}{2})^T B (A^\frac{1}{2})^T) \leq 1 \Rightarrow I_q \succeq_{\text{psd}} (A^\frac{1}{2})^T B (A^\frac{1}{2})^T \Rightarrow PBP^T \succeq_{\text{psd}} P (A^\frac{1}{2})^T B (A^\frac{1}{2})^T P^T \Rightarrow P \succeq_{\text{psd}} (A^\frac{1}{2})^T B (A^\frac{1}{2})^T$. The third equivalence holds because $(A^\frac{1}{2})^T B (A^\frac{1}{2})^T$ and $A^1 BPT$ share the same eigenvalues. Indeed, both matrices have nullspaces containing nullspace($A$), and every other eigenvector $x \in \text{range}(A)$ of $(A^\frac{1}{2})^T B (A^\frac{1}{2})^T$ has a unique corresponding eigenvector $(A^\frac{1}{2})^T x$ of $A^1 BPT$ with the same eigenvalue. The final equivalence holds because all eigenvalues of $A^1 B$ are non-negative. This completes the proof.

**APPENDIX D

**ALTERNATIVE FOURIER ANALYTIC PROOF OF CIRCLE CONDITION IN PROPOSITION 11**

*Proof.* Since all $\mathcal{X}$-circulant matrices are jointly diagonalized by a unitary “Fourier” matrix $F \in \mathbb{C}^{q \times q}$, we have:

$$W = F \text{diag} (\lambda_w) F^H$$

$$V = F \text{diag} (\lambda_v) F^H$$

where $\lambda_w \in \mathbb{C}^q$ and $\lambda_v \in \mathbb{C}^q$ are the eigenvalues of $W$ and $V$, respectively. This gives us:

$$WW^T = F \text{diag} (|\lambda_w|^2) F^H$$

$$VV^T = F \text{diag} (|\lambda_v|^2) F^H$$

where $|\lambda_w|^2 \in \mathbb{R}^q$ and $|\lambda_v|^2 \in \mathbb{R}^q$ denote the vectors that are the entry-wise squared magnitudes of $\lambda_w$ and $\lambda_v$, respectively. Since $W \succeq_{\text{psd}} V$, letting $P_X = u$ (which means that $P_X W = P_X V = u$) in part 3 of Theorem 1 gives:

$$WW^T \succeq_{\text{psd}} VV^T$$

$$\text{diag} (|\lambda_w|^2) \succeq_{\text{psd}} \text{diag} (|\lambda_v|^2)$$

$$\|\lambda_w\|^2_{\ell_2} \geq \|\lambda_v\|^2_{\ell_2}$$

$$\|w\|^2_{\ell_2} \geq \|v\|^2_{\ell_2}$$

$$\|w - u\|^2_{\ell_2} \geq \|v - u\|^2_{\ell_2}$$

where the second statement is a non-singular $*$-congruence using $F$, the fourth inequality follows from an analog of the vanilla Parseval-Plancherel theorem: $q \|w\|^2_{\ell_2} = \|W\|^2_{\text{Fro}} = \|\text{diag}(\lambda_w)\|^2_{\text{Fro}} = \|\lambda_w\|^2_{\ell_2}$ and $q \|v\|^2_{\ell_2} = \|\lambda_v\|^2_{\ell_2}$ (due to (9) and the unitary invariance of the Frobenius norm $\|\cdot\|_{\text{Fro}}$), and the final inequality holds because $\|w\|^2_{\ell_2} = \|w - u\|^2_{\ell_2} + \|u\|^2_{\ell_2}$ and $\|v\|^2_{\ell_2} = \|v - u\|^2_{\ell_2} + \|u\|^2_{\ell_2}$ as $w - u$ and $v - u$ are orthogonal to $u$. This completes the proof.

**Remark.** The circle condition is not sufficient to deduce whether $W \succeq_{\text{psd}} V$. For example, if $(\mathcal{X}, \oplus)$ is the cyclic Abelian group $\mathbb{Z}/q\mathbb{Z}$ and $F = F_q$ is the DFT matrix (in the preceding proof), there exist pmfs $w, v \in \mathcal{P}_q$ such that $\|w\|^2_{\ell_2} \geq \|v\|^2_{\ell_2}$ but the corresponding Fourier coefficients (or eigenvalues of $W$ and $V$) do not satisfy $|\lambda_w| \geq |\lambda_v|$ (entry-wise).
APPENDIX E

PROPERTIES OF DOMINATION FACTOR FUNCTION

Proposition 17 (Properties of Domination Factor Function). Given a channel $V \in \mathbb{R}^{q \times r}$, its domination factor function $\mu_V : (0, \frac{q - 1}{q}) \rightarrow \mathbb{R}^+$ (assuming finite codomain) is continuous, convex, and strictly increasing. Moreover, we have:

$$\lim_{\delta \rightarrow \frac{q - 1}{q}} \mu_V (\delta) = +\infty$$

which means that the domination factor function has a vertical asymptote at $\delta = \frac{q - 1}{q}$.

Proof. We first prove that $\mu_V : (0, \frac{q - 1}{q}) \rightarrow \mathbb{R}^+$ is strictly increasing. Observe that $W_{\delta'} \succeq_{soq} W_{\delta}$ for $0 < \delta' < \delta < \frac{q - 1}{q}$, because $W_{\delta} = W_{\delta'} W_{p}$ with:

$$p = \delta - \frac{\delta'}{1 - \delta' - \frac{\delta'}{q - 1}} + \frac{\delta \delta'}{1 - \delta' - \frac{\delta'}{q - 1}} + \frac{\delta \delta'}{1 - \delta' - \frac{\delta'}{q - 1}}$$

$$= \frac{\delta (1 - \delta' - \frac{\delta'}{q - 1}) - \delta' + \delta \delta' + \delta \delta'}{1 - \delta' - \frac{\delta'}{q - 1}}$$

$$= \frac{\delta - \delta' - \frac{\delta'}{q - 1}}{1 - \delta' - \frac{\delta'}{q - 1}} \in \left(0, \frac{q - 1}{q}\right)$$

where we use part 4 of Proposition 3, the proof of part 5 of Proposition 3 in Appendix B, and the fact that $W_p = W_{\delta'}^{-1} W_{\delta}$. As a result, we have for every $P_X, Q_X \in \mathcal{P}_q$:

$$D (P_X W_{\delta} || Q_X W_{\delta}) \leq \eta_{\mathcal{A}} (W_p) D (P_X W_{\delta'} || Q_X W_{\delta'})$$

using the SDPI for KL divergence, where part 4 of Proposition 14 reveals that $\eta_{\mathcal{A}} (W_p) \in (0, 1)$ since $p \in (0, \frac{q - 1}{q})$. Hence, we have for $0 < \delta' < \delta < \frac{q - 1}{q}$:

$$\mu_V (\delta') \leq \eta_{\mathcal{A}} (W_p) \mu_V (\delta) \quad (65)$$

using (40), and the fact that $0 < D (P_X W_{\delta'} || Q_X W_{\delta'}) < +\infty$ if and only if $0 < D (P_X W_{\delta} || Q_X W_{\delta}) < +\infty$. This implies that $\mu_V : (0, \frac{q - 1}{q}) \rightarrow \mathbb{R}^+$ is strictly increasing.

Next, we show that $\mu_V : (0, \frac{q - 1}{q}) \rightarrow \mathbb{R}^+$ is convex. For any fixed $P_X, Q_X \in \mathcal{P}_q$ such that $0 < D (P_X W_{\delta} || Q_X W_{\delta}) < +\infty$, consider the function:

$$\delta \mapsto \frac{D (P_X V || Q_X V)}{D (P_X W_{\delta} || Q_X W_{\delta})}$$

with domain $(0, \frac{q - 1}{q})$. This function is convex in $\delta$, because $\delta \mapsto D (P_X W_{\delta} || Q_X W_{\delta})$ is convex due to the convexity of KL divergence, and the reciprocal of a non-negative convex function is convex. Therefore, $\mu_V : (0, \frac{q - 1}{q}) \rightarrow \mathbb{R}^+$ is convex since (40) defines it as a pointwise supremum of a collection of convex functions. Furthermore, we note that $\mu_V : (0, \frac{q - 1}{q}) \rightarrow \mathbb{R}^+$ is also continuous since convex functions are continuous on the interior of their domain.

Finally, observe that:

$$\lim_{\delta \rightarrow \frac{q - 1}{q}} \mu_V (\delta) = \liminf_{\delta \rightarrow \frac{q - 1}{q}} \sup_{P_X, Q_X \in \mathcal{P}_q} \frac{D (P_X V || Q_X V)}{D (P_X W_{\delta} || Q_X W_{\delta})}$$

$$\geq \sup_{P_X, Q_X \in \mathcal{P}_q} \liminf_{\delta \rightarrow \frac{q - 1}{q}} \frac{D (P_X V || Q_X V)}{D (P_X W_{\delta} || Q_X W_{\delta})}$$

$$= \sup_{P_X, Q_X \in \mathcal{P}_q} \limsup_{\delta \rightarrow \frac{q - 1}{q}} \frac{D (P_X V || Q_X V)}{D (P_X W_{\delta} || Q_X W_{\delta})}$$

$$= +\infty$$

where the first equality follows from (40) (note that $0 < D (P_X W_{\delta} || Q_X W_{\delta}) < +\infty$ for $P_X \neq Q_X$ and $\delta$ close to $\frac{q - 1}{q}$) and the limit exists because the inferior limit is infinity, the second equality is the minimax inequality, the third equality is a straightforward real analysis computation, and the final equality holds because $P_X W_{(q - 1)/q} = \mathbf{u}$ for every $P_X \in \mathcal{P}_q$. This establishes the vertical asymptote of the domination factor function, and completes the proof. ■
Remark. Proposition 2 states that \( W_{\delta'} \geq_{\pi} W_{\delta} \Rightarrow W_{\delta'}/W_{\delta} \). So, the less noisy domination regions of \( W_{\delta} \) and \( W_{\delta'} \) satisfy \( W_{\delta'} \in W_{\delta} \), for \( 0 \leq \delta' < \delta \leq q^{-1} \). Moreover, we note that the weaker fact, \( W_{\delta'} \geq_{\pi} W_{\delta} \), is enough to deduce that \( \mu_Y : (0, q^{-1}) \rightarrow \mathbb{R}^+ \) is non-decreasing (but not enough to discern that it is strictly increasing).

APPENDIX F

PROOF OF PROPOSITION 14

Proof.
Part 1: We first recall from [20, Appendix, Theorem A.1] that the Markov chain \( \frac{1}{q} \Pi_{\mathcal{Q}} \in \mathcal{P}_q \) with uniform stationary distribution \( \pi = u \in \mathcal{P}_q \) has logarithmic Sobolev constant:

\[
\alpha \left( \frac{1}{q} \Pi_{\mathcal{Q}} \right) = \inf_{f \in L^2(X, \mu)} \frac{\mathcal{E}_{\text{std}} (f, f)}{\| \mathcal{D}(f^2 u) \| u = 0 },
\]

where \( \| \cdot \| = \| \cdot \|_2 \) denotes the \( L^2 \)-norm, and the second equality is Dobrushin’s two-point characterization of \( \alpha \).

Now using (56), we have:

\[
\alpha (W_{\delta}) = \frac{q(1-\delta)}{q-1} \alpha \left( \frac{1}{q} \Pi_{\mathcal{Q}} \right) = \left\{ \begin{array}{ll}
\frac{(q-2)\delta}{(q-1)\log(q-1)}, & q > 2, \\
\frac{\delta}{2}, & q = 2.
\end{array} \right.
\]

Part 2: Observe that \( W_{\delta} W_{\delta'} = W_{\delta} W_{\delta} = W_{\delta} \), where the first equality holds because \( W_{\delta} \) has uniform stationary distribution, and:

\[
\delta' = 1 - (1 - \delta)^2 - \frac{\delta^2}{q-1} = \delta \left( 2 - \frac{q\delta}{q-1} \right)
\]

using the proof of part 5 of Proposition 3. As a result, the discrete logarithmic Sobolev constant is:

\[
\alpha (W_{\delta} W_{\delta'}) = \alpha (W_{\delta'}) = \left\{ \begin{array}{ll}
\frac{(q-2)(2q-1-2q\delta)}{2q(1-\delta) \log(q-1)}, & q > 2, \\
\frac{2\delta}{2}, & q = 2.
\end{array} \right.
\]

from part 1 of this proposition.

Part 3: It is well-known in the literature that \( \rho_{\text{max}} (u, W_{\delta}) \) equals the second largest singular value of the divergence transition matrix \( \text{diag}(\sqrt{u})^{-1} W_{\delta} \text{diag}(\sqrt{u}) = W_{\delta} \) (see [27, Subsection I-B] and the references therein). Hence, from part 2 of Proposition 3, we have:

\[
\rho_{\text{max}} (u, W_{\delta}) = \left| 1 - \delta - \frac{\delta}{q-1} \right|.
\]

Part 4: First recall that the Dobrushin contraction coefficient (or the contraction coefficient for total variation distance) for any channel \( W \in \mathbb{R}^{q \times q} \) is given by:

\[
\eta_{\text{tv}} (W) \triangleq \sup_{P_X, P_{X'} \in \mathcal{P}_q, P_X \neq P_{X'}} \frac{\| P_X W - Q X W \|_{\ell_1}}{\| P_X - Q X \|_{\ell_1}}
\]

(66)

\[
= \frac{1}{2} \max_{x, x' \in [q]} \| W_{X'|X}(\cdot|x) - W_{Y'|X}(\cdot|x') \|_{\ell_1}.
\]

(67)

where \( \| \cdot \|_{\ell_1} \) denotes the \( \ell_1 \)-norm, and the second equality is Dobrushin’s two-point characterization of \( \eta_{\text{tv}} \) [43]. Using this characterization, we have:

\[
\eta_{\text{tv}} (W_{\delta}) = \frac{1}{2} \max_{x, x' \in [q]} \left\| w_{\delta} P_q^x - w_{\delta} P_q^{x'} \right\|_{\ell_1} = \left| 1 - \delta - \frac{\delta}{q-1} \right|
\]

where \( w_{\delta} \) is the noise pmf of \( W_{\delta} \) for \( \delta \in [0, 1] \), and \( P_q \in \mathbb{R}^{q \times q} \) is the generator cyclic permutation matrix defined in (10). It is well-known in the literature (see the introduction of [44] and the references therein) that:

\[
\rho_{\text{max}} (u, W_{\delta})^2 \leq \eta_{\text{tv}} (W_{\delta}) \leq \rho_{\text{max}} (u, W_{\delta}).
\]

(68)

So, the value of \( \eta_{\text{tv}} (W_{\delta}) \) and part 3 of this proposition give:

\[
\left( 1 - \delta - \frac{\delta}{q-1} \right)^2 \leq \eta_{\text{tv}} (W_{\delta}) \leq \left| 1 - \delta - \frac{\delta}{q-1} \right|.
\]

This completes the proof.
APPENDIX G

GRAMIAN LÖWNER DOMINATION IMPLIES SYMMETRIC PART LÖWNER DOMINATION

Lemma 3 (Gramian Löwner Domination implies Symmetric Part Löwner Domination). Given $A \in \mathbb{R}_{\geq 0}^{q \times q}$ and $B \in \mathbb{R}^{q \times q}$ that is normal, we have:

$$A^2 = AA^T \succeq_{PSD} BB^T \Rightarrow A = A + A^T \succeq_{PSD} B + B^T.$$  

Proof. Since $AA^T \succeq_{PSD} BB^T \succeq_{PSD} 0$, using the Löwner-Heinz theorem (presented in (60)) with $p = \frac{1}{2}$, we get:

$$A = (AA^T)^{\frac{1}{2}} \succeq_{PSD} (BB^T)^{\frac{1}{2}} \succeq_{PSD} 0$$

where the first equality holds because $A \in \mathbb{R}_{\geq 0}^{q \times q}$. It suffices to now prove that $(BB^T)^{1/2} \succeq_{PSD} (B + B^T)/2$, as the transitive property of $\succeq_{PSD}$ will produce $A \succeq_{PSD} (B + B^T)/2$. Since $B$ is normal, $B = UDU^H$ by the complex spectral theorem [31, Theorem 7.9], where $U$ is a complex unitary matrix, $U^H$ is its Hermitian transpose, and $D$ is a complex diagonal matrix. Using this unitary diagonalization, we have:

$$U|D|U^H = (BB^T)^{\frac{1}{2}} \succeq_{PSD} \frac{B + B^T}{2} = U \text{Re}(D) U^H$$

since $|D| \succeq_{PSD} \text{Re}(D)$, where $|D|$ and $\text{Re}(D)$ denote the element-wise absolute value and real part of $D$, respectively. This completes the proof. ■

REFERENCES


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