

# Notes on

## Optimal stopping of Markov processes

*Yury Polyanskiy*  
(Юрий Полянский)

This is a very short introduction to main concepts of the theory of optimal stopping, value functions and dynamic programming principles.

### 1 Notes on optimal stopping: Three main principles

Consider the Markov process  $X_k : \Omega \rightarrow E$  with kernel  $T$ , function  $g : E \rightarrow \bar{\mathbb{R}}$  and the problem:

$$s(x) = \sup_{\tau} \mathbb{E}^x g(X_{\tau}), \quad (1.1)$$

where sup is over all finite stopping moments  $\tau < \infty$ <sup>1</sup>.

We say that  $g \in L$  if  $g > -\infty$  and  $Tg^- < \infty$ . Say that  $g \in A^-$  if

$$\mathbb{E}^x[\sup_n g^-(X_n)] < \infty.$$

Say that  $g \in A^+$  if

$$\mathbb{E}^x[\sup_n g^+(X_n)] < \infty.$$

Function  $f \in L$  is excessive if

$$f \geq Tf.$$

#### 1.1 Principle 1: converses via excessive majorants

Notice that for excessive  $v$  the process  $v(X_n)$  is a supermartingale.

**Theorem 1** *If  $X_n$  is a supermartingale and  $Y_n$  is uniformly integrable martingale (uim) s.t.*

$$X_n \geq Y_n$$

*then  $X_n \rightarrow X_{\infty}$  (a.s. and  $L_1$ ) and for any (possibly infinite) stopping times  $S \leq T$  we have*

$$\mathbb{E}_S X_T \leq X_S.$$

---

<sup>1</sup>Shiryaev [1] seems to be very surprised by the fact that oftentimes (i.e. under some technical conditions) one moment  $\tau$  achieves the maximum simultaneously for all  $x$ . But this is very natural: since  $X_0 = x$  a.s. we can construct the optimal moment  $\tau^*$  by looking at  $X_0$  and then choosing the  $\tau_x$  that is optimal for  $x = X_0$ .

(I have written down the proof in the printout of Cinlar's lecture notes). Such supermartingales will be called SLUIMs.

Note that if  $v(x) \geq g(x)$  then

$$v(X_n) \geq g(X_n) \geq -\sup_k g^-(X_k)$$

and taking  $\mathbb{E}_n$  on both sides:

$$v(X_n) \geq Y_n,$$

where  $Y_n = -\mathbb{E}_n \sup_k g^-(X_k)$  is uim.

Consider any excessive  $v(x) \geq g(x)$  then

$$\mathbb{E}^x g(X_\tau) \leq \mathbb{E}^x v(X_\tau) \leq v(x).$$

Hence if  $g \in A^-$  we have

$$s(x) \leq v(x) \tag{1.2}$$

for any excessive majorant  $v$  of  $g$ .

It is stunning that for  $g \in A^-$  the bound (1.2) is tight as the next subsection shows.

In fact, we have proven even more: even if we have allowed infinite stopping moments  $\bar{\tau}$  and defined  $g(X_\infty) \triangleq \limsup g(X_n)$  we still must have

$$\sup_{\bar{\tau}} \mathbb{E}^x g(X_{\bar{\tau}}) \leq v(x) \tag{1.3}$$

because  $\mathbb{E}^x v(X_{\bar{\tau}}) \leq v(x)$  is true even if  $\bar{\tau}$  takes value  $+\infty$ .

Since the bound is tight there is no need to consider the complication of possibly infinite moments  $\bar{\tau}$ .

## 1.2 Principle 2: $s(x)$ is itself excessive

Indeed, take an  $\epsilon$ -optimal moment  $\tau$  such that

$$s(x) \leq \mathbb{E}^x g(X_\tau) + \epsilon$$

then we have

$$Ts(x) \leq \epsilon + \mathbb{E}^x \mathbb{E}^{X_1} g(X_{\tau+1}) = \epsilon + \mathbb{E}^x g(X_\tau) \circ \theta_1.$$

Now notice that  $X_\tau \circ \theta_1 = X_{\tau \circ \theta_1 + 1}$  and that the random variable  $\tau' = \tau \circ \theta_1 + 1$  is in fact a stopping moment. Therefore

$$Ts(x) \leq \epsilon + \mathbb{E}^x g(X_{\tau'}) \leq s(x) + \epsilon.$$

Since this is true for any  $\epsilon$  we must have

$$Ts(x) \leq s(x),$$

i.e.  $s(x)$  is an excessive majorant of  $g$ , according to (1.2) it is the least excessive majorant (l.e.m.):

$$s(x) = lem(g).$$

### 1.3 Principle 3: achievability

The main idea on the achievability side is a simple observation that if the optimal moment  $\tau^*$  really exists then it must be

$$g(X_{\tau^*}) = s(X_{\tau^*}).$$

Indeed, suppose that there is  $n$  such that  $\mathbb{P}^x[\tau^* = n, g(X_n) \leq s(X_n) - \epsilon] > 0$  then we could improve the performance of  $\tau^*$  by introducing the moment

$$\tau' = \tau^* 1_{A^c} + (\tau' + n) 1_A,$$

where  $A = \{\tau^* = n, g(X_n) \leq s(X_n) - \epsilon\}$  and  $\tau'$  is anything achieving  $s(x) - \epsilon/2$ .

Therefore, if the optimal stopping moment  $\tau^*$  exist, then  $\tau_0 \leq \tau^*$  where

$$\tau_0 = \inf\{n \geq 0 : g(X_n) = s(X_n)\}. \quad (1.4)$$

*Therefore, the first thing to check on the achievability side is that (1.4) is finite. If so, then the problem (1.1) is solved in the strongest possible sense. Namely, one fixed moment  $\tau_0$  achieves the sup for all  $x$  at the same time!*

In general, such  $\tau_0$  is not finite and hence solution is not satisfactory (however, if  $g \in A^+$  then this  $\tau_0$  is always optimal in the extended sense of (1.3)).

The natural next step is to consider:

$$\tau_\epsilon = \inf\{n \geq 0 : g(X_n) \geq s(X_n) - \epsilon\}. \quad (1.5)$$

Under condition  $g \in A^+$  these moments are always finite and hence are  $\epsilon$ -optimal *uniformly* for  $x \in E$ ! This is also rather surprising.

### 1.4 How to actually find $s(x)$ ?

So we know that  $s(x)$  is the l.e.m. of  $g(x)$ . How do we find it?

Consider the properties of excessive functions  $f, g$ :

1.  $f, g \geq 0$  then  $\min(f, g)$  is excessive.
2.  $Tf$  is excessive.
3.  $f_n \nearrow f_\infty$  then  $f_\infty$  is excessive.
4.  $f_n \searrow f_\infty$ ,  $Tf_1^+ < \infty$  and  $f_\infty > -\infty$ , then  $f_\infty$  is excessive.
5.  $\min(f, \text{const})$  is excessive.
6.  $\text{lem}(g \wedge c) \nearrow \text{lem}(g)$  when  $g \nearrow c$ . Properties 4, 5 fix the problems when  $g \notin A^+$ .
7. For any  $h \leq f$  we have:  $f_1 = \max(Tf, h)$  is excessive (and is in between  $h$  and  $f$ ).

Proof:  $f_1 \leq f$  hence  $Tf_1 \leq Tf \leq \max(Tf, h)$ .

Now, according to the last property we have that a  $lem(g)$  must satisfy

$$s(x) = \max(g, Ts). \quad (1.6)$$

This equation can be used to find  $s(x)$ .

### 1.5 $s(X_n)$ is a local martingale (has Doob property on $[0, \tau_0]$ ).

Note that from Bellman's equation (1.6) we see that on  $[0, \tau_0]$ , see (1.4) we have:  $s(X_n) > g(X_n)$  and therefore  $s(X_n) = Ts(X_n)$ , or in other words:

$$\mathbb{E}_n s(X_{n+1}) = s(X_n), \forall n \leq \tau_0.$$

This means that the process  $s(X_n)$  is a local martingale with localizing time  $\tau_0$ .

The natural interpretation of  $s(X_n)$  is the expected return (or the "fair price" for the option). Hence we see that in many cases the fair price of an option is actually a local martingale (and a global supermartingale).

### 1.6 Bounded stopping moments

What if in (1.1) we restring  $\tau$  to the class  $T_n$  s.t.

$$\tau \in T_n \iff \mathbb{P}_x[\tau \leq n] = 1.$$

Then define

$$s_n(x) = \sup_{\tau \in T_n} \mathbb{E}^x g(X_\tau).$$

The solution for  $s_0(x)$  is just  $g(x)$ . For  $s_1(x)$  it is obviously:

$$s_1(x) = \max(g(x), Tg(x)).$$

In general we have

$$s_n = \max(g, Ts_{n-1}).$$

In fact, Shiryaev [1] shows that for  $g \in A^-$  we have

$$s_n \nearrow s = lem(g).$$

In reality I think this is true for any  $g \in L$ , but too lazy to check his proof.

## 2 Technicalities

### 2.1 Main non-trivial tricks

**Lemma 2** *If  $g \in A^+$  then the following are well-defined excessive functions:*

$$\phi(x) \triangleq \mathbb{E}^x \sup_n g(X_n), \quad (2.1)$$

$$\phi^+(x) \triangleq \mathbb{E}^x \sup_n g^+(X_n). \quad (2.2)$$

**Lemma 3** *Any excessive function  $v$  satisfying*

$$v = \max(g, Tv)$$

*is a martingale on  $[0, \tau_\epsilon]$ ,  $\epsilon \geq 0$ , where*

$$\tau_\epsilon = \inf\{n \geq 0 : g(X_n) \geq v(X_n) - \epsilon\}. \quad (2.3)$$

*Rigorously this means that*

$$U_n \triangleq v(X_{\tau_\epsilon \wedge n})$$

*is a martingale.*

Proof is obvious since on  $[0, \tau_\epsilon]$  we have  $v(X_n) > g(X_n) + \epsilon$  and therefore

$$v(X_n) = Tv(X_n) = \mathbb{E}^{X_n} v(X_1) = \mathbb{E}_n v(X_{n+1}).$$

Unfortunately, this is not enough to conclude that

$$\mathbb{E}^x v(X_{\tau_\epsilon}) = v(x)$$

because in general  $U_n$  is not uim. However, the following lemma gives conditions for upper and lower-bounds.

**Lemma 4** *If in Lemma 3 we have  $g \in A^-$  then  $U_n$  is a SLUIM; in particular:*

$$\mathbb{E}^x v(X_{\tau_\epsilon}) \leq v(x). \quad (2.4)$$

*If in Lemma 3 we have  $g \in A^+$  and  $v \leq \phi^+$  (e.g. if  $v$  is the l.e.m. of  $g$  in the class of e.m.'s that contains  $\phi^+$ ), then  $U_n$  is submartingale upper-bounded by a uim; in particular:*

$$\mathbb{E}^x v(X_{\tau_\epsilon}) \geq v(x). \quad (2.5)$$

*All in all if  $g \in A^+ \cap A^-$  and  $v = \text{lem}(g)$  then<sup>1</sup>*

$$\mathbb{E}^x v(X_{\tau_\epsilon}) = v(x).$$

---

<sup>1</sup>Is it true that  $U_n$  is uim? I.e. is it true that  $v(X_n)$  is Doob on  $[0, \tau_\epsilon]$ ?

The dominating uims are  $-\mathbb{E}_n \sup_k g^-(X_k)$  and  $\mathbb{E}_n \sup_k g^+(X_k)$  correspondingly. Note that to prove just (2.4) or (2.5) only Fatou's lemma is enough since functions  $-\sup_k g^-(X_k)$  and  $\sup_k g^+(X_k)$  can be taken as the dominating integrables.

**Lemma 5** *If  $g \in A^+$  and any  $v$  (not nec. excessive) s.t.  $g \leq v \leq \phi$  then*

$$\limsup g(X_n) = \limsup v(X_n).$$

*In particular,  $\mathbb{P}[\tau_\epsilon < \infty] = 1$  for  $\epsilon > 0$  (as defined in (2.3)).*

Proof: We have

$$v(X_n) \leq \phi(X_n) = \mathbb{E}_n Y_n, \quad (2.6)$$

$$Y_n \triangleq \sup_{k \geq n} g(X_k), \quad (2.7)$$

$$Y_n \rightarrow Y_\infty = \limsup_{k \rightarrow \infty} g(X_k) \quad \text{a.s.} \quad (2.8)$$

By Hunt's Dominated Convergence Theorem ( $Y_n \leq \sup_n g^+(X_k)$ ) we have

$$\mathbb{E}_n Y_n \rightarrow \mathbb{E}_\infty Y_\infty = Y_\infty \quad \text{a.s. and in } L_1$$

Hence we have

$$\limsup v(X_n) \leq \limsup \mathbb{E}_n Y_n = \limsup_{n \rightarrow \infty} g(X_n).$$

The reverse inequality is obvious.

## 2.2 Case $g \in A^-$

**Theorem 6** *If  $g \in A^-$  then*

$$s = \text{lem}(g).$$

*It satisfies*

$$s = \max(g, Ts).$$

*It is a monotone limit of  $s_n$ :*

$$s_n \nearrow s.$$

*If  $g \in A^- \cap A^+$  then the moments  $\tau_\epsilon$  are finite and  $\epsilon$ -optimal; the moment  $\tau_0$  is the only candidate for the optimal moment (i.e. if  $\mathbb{P}^x[\tau_0 = \infty] > 0$  then there can be no moment  $\tau'$  s.t.  $s(x) = \mathbb{E}^x g(X_\tau)$ ).*

*Finally, the sup in (1.1) can be taken over hitting time of Borel sets  $\inf\{n \geq 0 : g(x_n) \in A\}$ .*

The heuristic argument above already proved  $s = lem(g)$  and we know l.e.m. should satisfy  $s = \max(g, Ts)$ . Finiteness of  $\tau_\epsilon$  follows from Lemma 5.

We only need to show how to achieve  $s(x)$  via hitting times. Consider

$$s_c(x) = lem(g \wedge c).$$

Obviously,  $s_c \nearrow s$  when  $c \rightarrow \infty$ . Since  $g \wedge c \in A^+$  we see that  $\tau_{c,\epsilon}$  is a finite hitting time; it achieves

$$\mathbb{E}^x g(X_{\tau_{c,\epsilon}}) \geq \mathbb{E}^x g(X_{\tau_{c,\epsilon}}) \wedge c \geq s_c(x) - \epsilon.$$

### 2.3 Case $g \in A^+$

This case is more delicate because now arbitrary e.m.'s can not be used to prove converse. I.e. if  $v \geq g$  and  $Tv \leq v$  then it is true that

$$\mathbb{E}^x g(X_\tau) \leq \mathbb{E}^x v(X_\tau),$$

but we can not in general conclude that  $\mathbb{E}^x v(X_\tau) \leq v(x)$  (i.e. now  $v(X_n)$  is not SLUIM).<sup>1</sup>

So in the case  $g \notin A^-$  we restrict attention to the regular e.m.'s (r.e.m.).

First, restrict allowable  $\tau$ 's to:

$$\tau \in \mathcal{N}_g^- \iff \mathbb{E}^x g^-(X_\tau) < \infty.$$

(obviously, among all  $\tau$ 's for which  $\mathbb{E}^x g(X_\tau)$  WLOG we can restrict maximization in (1.1) to  $\mathcal{N}_g^-$ ).

An excessive function  $f$  is called regular if

$$\forall \tau \in \mathcal{N}_g^- : \mathbb{E}^x f^-(X_\tau) < \infty \quad \text{and} \quad (2.9)$$

$$\mathbb{E}^x f(X_\tau) \leq f(x). \quad (2.10)$$

Note: In general it is a much harder problem to check if a particular  $f$  is r.e.m. (checking that it is an e.m. is easy). However, any SLUIM e.m. is automatically r.e.m.

**Theorem 7** *If  $g \in A^+$  then*

$$s = lrem(g).$$

*It satisfies*

$$s = \max(g, Ts).$$

*The moments  $\tau_\epsilon$  are finite and  $\epsilon$ -optimal; the moment  $\tau_0$  is the only candidate for the optimal moment (i.e. if  $\mathbb{P}^x[\tau_0 = \infty] > 0$  then there can be no moment  $\tau'$  s.t.  $s(x) = \mathbb{E}^x g(X_{\tau'})$ ).*

*Finally, the sup in (1.1) can be taken over hitting time of Borel sets  $\inf\{n \geq 0 : g(x_n) \in A\}$ .*

---

<sup>1</sup>Shiryaev's counter-example:  $X_n = X_{n-1} \cdot E_n$ ,  $E_n = 0, 1$  w.p.  $1/2$ ,  $X_0 = 2$ . Then if  $g = -x$ , we see  $g \notin A^-$ . Moreover,  $Tg \leq g$  and hence  $lem(g) = g$ . But  $s(x) = 0$ , achieved by  $\tau = \inf\{n : X_n = 0\}$ .

Proof: First, converse. Consider  $v(x)$  a r.e.m. of  $g$ . Then

$$\mathbb{E}^x g(X_\tau) \leq \mathbb{E}^x v(X_\tau) \leq v(x).$$

Therefore,

$$s(x) \leq \inf_{v\text{-r.e.m.}} v(x).$$

Second,  $s(x)$  is a r.e.m. We know it is always excessive, but it is NOT trivial to see that it is regular. Indeed, if we try to follow the same argument as before then we are led to the inequality  $s(x) \leq \epsilon + \mathbb{E}^x \mathbb{E}^{X_\tau} g(X_\sigma)$ ,  $\tau, \sigma \in \mathcal{N}_g^-$ . I do not know how to show that  $\tau + \sigma \circ \theta_\tau \in \mathcal{N}_g^-$ .

Instead, here is a direct proof (copy from Shiryaev [1] essentially), which constructs a sequence of r.e.m.'s converging to  $s(x)$ .

$$s_c(x) = lem(g \vee c).$$

(each such  $s_c$  exists since  $g \vee c$  is in  $A^-$ ).

Then  $s_c(x) \searrow s_*(x)$  when  $c \rightarrow -\infty$ . Since  $s_c(X_n)$  is a SLUIM we have:

$$\mathbb{E}^x s_c(X_\tau) \leq s_c(x). \quad (2.11)$$

Now, each  $s_c(x)$  is majorized by  $\phi^+(x)$ . Hence:

$$s_c(X_n) \leq \phi^+(X_n) \leq \sup_n g^+(X_n).$$

Then in (2.11) we can take  $\lim_{c \rightarrow \infty}$  inside expectation (DCT!) and then we get

$$\forall \tau \in \mathcal{N}_g^- : \quad \mathbb{E}^x s_*(X_\tau) \leq s_*(x).$$

That is  $s_*(x)$  is a r.e.m. of  $g$ . It is also achievable. Indeed, define

$$\tau_\epsilon = \inf\{n \geq 0 : g(X_n) \geq s_*(X_n) - \epsilon\}. \quad (2.12)$$

Then Lemma 5 shows  $\tau_\epsilon$  is finite and Lemma 4 that

$$\mathbb{E}^x s_*(X_{\tau_\epsilon}) \geq s_*(x).$$

The reverse inequality is also true: because  $\tau_\epsilon \in \mathcal{N}_g^-$  and  $s_*$  is a r.e.m. Therefore:

$$\mathbb{E}^x g(X_{\tau_\epsilon}) \geq s_*(x) - \epsilon.$$

Which implies

$$s(x) = s_*(x) = lrem(g).$$

**Corollary 8** *If  $g \in A^+$  and  $\lim g(X_n) = -\infty$  (a.s.) then  $\tau_0$  is a finite optimal moment:*

$$s(x) = \mathbb{E}^x g(X_{\tau_0}).$$

Proof: by Lemma 5 we know that

$$\limsup s(X_n) = \limsup g(X_n) = -\infty$$

and by Lemma 4 we know that

$$\mathbb{E}^x s(X_{\tau_0}) \geq s(x) > -\infty.$$

If  $\mathbb{P}[\tau_0 = \infty] > 0$  then we would have  $s(x) = -\infty$  a contradiction.

**Lemma 9** *If  $g \in A^+$  then among all excessive functions  $f$  between  $g$  and  $\phi$ :*

$$g \leq f \leq \phi$$

there is exactly one satisfying

$$f^* = \max(g, Tf^*). \quad (2.13)$$

*It is, of course, the l.r.e.m. of  $g$ :*

$$f^* = lrem(g).$$

Proof: Define operator  $G$ :

$$Ga = \max(g, Ta).$$

Then denote by  $f^*$  a (monotone) limit of  $G^n \phi$ . It is a routine check (applying MCT with dominating  $\sup_n g^+(X_n)$  etc) to see that  $f^*$  satisfies (2.13).

We need to show that it is actually a *lem* of  $g$ . Take any e.m.  $v$  and notice that for any stopping time  $\tau$  we have then

indeed, such  $f^*$  is by Lemma has the property:

$$\limsup f^*(X_n) = \limsup g(X_n),$$

and hence

$$\mathbb{P}[\tau_\epsilon < \infty] = 1.$$

On the other hand,

$$\mathbb{E}^x g(X_{\tau_\epsilon}) \geq \mathbb{E}^x f^*(X_{\tau_\epsilon}) - \epsilon.$$

Now since  $f^* \leq \phi^+$  we also have that:

$$\mathbb{E}^x f^*(X_{\tau_\epsilon}) \geq f^*(x).$$

Therefore,

$$\mathbb{E}^x g(X_{\tau_\epsilon}) \geq f^*(x) - \epsilon.$$

Finally, for any r.e.m.  $v$  we have

$$v(x) \geq \mathbb{E}^x g(X_{\tau_\epsilon}) \geq f^*(x) - \epsilon$$

or after taking  $\epsilon \rightarrow 0$ :

$$v(x) \geq f^*(x).$$

### 3 General case

In the general case, we can not say a lot about the solution of (1.1)

**Theorem 10** *If  $g \in L$  then*

$$s = lrem(g) .$$

*It satisfies*

$$s = \max(g, Ts) .$$

*The supremum in (1.1) can be taken over hitting times of Borel sets.*

Proof is easy:

$$s_c = lrem(g \wedge c)$$

Then  $s_c \nearrow s_*$  and hence  $s_*$  is a r.e.m. of  $g$  and satisfies  $s_* = \max(g, Ts_*)$ . Again, since  $s_*$  is a r.e.m. we have

$$s \leq s_* .$$

On the other hand, defining finite hitting times  $\tau_{c,\epsilon}$  we can achieve  $s_*$  as close as required:

$$s \geq s_* .$$

### 4 Cost constraints

Consider a generalization of (1.1):

$$s(x) = \sup_{\tau} \mathbb{E}^x \left[ \alpha^{\tau} g(X_{\tau}) - \sum_{j=0}^{\tau-1} \alpha^j c(X_j) \right] , \quad (4.1)$$

where  $\alpha \in (0, 1]$ .

This situation can be reduced to the previous case by defining a new markov process  $X'_n = (n, X_0, \dots, X_n)$ . Alternatively, we can redo all the steps above. In this case the excessiveness is defined as

$$v(x) \geq \alpha T v(x) - c(x) .$$

The condition  $g \in A^-$  becomes

$$\mathbb{E} \sup_n \left[ \alpha^{\tau} g(X_n) - \sum_{j=0}^{n-1} \alpha^j c(X_j) \right]^- < \infty$$

The only useful additional theorem is this one:

**Theorem 11** *If  $g \in A^+$  and  $\sum_{j=0}^{\infty} \alpha^j c(X_j) = +\infty$  (a.s.) then the moment  $\tau_0$  is optimal and finite.*

Proof: Corollary 8 applies.

## References

- [1] A. N. Shiryaev, *Optimal Stopping Rules*, Springer-Verlag: New York, 1978.