Fundamental limits of many-user MAC with finite payloads and fading

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Abstract

Consider a (multiple-access) wireless communication system where users are connected to a unique base station over a shared-spectrum radio links. Each user has a fixed number $k$ of bits to send to the base station, and his signal gets attenuated by a random channel gain (quasi-static fading). In this paper we consider the many-user asymptotics of Chen-Chen-Guo'2017, where the number of users grows linearly with the blocklength. In addition, we adopt a per-user probability of error criterion of Polyanskiy'2017 (as opposed to classical joint-error probability criterion). Under these two settings we derive bounds on the optimal required energy-per-bit for reliable multi-access communication. We confirm the curious behaviour (previously observed for non-fading MAC) of the possibility of perfect multi-user interference cancellation for user densities below a critical threshold. Further we demonstrate the suboptimality of standard solutions such as orthogonalization (i.e., TDMA/FDMA) and treating interference as noise (i.e. pseudo-random CDMA without multi-user detection).

Index Terms

Many-user MAC, fading, fundamental limits, no-CSI

I. INTRODUCTION

We clearly witness two recent trends in the wireless communication technology: the increasing deployment density and miniaturization of radio-equipped sensors. The first trend results in progressively worsening interference environment, while the second trend puts ever more stringent demands on communication energy efficiency. This suggests a bleak picture for the future networks, where a chaos of packet collisions and interference contamination prevents reliable connectivity.

This paper is part of a series aimed at elucidating the fundamental tradeoffs in this new “dense-networks” regime of communication, and on rigorously demonstrating suboptimality of state-of-the-art radio-access solutions (ALOHA, orthogonalization, or FDMA/TDMA and treating interference as noise, or TIN). This suboptimality will eventually lead to dramatic consequences. For example, environmental impact of billions of toxic batteries getting depleted at 1/10 or 1/100 of the planned service time is easy to imagine. In order to future-proof our systems, we should avoid locking in on outdated and unfixable multiple-access architectures causing tens of dB losses in energy efficiency. The information-theoretic analysis in this paper demonstrates that the latter is indeed unavoidable (with state-of-the-art schemes). However, our message is in fact optimistic, as we also demonstrate existence of protocols which are partially immune to the increase of the sensor density.

Specifically, in this paper we consider a problem of $K$ nodes communicating over a frame-synchronized multiple-access channel. When $K$ is fixed and the frame size $n$ (which we will also call “blocklength” or the “number of degrees of freedom”) is taken to infinity we get the classical regime [3], in which the fundamental limits are given by well-known mutual information expressions. A new regime, deemed many-access, was put forward by Chen, Chen and Guo [4]. In this regime the number of nodes $K$ grows with blocklength $n$. It is clear that the most natural scaling is linear: $K = \mu n, n \to \infty$, corresponding to the fact that in time $n$ there are linearly many users that will have updates/traffic to send [5]. That is, if each device wakes up once in every $T$ seconds and transmits over a frame of length $t$, then in time (proportional to) $t$ there are $K \approx t/T$ users where $t$ is large enough for this approximation to hold but small that no device wakes up twice. Further, asymptotic results obtained from this linear scaling have been shown to approximately predict behavior of the fundamental limit at finite blocklength, e.g. at $n = 30000$ and $K <= 300$ [5,6]. The analysis of [4] focused on the regime of infinitely large payloads (see also [7] for a related massive MIMO MAC analysis in this setting). In contrast [5] proposed to focus on a model where each of the $K = \mu n$ nodes has only finitely many bits to send. In this regime, it turned out, one gets the relevant engineering trade-offs. Namely, the communication with finite energy-per-bit is possible as $n \to \infty$ and the optimal energy-per-bit depends on the user density.

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Parts of the results on classical regime in this paper (theorems III.1-III.3) have appeared in the one of the authors’ Masters thesis [2].

1In this paper we do not distinguish between TDMA, FDMA or any other orthogonalization strategy. From here on, all orthogonalization strategies will be referred to as “TDMA”
\( \mu \). For this to happen, however, a second crucial departure from the classical MAC model was needed: the per-user probability of error, PUPE, criterion [5].

These two modifications (the scaling \( K = \mu n \) and the PUPE) were investigated in the case of the AWGN channel in [5] [6] [8]. We next describe the main discovery of that work. The channel model is:

\[
Y^n = \sum_{i=1}^{K} X_i + Z^n, \quad Z^n \sim \mathcal{CN}(0, I_n),
\]

and \( X_i = f_i(W_i) \in \mathbb{C}^n \) is the codeword of \( i \)-th user corresponding to \( W_i \in [2^k] \) chosen uniformly at random. The system is said to have PUPE \( \epsilon \) if there exist decoders \( \hat{W}_i = W_i(Y^n) \) such that

\[
P_{\epsilon,u} = \frac{1}{K} \sum_{i=1}^{K} \mathbb{P} \left[ W_i \neq \hat{W}_i \right] \leq \epsilon.
\]

The energy-per-bit is defined as \( \frac{E_b}{N_0} = \frac{1}{k} \sup_{i \in [K], w \in [2^k]} \| f_i(w) \|^2 \). The goal in [5] [8] was to characterize the asymptotic limit

\[
\mathcal{E}^*(\mu, k, \epsilon) \triangleq \lim_{n \to \infty} \inf_{\hat{E}_b} \sup_n \inf_{\hat{E}_b} \frac{E_b}{N_0}
\]

where infimum is taken over all possible encoders \( \{ f_i \} \) and decoders \( \{ \hat{W}_i \} \) achieving the PUPE \( \epsilon \) for \( K = \mu n \) users.

To predict how \( \mathcal{E}^*(\mu, \epsilon) \) behaves, first consider a naive Shannon-theoretic calculation [9]: if \( K \) users want to send \( k \) bits in \( n \) degrees of freedom, then their sum-power \( P_{\text{tot}} \) should satisfy

\[
n \log(1 + P_{\text{tot}}) = kK.
\]

In turn, the sum-power \( P_{\text{tot}} = \frac{kK E_b}{\pi N_0} \). Overall, we get

\[
\mathcal{E}^* \approx \frac{2^{\mu k} - 1}{k \mu}.
\]

This turns out to be a correct prediction, but only in the large-\( \mu \) regime. The true behavior of the fundamental limit is roughly given by

\[
\mathcal{E}^*(\mu, k, \epsilon) \approx \max \left( \frac{2^{\mu k} - 1}{k \mu}, \mathcal{E}_{\text{s.u.}} \right),
\]

where \( \mathcal{E}_{\text{s.u.}} = \mathcal{E}_{\text{s.u.}}(k, \epsilon) \) does not depend on \( \mu \) and corresponds to the single-user minimal energy-per-bit for sending \( k \) bits with error \( \epsilon \), for which a very tight characterization is given in [10]. In particular, with good precision for \( k \geq 10 \) we have

\[
\mathcal{E}_{\text{s.u.}}(k, \epsilon) = \frac{1}{2} \left( Q^{-1}(2^{-k}) - Q^{-1}(1 - \epsilon) \right)^2
\]

where \( Q \) is the complementary CDF of the standard normal distribution: \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt \).

In all, results of [5] [6] [8] suggest that the minimal energy-per-bit has a certain "inertia": as the user density \( \mu \) starts to climb from zero up, initially the energy-per-bit should stay the same as in the single-user \( \mu = 0 \) limit. In other words, optimal multiple-access architectures should be able to perfectly cancel all multi-user interference (MUI), achieving an essentially single-user performance for each user, provided the user density is below a critical threshold. Note that this is much better than orthogonalization, which achieves the same effect at the expense of shortening the available (to each user) blocklength by a factor of \( \frac{1}{K} \). Quite surprisingly, standard approaches to multiple-access such as TDMA and TIN [7] while having an optimal performance at \( \mu \to 0 \) demonstrated a significant suboptimality for \( \mu > 0 \) regime. In particular, no "inertia" was observed and the energy-per-bit for those suboptimal architectures is always a monotonically increasing function of the user density \( \mu \).

This opens the (so far open) quest for finding a future-proof MAC architecture that would achieve \( \mathcal{E}_{\text{s.u.}} \) energy-per-bit for a strictly-positive \( \mu > 0 \).

(We note that in this short summary we omitted another important part of [5]: the issue of random-access – i.e. communicating when the identities/codebooks of active users are unknown a priori. We mention, however, that for the random-access version of the problem, there are a number of low-complexity (and quite good performing) algorithms that are available [11][17].)

The contribution of this paper is in demonstrating the same perfect MUI cancellation effect in a much more practically relevant communication model, in which the ideal unit power-gains of [11] are replaced by random (but static) fading gain coefficients. We consider two cases of the channel state information: known at the receiver (CSIR) and no channel state information (noCSI).

2Note that pseudo-random CDMA systems without multi-user detection and large load factor provide an efficient implementation of TIN. So throughout our discussions, conclusions about TIN also pertain to CDMA systems of this kind.
Remark 1. Note that in considering the model the idea of PUPE is very natural, and implicitly appears in earlier works, e.g. [22, 23], where it is conflated with the outage probability. After this short warm-up we go to our main Section IV, which contains rigorous achievability and converse bounds for the scaling regime. Finally, we conclude with numerical evaluations and discussions in Section V, where we also compare our bounds with the TDMA and TIN.

A. Notations

Let $\mathbb{N}$ denote the set of natural numbers. For $n \in \mathbb{N}$, let $\mathbb{C}^n$ denote the $n$-dimensional complex Euclidean space. Let $S \subset \mathbb{C}^n$. We denote the projection operator or matrix on to the subspace spanned by $S$ as $P_S$ and its orthogonal complement as $P_S^\perp$. For $0 \leq p \leq 1$, let $h_2(p) = -p \log_2(p) - (1 - p) \log_2(1 - p)$ and $h(p) = -p \log(p) - (1 - p) \log(1 - p)$, with $0 \log 0$ defined to be 0. We denote by $\mathcal{N}(0, 1)$ and $\mathcal{CN}(0, 1)$ the standard normal and the standard circularly symmetric complex normal distributions, respectively. $\mathbb{P}$ and $\mathbb{E}$ denote probability measure and expectation operator respectively. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$. $\log$ denotes logarithm to base 2. Lastly, $\|\cdot\|$ represents the standard euclidean norm.

II. Definitions and System Model

Fix an integer $K \geq 1$ – the number of users. Let $\{P_{Y^n|X^n} = P_{Y^n|X^n_1, X^n_2, \ldots, X^n_K} : \prod_{i=1}^{K} X^n_i \rightarrow Y^n\}_{n=1}^{\infty}$ be a multiple access channel (MAC). In this work we consider only the quasi-static fading AWGN MAC: the channel law $P_{Y^n|X^n}$ is described by

$$Y^n = \sum_{i=1}^{K} H_i X^n_i + Z^n$$

where $Z^n \sim \mathcal{CN}(0, I_n)$, and $H_i \sim \mathcal{CN}(0, 1)$ are the fading coefficients which are independent of $\{X^n_i\}$ and $Z^n$. Naturally, we assume that there is a maximum power constraint:

$$\|X^n_i\|^2 \leq np.$$  

We consider two cases: 1) no channel state information (no-CSI): neither the transmitters nor the receiver knows the realizations of channel fading coefficients, but they both know the law; 2) channel state information only at the receiver (CSI): only the receiver knows the realization of channel fading coefficients. The special case of $[5]$ where $H_i = 1, \forall i$ is called the Gaussian MAC (GMAC).

In the rest of the paper we drop the superscript $n$ unless it is unclear.

Definition 1. An $((M_1, M_2, \ldots, M_K), n, \epsilon)_U$ code for the MAC $P_{Y^n|X^n}$ is a set of (possibly randomized) maps $\{f_i : [M_i] \rightarrow X^n_i\}_{i=1}^{K}$ (the encoding functions) and $g : Y^n \rightarrow \prod_{i=1}^{K} [M_i]$ (the decoder) such that if for $j \in [K]$, $X_j = f_j(W_j)$ constitute the input to the channel and $W_j$ is chosen uniformly (and independently of other $W_i$, $i \neq j$) from $[M_j]$ then the average (per-user) probability of error satisfies

$$P_{e,u} = \frac{1}{K} \sum_{j=1}^{K} \mathbb{P}[W_j \neq (g(Y))_j] \leq \epsilon$$

where $Y$ is the channel output.

We define an $((M_1, M_2, \ldots, M_K), n, \epsilon)_J$ code similarly, where $P_{e,u}$ is replaced by the usual joint error

$$P_{e,J} = \mathbb{P}\left[\bigcup_{j \in [K]} \{W_j \neq (g(Y))_j\}\right] \leq \epsilon$$

Further, if there are cost constraints, we naturally modify the above definitions such that the codewords satisfy the constraints.

Remark 1. Note that in (7), we only consider the average per-user probability. But in some situations, it might be relevant to consider maximal per-user error (of a codebook tuple) which is the maximum of the probability of error of each user. Formally, let $\mathcal{C}_{[K]} = \{C_1, \ldots, C_K\}$ denote the set of codebooks. Then

$$P^{\text{max}}_{e,u}(\mathcal{C}_{[K]}) = \max \left\{ \mathbb{P}\left[W_1 \neq \hat{W}_1\right], \ldots, \mathbb{P}\left[W_K \neq \hat{W}_K\right] \right\}$$


where the probabilities are with respect to the channel and possibly random encoding and decoding functions. In this paper we only consider the fundamental limits with respect to $P_{e,u}$ and PUPE always refers to this unless otherwise noted. But we note here that for both asymptotics and FBL, the difference is not important. See appendix C for a discussion on this – there we show that by random coding $\mathbb{E} [P_{e,u}^{\text{max}}]$ is asymptotically equal to $\mathbb{E} [P_{e,u}]$ (expectations are over random codebooks).

III. Classical regime: $K$ fixed, $n \to \infty$

In this section, we focus on the channel under classical asymptotics where $K$ is fixed (and large) and $n \to \infty$. Further, we consider two distinct cases of joint error and per-user error. We show that subspace projection decoder (12) achieves a) $\epsilon$–capacity region $(C_{\epsilon,J})$ for the joint error and b) the best known bound for $\epsilon$–capacity region $C_{\epsilon,PU}$ under per-user error. This motivates using projection decoder in the many-user regime.

A. Joint error

A rate tuple $(R_1, \ldots, R_K)$ is said to be $\epsilon$–achievable [24] for the MAC if there is a sequence of codes whose rates are asymptotically at least $R_i$ such that joint error is asymptotically smaller than $\epsilon$. Then the $\epsilon$–capacity region $C_{\epsilon,J}$ is the closure of the set of $\epsilon$–achievable rates. For our channel (5), the $C_{\epsilon,J}$ does not depend on whether or not the channel state information (CSI) is available at the receiver since the fading coefficients can be reliably estimated with negligible rate penalty as $n \to \infty$ [25][22]. Hence from this fact and using [24] Theorem 5 it is easy to see that, for $0 \leq \epsilon < 1$, the $\epsilon$–capacity region is given by

$$C_{\epsilon,J} = \{ R = (R_1, \ldots, R_K) : \forall i, R_i \geq 0 \text{ and } P_0(R) \leq \epsilon \}$$

(10)

where the outage probability $P_0(R)$ is given by

$$P_0(R) = \mathbb{P} \left[ \bigcup_{S \subseteq [K], S \neq \emptyset} \left\{ \log \left( 1 + P \sum_{i \in S} |H_i|^2 \right) \right\} \leq \sum_{i \in S} R_i \right\}$$

(11)

Next, we define a subspace projection based decoder, inspired from [18]. The idea is the following. Suppose there were no additive noise. Then the received vector will lie in the subspace spanned by the sent codewords no matter what the fading coefficients are. To formally define the decoder, let $C$ denote a set of vectors in $\mathbb{C}^n$. Denote $P_C$ as the orthogonal projection operator onto the subspace spanned by $C$. Let $P_C^\perp = I - P_C$ denote the projection operator onto the orthogonal complement of $\text{span}(C)$ in $\mathbb{C}^n$.

Let $C_1, \ldots, C_K$ denote the codebooks of the $K$ users respectively. Upon receiving $Y$ from the channel the decoder outputs $g(Y)$ which is given by

$$g(Y) = (f_1^{-1}(\hat{c}_1), \ldots, f_K^{-1}(\hat{c}_K))$$

$$\hat{c}_1, \ldots \hat{c}_K = \arg \max_{(c_1, \ldots, c_K) \in C} \left\| P_{[c_1, \ldots, c_K]} Y \right\|^2$$

where $f_i$ are the encoding functions.

In this section, we show that using spherical codebook with projection decoding, $C_{\epsilon,J}$ of the $K$–MAC is achievable. We prove the following theorem

Theorem III.1 (Projection decoding achieves $C_{\epsilon,J}$). Let $R \in C_{\epsilon,J}$ of (5). Then $R$ is $\epsilon$–achievable through a sequence of codes with the decoder being the projection decoder (12).

Proof. We generate codewords iid uniformly on the power sphere and show that (12) yields a small $P_{e,J}$. See appendix A for details.

B. Per-user error

In this subsection, we consider the case of per-user error under the classical setting. Further, we assume availability of CSI at receiver (CSIR) which again can be estimated with little penalty.

The $\epsilon$–capacity region for the channel under per-user error, $C_{\epsilon,PU}$ is defined similarly as $C_{\epsilon,J}$ but with per-user error instead of joint error. $C_{\epsilon,PU}$ is unknown, but the best lower bound is given by the Shamai-Bettesh capacity bound [22]: given a rate tuple $R = (R_1, \ldots, R_K)$, an upper bound on the per-user probability of error under the channel (5), as $n \to \infty$, is given by

$$P_{e,u} \leq P_{e}^S(R)$$

$$= 1 - \frac{1}{K} \mathbb{E} \sup \left\{ |D| : D \subseteq [K], \forall S \subseteq D, S \neq \emptyset \right\}$$

$$\sum_{i \in D} R_i \leq \log \left( 1 + \frac{P \sum_{i \in S} |H_i|^2}{1 + P \sum_{i \in D}^\perp |H_i|^2} \right)$$

(13)
where the maximizing set, among all those that achieve the maximum, is chosen to contain the users with largest fading coefficients. The corresponding achievability region is

\[ C_{e,P} = \{ R : P_e(R) \leq \epsilon \} \]  

(14)

and hence it is an inner bound on \( C_{e,P_U} \).

We note that, in [22], only the symmetric rate case i.e, \( R_i = R_j \forall i,j \) is considered. So (13) is the extension of that result to the general non-symmetric case.

Here, we show that the projection decoding (suitably modified to use CSIR) achieves the same asymptotics as (13) for per-user probability of error. Next we describe the modification to the projection decoder to use CSIR.

Let \( \{ C_i \}_{i=1}^{K} \) denote the codebooks of the \( K \) users with \( |C_i| = M_i \). We have a maximum power constraint given by (6). Using the idea of joint decoder from [22], our decoder works in 2 stages. The first stage finds the following set

\[ D \in \arg \max \left\{ |D| : D \subset [K], \forall S \subset D, S \neq \emptyset, \sum_{i \in S} R_i < \log \left( 1 + \frac{P \sum_{i \in S} |H_i|^2}{1 + P \sum_{i \in D^C} |H_i|^2} \right) \right\} \]  

(15)

where \( D \) is chosen to contain users with largest fading coefficients. The second stage is similar to (12) but decodes only those users in \( D \). Formally, let \( ? \) denote an error symbol. The decoder output \( g_D(Y) \in \prod_{i=1}^{K} C_i \) is given by

\[ (g_D(Y))_i = \begin{cases} f_{i}^{-1}(\hat{c}_i) & i \in D \\ ? & i \notin D \end{cases} \]

(16)

where \( f_i \) are the encoding functions. Our error metric is the average per-user probability of error [8].

The following theorem is the main result of this section.

**Theorem III.2.** For any \( R \in C_{e,P}^{S,B} \) there exists a sequence of codes with projection decoder (15) with asymptotic rate \( R \) such that the per-user probability of error is asymptotically smaller than \( \epsilon \)

**Proof.** We generate iid (complex) Gaussian codebooks \( \mathcal{CN}(0, P' I_n) \) with \( P' < P \) and show that for \( R \in C_{e,P}^{S,B} \), (16) gives small \( P_{e,u} \). See appendix A for details.

In the case of symmetric rate, an outer bound on \( C_{e,PU} \) can be given as follows.

**Proposition 1.** If the symmetric rate \( R \) is such that \( P_{e,u} \leq \epsilon \), then

\[ R \leq \min \left\{ \frac{1}{K(\theta - \epsilon)} \mathbb{E} \left[ \log_2 \left( 1 + P \min_{S \subset [K]} \sum_{i \in S} |H_i|^2 \right) \right], \log_2 \left( 1 - P \ln(1 - \epsilon) \right) \right\}, \forall \theta \in (\epsilon, 1] \]  

(17)

**Proof.** The first of the two terms in the min in (17) follows from Fano’s inequality (see [108], with \( \mu = K/n, M = 2^n R \) and taking \( n \to \infty \)). The second is a single-user based converse using a genie argument. See appendix A for details.

**Remark 2.** We note here that the second term inside the minimum in (17) is the same as the one we would obtain if we used strong converse for the MAC. To be precise, let \( \{ |H_{(1)}| > |H_{(2)}| > \ldots > |H_{(K)}| \} \) denote the order statistics of the fading coefficients. If \( R > \log (1 + P |H_{(1)}|^2) \) then, using a Genie that reveals the codewords (and fading gains) of \( t - 1 \) users corresponding to \( t - 1 \) largest fading coefficients, it can be seen that \( P_{e,u} \geq \frac{K - t + 1}{K} \). Setting \( t = \theta K \) and considering the limit as \( K \to \infty \) (with \( P = P_{tot}/K \)) we obtain \( S \leq -P_{tot} \log_2(1 - \epsilon) \) which is same as that obtained from the second term in (17) under these limits.

**C. Numerical evaluation**

First notice that \( C_{e,J} \) (under joint error) tends to \( \{ 0 \} \) as \( K \to \infty \) because, it can be seen, for the symmetric rate, by considering that order statistics of the fading coefficients that \( P_0(R) \to 1 \) for \( R_i = O(1/K) \). \( C_{e,PU} \), however, is more interesting. We evaluate trade-off between system spectral efficiency and the minimum energy-per-bit required for a target per-user error for the symmetric rate, in the limit \( K \to \infty \) and power scaling as \( O(1/K) \).
In the above figure we have also presented the performance of TDMA. That is, if we use orthogonalization then for any number of users $K$ (not necessarily large), we have
\[ \epsilon = P \left[ R > 1/K \log(1 + KP|H|^2) \right] \] (18)
where $\epsilon$ is the PUPE. Thus the sum-rate vs $E_b/N_0$ formula for orthogonalization is
\[ E_b/N_0 = \frac{2^S - 1}{S} \frac{1}{-\ln(1-\epsilon)} \] (19)
where $S$ is the sum-rate or the spectral efficiency.

We see that orthogonalization is suboptimal under the PUPE criterion. The reason is that it fails to exploit the multi-user diversity by allocating resources even to users in deep fades.

IV. MANY USER MAC: $K = \mu n$, $n \to \infty$

This is our main section. We consider the linear scaling regime where the number of users $K$ scales with $n$, and $n \to \infty$. We are interested in the tradeoff of minimum $E_b/N_0$ required for the PUPE to be smaller than $\epsilon$, with the user density $\mu$ ($\mu < 1$). So, we fix the message size $k$. Let $S = k\mu$ be the spectral efficiency.

We focus on the case of different codebooks, but under symmetric rate. So if $M$ denotes the size of the codebook, then $S = K \log M = \mu \log M$. Hence, given $S$ and $\mu$, $M$ is fixed. Let $P_{\text{tot}} = KP$ denote the total power. Therefore denoting by $\mathcal{E}$ the energy-per-bit, $\mathcal{E} = E_b/N_0 = \frac{nP}{\log_2 M} = \frac{P_{\text{tot}}}{S}$. For finite $E_b/N_0$, we need finite $P_{\text{tot}}$, hence we consider the power $P$ decaying as $O(1/n)$.

Let $C_j = \{c_{ij}^1, ..., c_{ij}^M \}$ be the codebook of user $j$, of size $M$. The power constraint is given by $\|c_{ij}\|^2 \leq nP = \mathcal{E} \log_2 M, \forall j \in [K], i \in [M]$. The collection of codebooks $\{C_j\}$ is called an $(n, M, \epsilon, \mathcal{E}, K)$-code if it satisfies the power constraint described before, and the per-user probability of error is smaller than $\epsilon$. Then, we can define the following fundamental limit for the channel
\[ \mathcal{E}^*(M, \mu, \epsilon) = \lim_{n \to \infty} \inf \{ \mathcal{E} : \exists(n, M, \epsilon, \mathcal{E}, K = \mu n) - \text{code} \} . \]

We make an important remark here that all the following results also hold for maximal per-user error (PUPE-max) as discussed in appendix C.

A. No-CSI

In this subsection, we focus on the no-CSI case. The difficulty here is that, a priori, we do not know which subset of the users to decode. We have the following theorem.

**Theorem IV.1.** Consider the channel (5) (no-CSI) with $K = \mu n$ where $\mu < 1$. Fix the spectral efficiency $S$ and target probability of error (per-user) $\epsilon$. Let $M = 2^{S/\mu}$ denote the size of the codebook and $P_{\text{tot}} = KP$ be the total power. Fix $\nu \in (1-\epsilon, 1]$. Let $\epsilon' = \epsilon - (1-\nu)$. Then if $\mathcal{E} > \mathcal{E}^*_\text{no-CSI} = \sup_{\nu < \theta \leq 1} \sup_{\xi \in [0, \nu(1-\theta)]} \frac{P_{\text{tot}}}{S}(\theta, \xi)$, there exists a sequence of $(n, M, \epsilon_n, \mathcal{E}, K = \mu n)$ codes such that $\limsup_{n \to \infty} \epsilon_n \leq \epsilon$, where, for $\frac{\epsilon'}{\nu} < \theta \leq 1$ and $\xi \in [0, \nu(1-\theta)]$,
\[ P_{\text{tot,nu}}(\theta, \xi) = \frac{1 - \hat{f}(\theta, \xi)}{1 - \hat{f}(\theta, \xi) \alpha(\xi + \nu \theta, \xi + 1 - \nu(1-\theta))} \] (20)
\[ \hat{f}(\theta, \xi) = \frac{f(\theta)}{\alpha(\xi, \xi + \nu \theta)} \] (21)

Fig. 1: $S$ vs $E_b/N_0$ for per-user error $\epsilon \leq 0.1$, $K$-fixed (but $K \gg 1$) $n \to \infty$
f(θ) = \frac{1 + \delta_1^2(1 - V_0)}{V_0} - 1 \quad (22)

V_0 = e^{-\theta}

\hat{V}_0 = \delta' + \frac{\theta\mu\ln M}{1 - \mu\nu} + \frac{1 - \mu(1 - \theta)}{1 - \mu(1 - \theta)} h\left(\frac{\theta\mu}{1 - \mu(1 - \theta)}\right) + \frac{\mu(1 - \nu(1 - \theta))}{1 - \mu(1 - \theta)} h\left(\frac{\theta\nu}{1 - \mu(1 - \theta)}\right) \quad (24)

\delta' = \frac{\mu h(1 - \nu(1 - \theta))}{1 - \mu

\delta^* = \frac{\mu h(1 - \nu(1 - \theta))}{1 - \mu \nu} \quad (25)

\delta^* = \frac{\mu h(1 - \nu(1 - \theta))}{1 - \mu \nu} \quad (25)

\delta^* \leq E^* \leq E^*_{\text{no-CSI}}.

**Proof.** The proof uses random coding. Let each user generate a Gaussian codebook of size $M$ and power $P' < P$ independently such that $KP' = P_{\text{tot}} < P_{\text{tot}}$. Let $W_j$ denote the random (in $[M]$) message of user $j$. So, if $C_j = \{c^{i_j}_j : i \in [M]\}$ is the codebook of user $j$, he transmits $X_j = c^{i_j}_j, \{\|c^{i_j}_j\|^2 \leq nP\}$. For simplicity let $(c_1, c_2, ..., c_K)$ be the sent codewords. Fix $\nu \in (1 - \epsilon, 1]$. Let $K_1 = \nu K$ be the number of users that are decoded. Since there is no knowledge of CSIR, it is not possible to, a priori, decide what set to decode. Instead, the decoder searches of all $K_1$ sized subsets of $[M]$. Formally, let $\hat{?}$ denote an error symbol. The decoder output $g_D(Y) \in \prod_{i=1}^K C_i$ is given by

$$\hat{S}, (\hat{i}_j)_{i \in \hat{S}} = \arg \max_{S \subseteq [K]} \max_{(c_i)_{i \in S}} \|P_{(c_i)_{i \in S}}Y\|^2$$

$$(g_D(Y))_i = \begin{cases} f^{-1}_i(\hat{i}_j) & i \in \hat{S} \\ ? & i \notin \hat{S} \end{cases} \quad (31)$$

where $f_i$ are the encoding functions. The probability of error (averaged over random codebooks) is given by

$$P_e = \frac{1}{K} \sum_{j=1}^K \mathbb{P} [W_j \neq \hat{W}_j] \quad (32)$$

where \(\hat{W}_j = (g(Y))_j\) is the decoded message of user $j$.

We perform a change of measure to $X_j = c^{i_j}_j$. Since $P_e$ is the expectation of a non-negative random variable bounded by 1, this measure change adds a total variation distance which can bounded by $P_0 = KP_h \left(\frac{\chi^2(2n)}{2n} > \frac{P}{P'}\right) \rightarrow 0$ as $n \rightarrow \infty$, where $\chi^2(d)$ is the distribution of sum of squares of $d$ iid standard normal random variables (the chi-square distribution). The reason is as follows. If we have two random vectors $U_1$ and $U_2$ on a the same probability space such that $U_1 = U_21[U_2 \in E]$, where $E$ is a Borel set, then for any Borel set $A$, we have

$$|\mathbb{P} [U_1 \in A] - \mathbb{P} [U_2 \in A]| = |\mathbb{P} [U_2 \in A \cap E^c] \mathbb{P} [U_2 \in E^c] - \mathbb{P} [U_2 \in A \cap E^c] |$$

$$\quad \leq \mathbb{P} [U_2 \in E^c]. \quad (33)$$

Henceforth we only consider the new measure.
Let $\epsilon > 1 - \nu$ and $\epsilon' = \epsilon - (1 - \nu)$. Now we have

$$P_e \leq \epsilon + \mathbb{P} \left[ \frac{1}{K} \sum_{j=1}^{K} 1[W_j \neq \hat{W}_j] > \epsilon \right]$$

$$= \epsilon + \mathbb{P} \left[ \sum_{j=1}^{K} 1[W_j \neq \hat{W}_j] > K\epsilon' + K - K_1 \right]$$

$$= \epsilon + p_1.$$  \hspace{1cm} (34)

where $p_1 = \mathbb{P} \left[ \bigcup_{t=\epsilon'K}^{K} \left\{ \sum_{j=1}^{K} 1[W_j \neq \hat{W}_j] = K - K_1 + t \right\} \right]$.

Let $F_t = \left\{ \sum_{j=1}^{K} 1[W_j \neq \hat{W}_j] = K - K_1 + t \right\}$. Let $c_{[S]} = \{ c_i : i \in S \}$ and $H_{[S]} = \{ H_i : i \in S \}$, where $S \subset [K]$.

Conditioning on $c_{[K]}, H_{[K]}$ and $Z$, we have

$$\mathbb{P} \left[ F_t | c_{[K]}, H_{[K]}, Z \right] \leq \mathbb{P} \left[ \exists S \subset [K] : |S| = K - K_1 + t, \exists S_1 \subset S : |S_1| = t, \exists c' \in c_i : i \in S_1, c' \neq c_i \right]:$$

$$\mathbb{E} \left[ \frac{P_{c_{[S_2]}}, c_{[S_1]} Y} {\max_{S_2 \subset S} \left( \frac{P_{c_{[S_2]}, c_{[S_1]} Y}} {c_{[K]}, H_{[K]}, Z} \right)} \right]$$

$$\leq \mathbb{P} \left[ \bigcup_{S_2} \bigcup_{S_1} \bigcup_{c' \in c_i : i \in S_1, c' \neq c_i} F(S, S_2, S_1, t) \big| c_{[K]}, H_{[K]}, Z \right].$$  \hspace{1cm} (35)

where $F(S, S_2, S_1, t) = \left\{ \left( \frac{P_{c_{[S_2]}}, c_{[S_1]} Y} {\max_{S_2 \subset S} \left( \frac{P_{c_{[S_2]}, c_{[S_1]} Y}} {c_{[K]}, H_{[K]}, Z} \right)} \right) > \left( \frac{P_{c_{[S_2]}}, c_{[S_1]} Y} {\max_{S_2 \subset S} \left( \frac{P_{c_{[S_2]}, c_{[S_1]} Y}} {c_{[K]}, H_{[K]}, Z} \right)} \right) \right\}$, and $S_2 \subset S$ is a possibly random (depending only on $H_{[K]}$) subset of size $t$, to be chosen later. Next we will bound $\mathbb{P} \left[ F(S, S_2, S_1, t) \big| c_{[K]}, H_{[K]}, Z \right]$.

For the sake of brevity, let $A_0 = c_{S_2} \cup c_{[K] \setminus S}$, $A_1 = c_{[K] \setminus S}$ and $B_1 = c_{[S_1]}$. We have the following claim.

Claim 1. For any $S_1 \subset S$ with $|S_1| = t$, conditioned on $c_{[K]}$, $H_{[K]}$ and $Z$, the law of $\left( \frac{P_{c_{[S_2]}}, c_{[S_1]} Y} {\max_{S_2 \subset S} \left( \frac{P_{c_{[S_2]}, c_{[S_1]} Y}} {c_{[K]}, H_{[K]}, Z} \right)} \right)$ is the same as the law of $\|P_\mathcal{A} Y\|^2 + \|(I - P_\mathcal{A}) Y\|^2 \text{Beta}(t, n - K_1)$ where Beta$(a, b)$ is a beta distributed random variable with parameters $a$ and $b$.

Proof. Let us write $V = \text{span}(A_1, B_1) = \mathcal{A} \oplus \mathcal{B}$ where $A \perp B$ are subspaces of dimension $K_1 - t$ and $t$ respectively, with $A = \text{span}(A_1)$ and $B$ is the orthogonal complement of $A_1$ in $V$. Hence $\|P_\mathcal{A} Y\|^2 = \|P_\mathcal{A} Y\|^2 + \|P_\mathcal{A} Y\|^2$ (by definition, $P_\mathcal{A} = P_{A_1}$). Now we analyze $\|P_{\mathcal{B}} Y\|^2$. We can further write $P_{\mathcal{B}} Y = P_{\mathcal{B}} P_{\mathcal{A}} Y$. Observe that the subspace $B$ is the span of $P_{\mathcal{A}} B_1$, and, conditionally, $P_{\mathcal{A}} B_1 \sim \mathcal{CN}^{\|S\|}(0, P_{\mathcal{A}} P_{\mathcal{B}})$ which is the product measure of $|S|$ complex normal vectors in a subspace of dimension $n - K_1 + t$. Hence, the conditional law of $\|P_{\mathcal{B}} P_{\mathcal{A}} Y\|^2$ is the law of squared length of projection of a fixed $n - K_1 + t$ dimensional vector of length $\|(I - P_\mathcal{A}) Y\|^2$ onto a uniformly random $t$ dimensional subspace.

Further, the law of the squared length of the orthogonal projection of a fixed unit vector in $\mathbb{C}^d$ onto a random $t$-dimensional subspace is the same as the law of the squared length of the orthogonal projection of a random unit vector in $\mathbb{C}^d$ onto a fixed $t$-dimensional subspace, which is Beta$(t, d - t)$ (see e.g., [29], Eq. 79)): that is, if $u$ is a unit random vector in $\mathbb{C}^d$ and $L$ is a fixed $t$ dimensional subspace, then $P \left[ \|P_L u\|^2 \leq x \right] = \int_0^1 \frac{1}{t} \int_0^x \frac{1}{t} \int_0^x \cdots \int_0^x \beta_{t-1} (|Z_1|^2, \ldots, |Z_t|^2) \frac{\prod_{i=1}^t |Z_i|^2}{\sum_{i=1}^t |Z_i|^2} \frac{dwdw}{\cdots dw}$. Hence the conditional law of $\|P_{\mathcal{B}} P_{\mathcal{A}} Y\|^2$ is $\|(I - P_\mathcal{A}) Y\|^2 \text{Beta}(t, n - K_1)$.

Therefore we have,

$$\mathbb{P} \left[ F(S, S_2, S_1, t) \big| c_{[K]}, H_{[K]}, Z \right] = \mathbb{P} \left[ \text{Beta}(n - K_1, t) < G_S(c_{[K]}, H_{[K]}, Z) \right] = F_B \left( G_S; n - K_1, t \right)$$  \hspace{1cm} (36)

where

$$G_S = \frac{\|Y\|^2 - \|P_{A_1} Y\|^2}{\|Y\|^2 - \|P_{A_1} Y\|^2}.$$  \hspace{1cm} (37)
Since \( t \geq 1 \), we have \( F_{\beta}(G_S; n - K_1, t) \leq \binom{n-K_1+t-1}{t-1}G^{n-K_1}_S \).

Let us denote \( \bigcup_{t=\ell+1}^{\ell+K} \) as \( \bigcup_{\ell} \bigcup_{S \subseteq [K]} \) as \( \bigcup_{S \subseteq [K]} \), and \( \bigcup_{\ell} \bigcup_{S \subseteq [K]} \) as \( \bigcup_{\ell, S, K_1} \); similarly for \( \sum \) and \( \cap \) for the ease of notation. Using the above claim, we get,

\[
\mathbb{P} \left[ F_t | c_K, H[K], Z \right] \leq \sum_{S, K_1} \binom{K - K_1 + t}{t} M^t \binom{n - K_1 + t - 1}{t - 1} G^{n-K_1}_S. \tag{38}
\]

Therefore \( p_1 \) can be bounded as

\[
p_1 = \mathbb{P} \left[ \bigcup_t F_t \right] \leq \mathbb{E} \left[ \min \left\{ 1, \sum_{t, S, K_1} \binom{K - K_1 + t}{t} M^t \binom{n - K_1 + t - 1}{t - 1} G^{n-K_1}_S \right\} \right] = \mathbb{E} \left[ \min \left\{ 1, \sum_{t, S, K_1} e^{(n-K_1)s_t} M^t G^{n-K_1}_S \right\} \right] \tag{39}
\]

where \( s_t = \ln(\binom{K - K_1 + t}{t}) \sum_{t=\ell+1}^{\ell+K} \binom{n-K_1+t-1}{t-1} G^{n-K_1}_S \).

Now we can bound the binomial coefficient \([27] \), Ex. 5.8 as

\[
\binom{n - K_1 + t - 1}{t - 1} = \sqrt{\frac{n-K_1+t-1}{2\mu(t-1)(n-K_1)}} e^{(n-K_1+t-1)h(\frac{\mu}{\mu-\mu})} = O \left( \frac{1}{\sqrt{n}} \right) e^{\mu(1-\mu)(1-\mu)h(\frac{\mu}{1-\mu})}. \tag{40}
\]

Similarly, \( \binom{K - K_1 + t}{t} \leq O \left( \frac{1}{\sqrt{n}} \right) e^{\mu(1-\mu)(1-\mu)h(\frac{\mu}{1-\mu})} \tag{41} \)

Let \( r_t = s_t + \frac{t \ln M}{n-K_1} \). For \( \delta > 0 \), define \( \tilde{V}_{n,t} = r_t + \delta \) and \( V_{n,t} = e^{-\tilde{V}_{n,t}} \). Let \( E_1 \) be the event

\[
E_1 = \bigcap_{t, S, K_1} \left\{ - \ln G_S - r_t > \delta \right\} = \bigcap_{t, S, K_1} \left\{ G_S < V_{n,t} \right\}. \tag{42}
\]

Let \( p_2 = \mathbb{P} \left[ \bigcup_{t, S, K_1} \left\{ G_S > V_{n,t} \right\} \right] \). Then

\[
p_1 \leq \mathbb{E} \left[ \min \left\{ 1, \sum_{t, S, K_1} e^{(n-K_1)r_t} G^{n-K_1}_S \right\} (1[E_1] + 1[E_1^c]) \right] \leq \mathbb{E} \left[ \sum_{t, S, K_1} e^{-(n-K_1)\delta} \right] + p_2 \]

\[
= \sum_t \left( K - K_1 + t \right) e^{-(n-K_1)\delta} + p_2. \tag{43}
\]

Observe that, for \( t = \theta K_1 + \theta \nu K_1, s_t = \frac{\theta \nu h}{1-\mu} \left( \frac{\theta \nu}{1-\mu} \right) + \frac{\nu(1-\nu)(1-\mu)h}{1-\nu(1-\mu)} = O \left( \frac{\ln(n)}{n} \right) \) and \( r_t = s_t + \frac{\theta \nu}{1-\mu} \ln M \). Therefore \( n \to \infty \) with \( \theta \) fixed, we have

\[
\lim_{n \to \infty} \tilde{V}_{n, \theta \nu, M} = \tilde{V}_{\theta} \tag{44}
\]

where \( \tilde{V}_{\theta} \) is given in \([24] \).
Now, note that, for $1 < t < K_1$, 
\[ \left( \frac{K}{K - K_1 + t} \right) \leq \sqrt{\frac{K}{2\pi(K - K_1 + t)(K_1 - t)}} e^{K_h \left( \frac{K - K_1 + t}{K} \right)}. \] 
(45)

Hence choosing $\delta > \frac{K_h(K - K_1 + t)}{n - K_1}$ will ensure that the first term in (43) goes to 0 as $n \to \infty$. So for $t = \theta K_1 = \theta \nu K$, we need to have
\[ \delta > \delta^*. \] 
(46)

where $\delta^*$ is given in (25).

Let us bound $p_2$. Let $Z = Z + \sum_{i \in S \setminus S_2} H_i c_i$. We have

Claim 2.
\[ p_2 = \mathbb{P} \left[ \bigcup_{t,S,K_1} \{ G_S > V_{n,t} \} \right] \leq \mathbb{P} \left[ \bigcup_{t,S,K_1} \left\{ (1 - V_{n,t}) P_{A_1}^\perp \dot{Z} - V_{n,t} P_{A_1}^\perp \sum_{i \in S_2^\perp} H_i c_i \right\}^2 \geq V_{n,t} \left\{ P_{A_1}^\perp \sum_{i \in S_2^\perp} H_i c_i \right\}^2 \right]. \] 
(47)

Proof. See appendix B

Let $\chi^2_2(\lambda, d)$ denote the non-central chi-squared distributed random variable with non-centrality $\lambda$ and degrees of freedom $d$. That is, if $W_i \sim \mathcal{N}(\mu_i, 1), i \in [d]$ and $\lambda = \sum_{i \in [d]} \mu_i^2$, then $\chi^2_2(\lambda, d)$ has the same distribution as that of $\sum_{i \in [d]} W_i^2$. We have the following claim.

Claim 3. Conditional on $H_{[K]}$ and $A_0$,
\[ \left\| P_{A_1}^\perp \left( \dot{Z} - \frac{V_{n,t}}{1 - V_{n,t}} \sum_{i \in S_2^\perp} H_i c_i \right) \right\|^2 \sim \left( 1 + P' \sum_{i \in S \setminus S_2} |H_i|^2 \right) \frac{1}{2} \chi^2_2(2F, 2n') \] 
(48)

where
\[ F = \frac{\left\| \frac{V_{n,t}}{1 - V_{n,t}} P_{A_1}^\perp \sum_{i \in S_2^\perp} H_i c_i \right\|^2}{1 + P' \sum_{i \in S \setminus S_2^\perp} |H_i|^2} \] 
(49)

\[ n' = n - K_1 + t. \] 
(50)

Hence its conditional expectation is
\[ \mu = n' + F. \] 
(51)

Proof. See appendix B

Now let
\[ T = \frac{1}{2} \chi^2_2(2F, 2n') - \mu \] 
(52)

\[ U = \frac{V_{n,t}}{1 - V_{n,t}} \frac{\left\| P_{A_1}^\perp \sum_{i \in S_2^\perp} H_i c_i \right\|^2}{1 + P' \sum_{i \in S \setminus S_2^\perp} |H_i|^2} - n' \] 
(53)

\[ U^1 = \frac{1}{1 - V_{n,t}} (V_{n,t} W_S - 1) \] 
(54)

where $W_S = \left( 1 + \frac{\left\| P_{A_1}^\perp \sum_{i \in S \setminus S_2} H_i c_i \right\|^2}{n'(1 + P' \sum_{i \in S \setminus S_2^\perp} |H_i|^2)} \right)$. Notice that $U = n' U^1$ and $F = \frac{V_{n,t}}{1 - V_{n,t}} n'(1 + U^1)$. Then we have
Lemma IV.2 \cite{28}. Let $\chi = \chi_2^2(\lambda, d)$ be a non-central chi-squared distributed variable with $d$ degrees of freedom and non-centrality parameter $\lambda$. Then $\forall x > 0$

\[
P\left[ \chi - (d + \lambda) \geq 2\sqrt{(d + 2\lambda)x + 2x} \right] \leq e^{-x} \]

\[
P\left[ \chi - (d + \lambda) \leq -2\sqrt{(d + 2\lambda)x} \right] \leq e^{-x} \]

(57)

Hence, for $x > 0$, we have

\[
P\left[ \chi - (d + \lambda) \geq x \right] \leq e^{-\frac{1}{4}(x + d + 2\lambda - \sqrt{d + 2\lambda}\sqrt{d x + d + 2x})}. \]

(58)

and for $x < (d + \lambda)$, we have

\[
P\left[ \chi \leq x \right] \leq e^{-\frac{1}{4}\left(\frac{(d + \lambda - x)^2}{d + 2x}\right)}. \]

(59)

Observe that, in (58), the exponent is always negative for $x > 0$ and finite $\lambda$ due to AM-GM inequality. When $\lambda = 0$, we can get a better bound for the lower tail in (59) by using \cite{21} Lemma 25.

Lemma IV.3 \cite{21}. Let $\chi = \chi_2(d)$ be a chi-squared distributed variable with $d$ degrees of freedom. Then $\forall x > 1$

\[
P\left[ \chi \leq \frac{d}{x} \right] \leq e^{-\frac{d}{2}\left(\ln x + \frac{1}{x} - 1\right)} \]

(60)

Therefore, from (47), (55), (56) and (58), we have

\[
p_2 \leq \sum_{t, S, K_1} E\left[ e^{-n'f_n(U^1)} 1[U^1 > \delta_1] \right] + P\left[ \bigcup_{t, S, K_1} \{U^1 \leq \delta_1\} \right] \]

(61)

where $f_n$ is given by

\[
f_n(x) = x + 1 + \frac{2V_{n,t}}{1 - V_{n,t}}(1 + x) - \sqrt{1 + \frac{2V_{n,t}}{1 - V_{n,t}}(1 + x)} \sqrt{2x + 1 + \frac{2V_{n,t}}{1 - V_{n,t}}(1 + x)}. \]

(62)

Next, we have the following claim.

Claim 4. For $0 < V_{n,t} < 1$ and $x > 0$, $f_n(x)$ is a monotonically increasing function of $x$.

Proof. See appendix B.

From this claim, we get

\[
p_2 \leq \sum_{t, S, K_1} e^{-n'f_n(\delta_1)} + p_3 \]

(63)
where \( p_3 = P[E_2^c] \).

Now, if, for each \( t, \delta_1 \) is chosen such that \( f_n(\delta_1) > \frac{K_h(K-1+t)}{n-K_h+t} \), then the first term in (63) goes to 0 as \( n \to \infty \). Therefore, for \( t = \theta K_1 \), setting \( c_0 \) and \( q_0 \) as in (26) and (27) respectively, and choosing \( \delta_1 \) such that

\[
\delta_1 > \delta_1^* \tag{64}
\]

with \( \delta_1^* \) given by (28), will ensure that the first term in (63) goes to 0 as \( n \to \infty \).

Note that

\[
p_3 = P[E_2^c] = P \left[ \bigcup_{t,S,K_1} \left\{ V_{t,S} - 1 \leq \delta_1(1-V_{n,t}) \right\} \right]. \tag{65}
\]

Conditional on \( H[K] \), \( \left\| P \sum_{i \in S_2} H_i c_i \right\|^2 \sim \frac{1}{2} P \sum_{i \in S_2} |H_i|^2 \chi_2^2(2n') \), where \( \chi_2(2n') \) is a chi-squared distributed random variable with \( 2n' \) degrees of freedom (here the superscript \( S_2^* \) denotes the fact that this random variable depends on the codewords corresponding to \( S_2^* \)). For \( 1 > \delta_2 > 0 \), consider the event \( E_4 = \bigcap_{t,S,K_1} \left\{ \frac{\chi_2^2(2n')}{2n'} > 1 - \delta_2 \right\} \). Using (60), we can bound \( p_3 \) as

\[
p_3 \leq \sum_t \left( \frac{K}{K_1 + t} \right) e^{-n(1-\delta_2)\delta_2} + p_4 \tag{66}
\]

where

\[
p_4 = P[E_3^c] = P \left[ \bigcup_{t,S,K_1} \left\{ V_{n,t} \left( 1 + \frac{P' \sum_{i \in S_2} |H_i|^2(1-\delta_2)}{1 + P' \sum_{i \in S \setminus S_2^*} |H_i|^2} \right) \leq 1 + \delta_1(1-V_{n,t}) \right\} \right]. \tag{67}
\]

Again, it is enough to choose \( \delta_2 \) such that

\[
\delta_2 > \delta_2^* \tag{68}
\]

with \( \delta_2^* \) given by (29), to make sure that the first term in (66) goes to 0 as \( n \to \infty \).

Note that the union bound over \( S \) is the minimum over \( S \), and this minimizing \( S \) should be contiguous amongst the indices arranged according the decreasing order of fading powers. Further, \( S_2^* \) is chosen to be corresponding to the top \( t \) fading powers in \( S \). Hence, we get

\[
p_4 = P \left[ \bigcup_t \left\{ \min_{0 \leq j \leq K_1-t} \left( \frac{P \sum_{i=j+1}^{j+t} |H(i)|^2(1-\delta_2)}{1 + P \sum_{i=j+t+1}^{K_1} |H(i)|^2} \right) \leq 1 + \delta_1(1-V_{n,t}) \right\} \right]. \tag{69}
\]

We make the following claim

**Claim 5.**

\[
\limsup_{n \to \infty} p_4 \leq 1 \left[ \bigcup_{\theta \in (0,1/2]} \inf_{\xi \in [0,\nu(1-\theta)]} \left( \frac{(1-\delta_2)P_{tot} \alpha(\xi, \xi + \nu \theta)}{1 + P_{tot} \alpha(\xi + \nu \theta, \xi + 1 - \nu(1-\theta))} \right) \leq 1 + \delta_1(1-V_{n,t}) \right] \tag{70}
\]

where \( \alpha(a, b) \) is given by (30).

**Proof.** We have \( |H_1|^2, \ldots, |H_K|^2 \) with CDF \( F(x) = (1-e^{-x})1[x \geq 0] \). Let \( \hat{F}_K(x) = \frac{1}{K} \sum_{i=1}^{K} 1[|H_i|^2 \leq x] \) be the empirical CDF (ECDF). Then standard Chernoff bound gives, for \( 0 < r < 1 \),

\[
P \left[ |\hat{F}_K(x) - F(x)| > r F(x) \right] \leq 2 e^{-KcF(x)r^2} \tag{71}
\]

where \( c \) is some constant.

From (29), we have the following representation. Let \( 0 < \gamma < 1 \). Then

\[
|H_{(n \gamma)}|^2 = F^{-1}(1-\gamma) - \frac{\hat{F}_K(F^{-1}(1-\gamma)) - (1-\gamma)}{f(F^{-1}(1-\gamma))} + R_K \tag{72}
\]

where \( f \) is the pdf corresponding to \( F \), and with probability 1, we have \( R_K = O(n^{-3/4} \log(n)) \) as \( n \to \infty \).
Let $\tau > 0$. Then using (71) and (72), we have

$$||H_{(n,\gamma)}||^2 - F^{-1}(1 - \gamma) \leq O\left(\frac{1}{n^{1-\tau}}\right)$$

(73)

with probability at least $1 - e^{-O(n^{1-\tau})}$.

Hence, for $0 < \xi < \zeta < 1$, we have, with probability $1 - e^{-O(n^{1-\tau})}$,

$$\frac{1}{K} \sum_{i=\lfloor nK \rfloor}^{\lceil \beta K \rceil} |H_{(i)}|^2 = \left[\frac{1}{K} \sum_{i=1}^{K} |H_i|^2 \mathbf{1} \left[ b \leq |H_i|^2 \leq a \right]\right] + o(1)$$

(74)

where $a = F^{-1}(1 - \xi)$ and $b = F^{-1}(1 - \zeta)$. Now, by law of large numbers (and Bernstein’s inequality [30]), with overwhelming probability (exponentially close to 1), we have

$$\frac{1}{K} \sum_{i=1}^{K} |H_i|^2 \mathbf{1} \left[ b \leq |H_i|^2 \leq a \right] = \int_{b}^{a} x dF(x) + o(1)$$

(75)

and $\int_{b}^{a} x dF(x) = \int_{\xi}^{\zeta} F^{-1}(1 - \gamma) d\gamma = \alpha(\xi, \zeta)$.

Define the events

$$J_{n,\theta,\xi} = \left\{ \begin{array}{l}
\left( P' \sum_{i=\lfloor \xi(1+\nu)K \rfloor+1}^{\lfloor (\xi+\nu\theta)K \rfloor} |H_{(i)}|^2 (1 - \delta_2) \right)
\leq 1 + \delta_1 \left(1 - V_{n,[\theta\nu K]}\right) - 1
\end{array} \right\}$$

(76)

$$I_{n,\theta,\xi} = \left\{ \begin{array}{l}
\left( 1 - \delta_2 \right) P' \sum_{i=\lfloor (\xi+\nu\theta)K \rfloor+1}^{\lfloor (\xi+\nu(1-\theta))K \rfloor} |H_{(i)}|^2 
\leq 1 + \delta_1 \left(1 - V_{n,[\theta\nu K]}\right) - 1
\end{array} \right\}$$

(77)

$$I_{\theta,\xi} = \left\{ \begin{array}{l}
\left( 1 - \delta_2 \right) P' \sum_{i=\lfloor (\xi+\nu\theta)K \rfloor+1}^{\lfloor (\xi+\nu(1-\theta))K \rfloor} |H_{(i)}|^2 
\leq 1 + \delta_1 \left(1 - V_{\theta}\right) - 1
\end{array} \right\}$$

(78)

$$E_{n,\theta,\xi} = \left\{ \begin{array}{l}
\frac{1}{K} \sum_{i=\lfloor \xi K \rfloor+1}^{\lceil (\xi+\nu\theta)K \rceil} |H_{(i)}|^2 - \alpha \left( \xi, \xi + \nu\theta \right) \leq o(1)
\end{array} \right\} \bigcap \left\{ \begin{array}{l}
\frac{1}{K} \sum_{i=\lfloor (\xi+\nu(1-\theta))K \rfloor+1}^{\lceil (\xi+\nu(1-\theta))K \rceil} |H_{(i)}|^2 - \alpha \left( \xi, \xi + \nu(1-\theta) \right) \leq o(1)
\end{array} \right\}$$

(79)

$$E_n = \bigcap_{\theta \in A_n, \xi \in B_{K,\theta}} E_{n,\theta,\xi}$$

(80)

where $A_n = \left\{ \frac{\xi}{2}, 1 \right\} \cap \left\{ \frac{K_i}{K} : i \in [K_1] \right\}$ and $B_{K,\theta} = [0, (1 - \theta)] \cap \left\{ \frac{1}{K} : i \in [K] \right\}$. Note that, from (74) and (75), $P\left[ E_{n,\theta,\xi}^c \right]$ is exponentially small in $n$.

Then we have

$$p_4 = P \left[ \bigcup_{\theta \in A_n, \xi \in B_{K,\theta}} J_{n,\theta,\xi} \right]$$

$$\leq P \left[ \bigcup_{\theta \in A_n, \xi \in B_{K,\theta}} J_{n,\theta,\xi} \cap E_{n,\theta,\xi} \right] + \sum_{\theta \in A_n, \xi \in B_{K,\theta}} P \left[ E_{n,\theta,\xi}^c \right]$$

$$\leq 1 \left[ \bigcup_{\theta \in A_n, \xi \in B_{K,\theta}} I_{n,\theta,\xi} \right] + o(1)$$

$$\leq 1 \left[ \bigcup_{\theta \in [0,1] : \xi \in [0,\nu(1-\theta)]} I_{n,\theta,\xi} \right] + o(1).$$

(81)
Since the receives knows $H$, let $W$ output the number of users that are decoded. Fix a decoding set $D$. X

\[
\hat{W} = \arg \min_{\hat{W} \in \mathcal{W}} \left\{ \left\| Y - \sum_{i \in D} H_i c_i \right\|^2 \right\}.
\]

The probability of error is given by

\[
P_e = \frac{1}{K} \sum_{j=1}^{K} \mathbb{P} \left[ W_j \neq \hat{W}_j \right]
\]

where $\hat{W}_j = (g(Y))_j$ is the decoded message of user $j$. Similar to the no-CSI case, we perform a change of measure to $X_j = c_{W_j}$ by adding a total variation distance bounded by $p_0 = K \mathbb{P} \left[ \frac{\chi^2(2n)}{2n} > \frac{p}{f} \right] \to 0$ as $n \to \infty$.

Let $\epsilon' = \epsilon - (1 - \nu)$. Now we have

\[
P_e = \mathbb{E} \left[ \frac{1}{K} \sum_{j=1}^{K} 1 \{ W_j \neq \hat{W}_j \} \right]
\]
\[ \frac{K - K_1}{K} + \mathbb{E} \left[ \frac{1}{K} \sum_{j \in D} 1\{W_j \neq \hat{W}_j\} \right] \]
\[ \leq (1 - \nu) + \epsilon' + \nu \mathbb{P} \left[ \frac{1}{K} \sum_{j \in D} 1\{W_j \neq \hat{W}_j\} \geq \epsilon' \right] \]
\[ = \epsilon + \nu p_1 \]  
(86)

where \( p_1 = \mathbb{P} \left[ \bigcup_{i \in \mathcal{K}} \{ \sum_{j \in D} 1\{W_j \neq \hat{W}_j\} = t \} \right] \).

From now on, we just write \( \bigcup_i \) to denote \( \bigcup_{i \in \mathcal{K}} \cdot \sum_t \) for \( \sum_{i \in \mathcal{K}} \), and \( \sum_S \) for \( \sum_{S \subseteq D} \). Let \( c_i[S] \equiv \{ c_i : i \in [S] \} \) and \( H[K] = \{ H_i : i \in [K] \} \).

Let \( F_t = \{ \sum_{j \in D} 1\{W_j \neq \hat{W}_j\} = t \} \). Let \( \rho \in [0, 1] \). We bound \( \mathbb{P}[F_t] \) using Gallager’s rho trick similar to [5] as

\[ \mathbb{P}[F_t | Z, c[K], H[K]] \leq \mathbb{P}[\exists S \subset D : |S| = t, \exists \{ c'_i \in C_i : i \in S, c'_i \neq c_i \} : \]
\[ \left\| Y - \sum_{i \in S} H_i c_i - \sum_{i \in D \setminus S} H_i c_i \right\|^2 < \left\| Y - \sum_{i \in D} H_i c_i \right\|^2 \mathbb{E}[Z_i c[K], H[K]] \]
\[ \leq \sum_S \mathbb{P} \left[ \bigcup_{c'_i \in C_i, c_i \in S, c'_i \neq c_i} \left\{ \left\| Z_D + \sum_{i \in S} H_i c_i - \sum_{i \in S} H_i c'_i \right\|^2 < \left\| Z_D \right\|^2 \right\} \right] \]
\[ \leq \sum_S M^{\rho t} \mathbb{P} \left[ \left\| Z_D + \sum_{i \in S} H_i c_i - \sum_{i \in S} H_i c'_i \right\|^2 < \left\| Z_D \right\|^2 \mathbb{E}[Z_i c[K], H[K]] \right]^\rho \]  
(87)

where \( Z_D = Z + \sum_{i \in [K] \setminus D} H_i c_i \) and \( c'_i[S] \) in the last display denotes a generic set of unsent codewords corresponding to codebooks of users in set \( S \).

We use the following simple lemma which is a trivial extension of a similar result used in [5] to compute the above probability.

**Lemma IV.5.** Let \( Z \sim \mathcal{CN}(0, I_n) \) and \( u \in \mathbb{C}^n \). Let \( D = \text{diag}(d_1, \ldots, d_n) \in \mathbb{C}^{n \times n} \) be a diagonal matrix. If \( \gamma > \sup_{j \in [n]} -\frac{1}{|d_j|^2} \), then

\[ \mathbb{E} \left[ e^{-\gamma \|DZ + u\|^2} \right] = \frac{1}{\prod_{j \in [n]} (1 + \gamma |d_j|^2)} e^{-\gamma \sum_{j \in [n]} |u_j|^2} \frac{|u_j|^2}{1 + \gamma |d_j|^2} \]

**Proof.** Omitted.

\[ \square \]

So, using the above lemma, we have, for \( \lambda_1 > 0 \),

\[ \mathbb{E}(c_S) \left[ \mathbb{P} \left[ \left\| Z_D + \sum_{i \in S} H_i c_i - \sum_{i \in S} H_i c'_i \right\|^2 < \left\| Z_D \right\|^2 \mathbb{E}[Z_i c[K], H[K]] \right] \right]^\rho \]
\[ = \mathbb{E}(c_S) \left[ \mathbb{P} \left[ \exp \left( -\lambda_1 \left\| Z_D + \sum_{i \in S} H_i c_i - \sum_{i \in S} H_i c'_i \right\|^2 \right) > \exp \left( -\lambda_1 \left\| Z_D \right\|^2 \right) \mathbb{E}[Z_i c[K], H[K]] \right] \right]^\rho \]
\[ \leq \frac{e^{\rho \lambda_1} \left\| Z_D \right\|^2}{(1 + \lambda_1 P' \sum_{i \in S} |H_i|^2)^{\rho t} e^{-\rho \lambda_1 \sum_{i \in S} |H_i|^2}} \]
\[ \leq \frac{e^{\rho \lambda_1} \left\| Z_D \right\|^2}{(1 + \lambda_1 P' \sum_{i \in S} |H_i|^2)^{\rho t} e^{-\rho \lambda_1 \sum_{i \in S} |H_i|^2}} \]
(88)

where \( \mathbb{E}(c_S) \) denotes taking expectation with respect to \( \{ c'_i : i \in S \} \) alone, and \( 1 + \lambda_1 P' \sum_{i \in S} |H_i|^2 > 0 \).

Let \( \lambda_2 = \frac{\rho \lambda_1}{1 + \lambda_1 P' \sum_{i \in S} |H_i|^2} \). Note that \( \lambda_2 \) is a function of \( H_S \). Now using lemma [IV.5] again to take expectation over \( c_S \), we get

\[ \mathbb{E}(c_S) \left[ \frac{e^{\rho \lambda_1} \left\| Z_D \right\|^2}{(1 + \lambda_1 P' \sum_{i \in S} |H_i|^2)^{\rho t} e^{-\rho \lambda_1 \sum_{i \in S} |H_i|^2}} \right] \]

15
\[ \frac{1}{(1 + \lambda_1 P') \sum_{i \in S} |H_i|^2} \left( 1 + \frac{1}{1 + \lambda_2 P'} \sum_{i \in S} |H_i|^2 \right) e^{\left( \rho_{\lambda_1} - \frac{\lambda_2}{1 + \lambda_2 P'} \sum_{i \in S} |H_i|^2 \right) \|Z_D\|^2} \]  

(89)

with \( 1 + \lambda_2 P' \sum_{i \in S} |H_i|^2 > 0 \). Finally, taking expectation over \( Z \), we get

\[ P \left[ F_i | H_{[K]} \right] \leq \sum_S M^{pl} e^{-n E_0(\lambda_1; \rho, H_{[K]}, S)} \]  

(90)

where

\[ E_0(\lambda_1; \rho, H_{[K]}, S) = \rho \ln \left( 1 + \lambda_1 P' \sum_{i \in S} |H_i|^2 \right) + \ln \left( 1 + \lambda_2 P' \sum_{i \in S} |H_i|^2 \right) + \ln \left( 1 - (1 + P' \sum_{i \in D^e} |H_i|^2) \left( \rho \lambda_1 - \frac{\lambda_2}{1 + \lambda_2 P'} \sum_{i \in S} |H_i|^2 \right) \right) \]  

(91)

with \( 1 - (1 + P' \sum_{i \in D^e} |H_i|^2) \left( \rho \lambda_1 - \frac{\lambda_2}{1 + \lambda_2 P'} \sum_{i \in S} |H_i|^2 \right) > 0 \).

It is easy to see that the optimum value of \( \lambda_1 \) that maximizes \( E_0 \) is given by

\[ \lambda_1^* = \frac{1}{(1 + P' \sum_{i \in D^e} |H_i|^2) (1 + \rho)} \]  

(92)

and hence the maximum value of the exponent \( E_0(\rho, H_{[K]}, S) = E_0(\lambda_1^*; \rho, H_{[K]}, S) \) is given by

\[ E_0(\rho, H_{[K]}, S) = \rho \ln \left( 1 + \frac{P' \sum_{i \in S} |H_i|^2}{(1 + \rho) \left( 1 + P' \sum_{i \in D^e} |H_i|^2 \right)} \right). \]  

Therefore, we have

\[ p_1 \leq \mathbb{E} \left[ \sum_t \sum_S e^{\rho t \ln M} e^{-n E_0(\rho, H_{[K]}, S)} \right]. \]  

(93)

Since we want an upper bound for \( \text{(93)} \), we would like to take minimum over \( S \subseteq D : |S| = t \). For a given choice of \( D \), this corresponds to minimizing \( P' \sum_{i \in S} |H_i|^2 \) which mean we take \( S \) to contain indices in \( D \) which correspond to \( t \) smallest fading coefficients (within \( D \)). Then, the best such bound is obtained by choosing \( D \) that maximizes \( \frac{P' \sum_{i \in S} |H_i|^2}{(1 + P' \sum_{i \in D^e} |H_i|^2)} \). Clearly this corresponds to choosing \( D \) to contain indices corresponding to top \( K_1 \) fading coefficients.

Therefore, we get

\[ p_1 \leq \mathbb{E} \left[ \sum_t \left( \frac{K_1}{t} \right) e^{\rho t \ln M} e^{-n \rho \ln \left( 1 + \frac{P' \sum_{i \in K_{1-t} \cup H_{[K]} \setminus K_{1-t}} |H_i|^2}{(1 + \rho)(1 + P' \sum_{i \in D^e} |H_i|^2)} \right)} \right]. \]

Let \( A_n = \left\{ \frac{i}{n} \right\} \cap \left\{ \frac{i}{K_1} : i \in [K_1] \right\} \). For \( \theta \in A_n \) and \( t = \theta K_1 \), using [27] Ex. 5.8] again, we have

\[ \left( \frac{K_1}{t} \right) \leq \sqrt{\frac{K_1}{2\pi(t(K_1 - t))}} e^{K_1 \frac{h(\theta)}{1 - \theta}} = O \left( \frac{1}{\sqrt{n}} \right) e^{n \mu \nu h(\theta)}. \]  

(94)

The choice of \( \rho \) was arbitrary, and hence,

\[ p_1 \leq \mathbb{E} \left[ \min \left\{ 1, \sum_{\theta \in A_n} e^{-n \sup_{\rho \in [0, 1]} \rho \ln \left( 1 + \frac{P' \sum_{i \in \theta K_{1-t} \cup H_{[K]} \setminus K_{1-t}} |H_i|^2}{(1 + \rho)(1 + P' \sum_{i \in D^e} |H_i|^2)} \right) - \mu \theta \ln M \right\} \right] \]

\[ \leq \mathbb{E} \left[ \min \left\{ 1, \sup_{\rho \in [0, 1]} e^{-n \sup_{\rho \in [0, 1]} \rho \ln \left( 1 + \frac{P' \sum_{i \in \theta K_{1-t} \cup H_{[K]} \setminus K_{1-t}} |H_i|^2}{(1 + \rho)(1 + P' \sum_{i \in D^e} |H_i|^2)} \right) - \mu \theta \ln M \right\} \right] \]

(95)

where we have used min since \( p_1 \leq 1 \). Now, using similar arguments as in the proof of claim \[8\] and taking limits, we can see that
with exponentially high probability. Hence, 

\[ p_1 \leq E \left[ |A_n|^e - n \inf_{\theta \in A_n} \sup_{\rho \in [0, 1]} \left( \rho \ln \left( 1 + \frac{P_{\text{tot}}(\nu(1-\theta), \nu)}{1 + \frac{P_{\text{tot}}(\nu(1-\theta), \nu)}{1 + \frac{P_{\text{tot}}(\nu, 1)}}} \right) - \mu \log (1 + \mu \rho) - \mu \theta \ln M \right) \right] + o(1) \]

(97)

where \( A = \left[ \frac{e}{p}, 1 \right] \).

Therefore, choosing \( P_{\text{tot}} > \sup_{\theta \in A} \inf_{\rho \in [0, 1]} P_{\text{tot}}(\theta, \rho) \) will ensure that \( \lim sup_{n \to \infty} p_1 = 0 \).

\[ \square \]

C. Converse

In this section we derive a converse for \( \mathcal{E}^* \), based on the Fano inequality and the results from [31].

**Theorem IV.6.** Let \( M \) be the codebook size. Given \( \epsilon \) and \( \mu \), let \( S = \mu \log M \). Then assuming that the distribution of \( |H|^2 \) has a density with \( \mathbb{E} [|H|^2] = 1 \) and \( \mathbb{E} [|H|^4] < \infty \), \( \mathcal{E}^*(M, \mu, \epsilon) \) satisfies the following two bounds

1) \( \mathcal{E}^*(M, \mu, \epsilon) \geq \inf \frac{P_{\text{tot}}}{S} \)

(98)

where infimum is taken over all \( P_{\text{tot}} > 0 \) that satisfies

\[ \theta S - \epsilon \mu \log \left( \frac{2^{S/\mu}}{\mu - 1} \right) - \mu h_2(\epsilon) \leq \log \left( 1 + P_{\text{tot}}(1 - \theta, 1) \right), \forall \theta \in [0, 1] \]

(99)

where \( \alpha(a, b) = \int_a^b F_{|H|^2}(1 - \gamma)d\gamma \), and \( F_{|H|^2} \) is the CDF of squared absolute value of the fading coefficients.

2) \( \mathcal{E}^*(M, \mu, \epsilon) \geq \inf \frac{P_{\text{tot}}}{S} \)

(100)

where infimum is taken over all \( P_{\text{tot}} > 0 \) that satisfies

\[ \epsilon \geq 1 - E \left[ Q \left( Q^{-1} \left( \frac{1}{M} \right) - \sqrt{\frac{2P_{\text{tot}}}{\mu}} |H|^2 \right) \right] \]

(101)

where \( Q \) is the complementary CDF function of the standard normal distribution.

**Proof.** First, we use the Fano inequality.

Let \( W = (W_1, ..., W_K) \), where \( W_i \sim \text{Unif}[M] \) denote the sent messages of \( K \) users. Let \( X = (X_1, ..., X_K) \) where \( X_i \in \mathbb{C}^n \) be the corresponding codewords, \( Y \in \mathbb{C}^n \) be the received vector. Let \( W = (\hat{W}_1, ..., \hat{W}_K) \) be the decoded messages. Then \( W \to X \to Y \to \hat{W} \) forms a Markov chain. Then \( \epsilon = P_e = \frac{1}{K} \sum_{i \in [K]} P \left[ W_i \neq \hat{W}_i \right] \).

Suppose a genie \( G \) reveal a set \( S_1 \subset [K] \) for transmitted messages \( W_{S_1} = \{ W_i : i \in S_1 \} \) and the corresponding fading coefficients \( H_{S_2} \) to the decoder. So, a converse bound in the Genie case is a converse bound for our problem (when there is no Genie). Further, the equivalent channel at the receiver is

\[ Y_G = \sum_{i \in S_2} H_i X_i + Z \]

(102)

where \( S_2 = [K] \setminus S_1 \), and the decoder outputs a \([K]\) sized tuple. So, PUPE with Genie is given by

\[ P_{e_G}^\epsilon = \frac{1}{K} \sum_{i \in [K]} P \left[ W_i \neq \hat{W}_i^G \right] \]

(103)
Now, it can be seen that the optimal decoder must have the codewords revealed by the Genie in the corresponding locations in the output tuple, i.e., if $\hat{W}_i^G$ denotes the output tuple (in the Genie case), for $i \in S_1$, we must have that $W_i = \hat{W}_i^G$. Otherwise, PUPE can be strictly decreased by including these Genie revealed codewords.

So, letting $E_i = 1[|W_i \neq \hat{W}_i^G|]$ and $\epsilon_i^G = \mathbb{E}[E_i]$, we have that $\epsilon_i^G = 0$ for $i \in S_1$. For $i \in S_2$, a Fano type argument gives

$$I(W_i; \hat{W}_i^G) \geq \log M - \epsilon_i^G \log(M - 1) - \log(\epsilon_i^G).$$  \hspace{1cm} (104)

So, using the fact that $\sum_{i \in S_2} I(W_i; \hat{W}_i^G) \leq I(W_{S_2}; \hat{W}_{S_2}^G) \leq n \mathbb{E}\left[\log(1 + P \sum_{i \in S_2} |H_i|^2)\right]$, we have

$$|S_2| \log M - \sum_{i \in S_2} \epsilon_i^G \log(M - 1) - \sum_{i \in S_2} h_2(\epsilon_i^G) \leq n \mathbb{E}\left[\log(1 + P \sum_{i \in S_2} |H_i|^2)\right].$$ \hspace{1cm} (105)

By concavity of $h_2$, we have

$$\frac{1}{R} \sum_{i \in S_2} h_2(\epsilon_i^G) = \frac{1}{R} \sum_{i \in |K|} h_2(\epsilon_i^G) \leq h_2(P_e^G).$$ \hspace{1cm} (106)

Hence we get

$$\frac{|S_2|}{K} \log M - P_e^G \log(M - 1) - h_2(P_e^G) \leq \frac{n}{K} \mathbb{E}\left[\log(1 + P \sum_{i \in S_2} |H_i|^2)\right].$$ \hspace{1cm} (107)

Next, notice that $P_e^G \leq P_e \leq 1 - \frac{1}{M}$ and hence $P_e^G \log(M - 1) + h_2(P_e^G) \leq P_e \log(M - 1) + h_2(P_e)$. Further the inequality above holds for all $S_2 \subset |K|$ (which can depend of $H_{|K|}$ as well). Hence, letting $|S_2| = \theta K$

$$\theta \log M - P_e \log(M - 1) - h_2(P_e) \leq \frac{1}{I} \mathbb{E}\left[\log \left(1 + \inf_{S_2:|S_2| = \theta K} \frac{P_{tot}}{K} \sum_{i \in S_2} |H_i|^2\right)\right].$$ \hspace{1cm} (108)

Now, taking limit as $K \rightarrow \infty$ and using results on strong laws of order statistics [32] Theorem 2.1], we get that

$$\log \left(1 + \inf_{S_2:|S_2| = \theta K} \frac{P_{tot}}{K} \sum_{i \in S_2} |H_i|^2\right) \rightarrow \log \left(1 + P_{tot} \alpha(1 - \theta, 1)\right).$$ \hspace{1cm} (109)

For any $a, b \in [0, 1]$ with $a < b$, let $S_K = S_K(a, b) = \frac{1}{K} \sum_{i=aK}^{bK} |H(i)|^2$. Note that $S_K \rightarrow \alpha(a, b)$ as $K \rightarrow \infty$. Then

$$\mathbb{E}\left[S_K^2\right] \leq \mathbb{E}\left[\left(\frac{1}{K} \sum_{i=1}^{K} |H_i|^2\right)^2\right] = 1 + \frac{\mathbb{E}\left[|H|^4\right] - 1}{K} \leq \mathbb{E}\left[|H|^4\right].$$ \hspace{1cm} (110)

Hence the family of random variables $\{S_K : K \in \mathbb{N}\}$ is uniformly integrable. Further $0 \leq \log(1 + P_{tot} S_K) \leq P_{tot} S_K$. Hence the family $\{\log(1 + P_{tot} S_K) : K \geq 1\}$ is also uniformly integrable. Then from theorem [33] Theorem 9.1.6] $\mathbb{E}\left[\log(1 + P_{tot} S_K)\right] \rightarrow \log(1 + P_{tot} \alpha(a, b))$. Using this in (108) with $a = 1 - \theta$ and $b = 1$, we obtain (99).

Next we use the result from [31] to get another bound.

Using the fact that $S/\mu$ bits are needed to be transmitted under a per-user error of $\epsilon$, we can get a converse on the minimum $E_b/N_0$ required by deriving the corresponding results for a single user quasi-static fading MAC. In [31], the authors gave the following non-asymptotic converse bound on the minimum energy required to send $k$ bits for an AWGN channel. Consider the single user AWGN channel $Y = X + Z$, $X, Z \in \mathbb{R}^\infty$, $Z \sim \mathcal{N}(0, 1)$. Let $M_\epsilon^*(E, \epsilon)$ denote the largest $M$ such that there exists a $(E, M, \epsilon)$ code for this channel: codewords $(e_1, ..., e_M)$ with $|e_i|^2 \leq E$ and a decoder such that probability of error is smaller than $\epsilon$. The following is a converse bound from [31].

**Lemma IV.7** [31]. Any $(E, M, \epsilon)$ code satisfies

$$\frac{1}{M} \geq Q \left(\sqrt{2E} + Q^{-1}(1 - \epsilon)\right)$$ \hspace{1cm} (111)

Translating to our notations, for the channel $Y = HX + Z$, conditioned on $H$, if $\epsilon(H)$ denotes the probability of error for each realization of $H$, then we have
\[
\frac{1}{M} \geq Q\left(\sqrt{\frac{2P_{\text{tot}}}{\mu}|H|^2} + Q^{-1}(1 - \epsilon(H))\right) .
\]\(\text{(112)}\)

Further \(E[\epsilon(H)] = \epsilon\). Therefore we have
\[
\epsilon \geq 1 - \mathbb{E}\left[Q\left(Q^{-1}\left(\frac{1}{M}\right) - \sqrt{\frac{2P_{\text{tot}}}{\mu}|H|^2}\right)\right] .
\]\(\text{(113)}\)

Hence we have the required converse bound.

**Remark 4.** We also get the following converse from [18, theorem 7] by taking the appropriate limits \(P = \frac{P_{\text{tot}}}{\mu n}\) and \(n \rightarrow \infty\).
\[
\log M \leq -\log \left(\mathbb{E}\left[Q\left(c + \frac{P_{\text{tot}}|H|^2}{2\mu}\right)\right]\right)
\]\(\text{(114)}\)
where \(c\) satisfies
\[
\mathbb{E}\left[Q\left(c + \frac{P_{\text{tot}}|H|^2}{2\mu}\right)\right] = 1 - \epsilon .
\]\(\text{(115)}\)

But this is strictly weaker than (113). This is because, using lemma [IV.7] we perform hypothesis testing (in the meta-converse) for each realization of \(H\) but in the bound used in [18], hypothesis testing is performed over the joint distribution (including the distribution of \(H\)). This is to say that if \(H\) is presumed to be constant (and known), then in [114] and [115] we can remove the expectation over \(H\) and this gives precisely the same bound as (112). \(\Box\)

Bounds tighter than (99) can be obtained if further assumptions are made on the codebook. For instance, if we assume that each codebook consists of iid entries of the form \(
\frac{C}{R}
\) where \(C\) is sampled from a distribution with zero mean and finite variance, then using ideas similar to [34, Theorem 3] we have the following converse bound.

**Theorem IV.8.** Let \(M\) be the codebook size, and let \(\mu n\) users (\(\mu < 1\)) generate their codebooks independently with each code symbol iid of the form \(
\frac{C}{R}
\) where \(C\) is of zero mean and variance \(P_{\text{tot}}\). Then in order for the iid codebook to achieve PUPE \(\epsilon\) with high probability, the energy-per-bit \(E\) should satisfy
\[
\mathcal{E} \geq \inf \frac{P_{\text{tot}}}{\mu \log M}
\]\(\text{(116)}\)
where infimum is taken over all \(P_{\text{tot}} > 0\) that satisfies
\[
\ln M - \epsilon \ln(M - 1) - h(\epsilon) \leq \left(M\mathcal{V}\left(\frac{1}{\mu M}, P_{\text{tot}}\right) - \mathcal{V}\left(\frac{1}{\mu}, P_{\text{tot}}\right)\right)
\]\(\text{(117)}\)
where \(\mathcal{V}\) is given by [34]
\[
\mathcal{V}(r, \gamma) = r \ln (1 + \gamma - F(r, \gamma)) + \ln (1 + r\gamma - F(r, \gamma)) - \frac{F(r, \gamma)}{\gamma}
\]\(\text{(118)}\)
\[
F(r, \gamma) = \frac{1}{4} \left(\sqrt{\gamma \left(\sqrt{r} + 1\right)^2 + 1} - \sqrt{\gamma \left(\sqrt{r} - 1\right)^2 + 1}\right)^2
\]\(\text{(119)}\)

**Proof sketch.** The proof is almost the same as in [34, Theorem 3] (see [34, Remark 3] as well). We will highlight the major differences here. First, our communication system can be modeled as a support recovery problem as follows. Let \(A\) be the \(n \times K\) matrix consisting of \(n \times M\) blocks of codewords of users. Let \(H\) be the \(K\) block diagonal matrix with block \(i\) being a diagonal \(M \times M\) matrix with all diagonal entries being equal to \(H_i\). Finally let \(W \in \{0, 1\}^{KM}\) with \(K\) blocks of size \(M\) each and within each \(M\) sized block, there is exactly one 1. So the position of 1 in block \(i\) of \(W\) denotes the message or codeword corresponding to the user \(i\) which is the corresponding column in block \(i\) of matrix \(A\). Hence our channel can be represented as
\[
Y = AHW + Z
\]\(\text{(120)}\)
with the goal of recovering \(W\).
Next the crucial step is bound $R^K(\epsilon, M)$ in \cite{104} as
\begin{equation}
R^K(\epsilon, M) \leq I(W; Y | A) = I(HW; Y | A) - I(HW; Y | A, W)
\end{equation}
where the equality in the above display follows from \cite{34} equation (78). The first term in above display is bounded as
\begin{equation}
I(HW; Y | A = A_1) = I(HW; A_1 HW + Z) \leq \sup_U I(U; A_1 U + Z)
\end{equation}
where $A_1$ is a realization of $A$ and supremum is over random vectors $U \in \mathbb{C}^{K \times M}$ such that $\mathbb{E}[U] = 0$ and $\mathbb{E}[UU^*] = \mathbb{E}[(HW)(HW)^*] = \frac{\mathbb{E}[|H_1|^2]}{M} I_{KM \times KM}$. Now similar to \cite{34}, the supremum is achieved when $U \sim \mathcal{CN}(0, \frac{\mathbb{E}[|H_1|^2]}{M} I_{KM \times KM})$. Hence
\begin{equation}
I(HW; Y | A = A_1) \leq \log \det \left( I_{n \times n} + \frac{1}{M} A A^* \right).
\end{equation}

Next, for any realization $A_1$ and $W_1$ of $A$ and $W$ respectively, we have
\begin{align}
I(HW; Y | A = A_1, W = W_1) &= I(HW_1; A_1 HW_1 + Z) \\
&= I(\tilde{H}; (A_1)_{W_1} \tilde{H} + Z) \\
&\geq I(\tilde{H}; \tilde{H} + (A_1)_{W_1}^* Z)
\end{align}
where $\tilde{H} = [H_1, \ldots, H_K]^T$ and $(A_1)_{W_1}$ is the $n \times K$ submatrix of $A_1$ formed by columns corresponding to the support of $W_1$ and $\dagger$ denotes the Moore-Penrose inverse (pseudoinverse). The last equality in the above follows from the data processing inequality. Now, by standard mutual information of Gaussians, we have
\begin{equation}
I(\tilde{H}; \tilde{H} + (A_1)_{W_1}^* Z) = \log \det \left( I_{K \times K} + ((A_1)_{W_1})^* (A_1)_{W_1} \right).
\end{equation}

Hence
\begin{equation}
I(HW; Y | A, W) = \mathbb{E}[\log \det (I_{K \times K} + A_W^* A_W)].
\end{equation}

Hereafter, the we can proceed similarly to the proof of \cite{34} Theorem 3 using results from random-matrix theory \cite{35, 36} to finish the proof. \hfill \Box

We remark here that for a general fading distribution, the term $I(\tilde{H}; \tilde{H} + (A_1)_{W_1}^* Z)$ can be lower bounded similar to the proof of \cite{34} Theorem 3 using EPI (and its generalization \cite{37}) to get
\begin{equation}
I(\tilde{H}; \tilde{H} + (A_1)_{W_1}^* Z) \geq K \log \left( 1 + N_H \left( \det ((A_1)_{W_1})^* (A_1)_{W_1} \right)^{\frac{1}{2K}} \right)
\end{equation}
where $N_H = \frac{1}{\epsilon^2} \exp(h(H))$ is the entropy power of fading distribution. Hence
\begin{equation}
I(HW; Y | A, W) \geq K \mathbb{E}\left[ \log \left( 1 + K H \left( \det (A_W^* A_W) \right)^{\frac{1}{2K}} \right) \right].
\end{equation}

Again, we can use results from random-matrix theory \cite{36} and proceed similarly to the proof of \cite{34} Theorem 3 to get a converse bound with the second term in \cite{117} replaced by $\mathcal{V}_{LB} \left( \frac{1}{\mu}, P_{tot} \right)$ and
\begin{equation}
\mathcal{V}_{LB}(\tau, \gamma) = \ln \left( 1 + \gamma r \left( \frac{r}{r - 1} \right)^{r-1} \frac{1}{e} \right)
\end{equation}

We make a few observations regarding the preceding theorem. First and foremost, this hold only for the case of no-CSI because the term analogous to $I(HW; Y | A, W)$ in the case of CSIR is $I(HW; Y | A, \tilde{H}, W)$ which is zero. Next, it assumes that the codebooks have iid entries with variance scaling $\Theta(1/n)$. This point is crucial to lower bounding $I(HW; Y | A, W)$, and this is where a significant improvement comes when compared to \cite{59}. Indeed, EPI and results from random matrix theory give $O(n)$ lower bound for $I(HW; Y | A, W)$. This once again brings to focus the the difference between classical regime and the scaling regime, where in the former, this term is negligible. Further this leaves open the question of whether we could improve performance in the high-density of users case by using non-iid codebooks.

Now, as to what types of codebooks give a $\Theta(n)$ lower bound for $I(HW; Y | A, W)$, a partial answer can be given by carefully analyzing the full proof of the theorem. In particular, if $S = \text{supp} W$ i.e. the support of $W$, then as seen from \cite{34} equation (85), any non zero lower bound on $\det(A_S^* A_S)^{1/K}$ is enough. So if the matrix $A_S^* A_S$ is strongly diagonally dominant or if there is a $O(n)$ constraint on $\text{Tr}(A_S^* A_S)^2$ one can lower bound the determinant using Hadamard type lower bounds (see \cite{58} for instance). These boil down to the codewords being overwhelmingly close to orthogonal.
V. NUMERICAL EVALUATION AND DISCUSSION

In this section, we provide the results of numerical evaluation of the bounds in the paper. We focus on the trade-off of user density $\mu$ with the minimum energy-per-bit $\mathcal{E}^*$ for a given message size $k$ and target probability of error $P_e$.

For $k = 100$ bits, we evaluate the trade-off from the bounds in this paper for $P_e = 0.1$ and $P_e = 0.001$ in figures 3 and 2 respectively. For TDMA, we split the frame of length $n$ equally among $K$ users, and compute the smallest $P_{tot}$ the ensures the existence of a single user quasi-static AWGN code of rate $S$, blocklength $\frac{n}{\mu}$ and probability of error $\epsilon$ using the bound from [18]. The simulations of the single user bound is performed using codes from [39]. TIN is computed using a method similar to theorem IV.4.

From these figures, we clearly observe the perfect MUI cancellation effect mentioned in the introduction. As $\mu$ increases from 0, the $\mathcal{E}^*$ is almost a constant (slightly increasing for the achievability bounds) but then undergoes a “phase transition” where $\mathcal{E}^*$ increases sharply. Hence this suggests there is a certain $\mathcal{E}_{\text{s.u.}} = \mathcal{E}_{\text{s.u.}}(k, \epsilon)$ and $\mu_{\text{s.u.}} > 0$ such that $\mathcal{E}^* = \mathcal{E}_{\text{s.u.}}$ for all $\mu < \mu_{\text{s.u.}}$. Further, standard schemes for multiple-access like TDMA and TIN do not have this behavior. Moreover, although these suboptimal schemes have an optimal trade-off at $\mu \to 0$ they show a significant suboptimality at higher $\mu$. We note again that this perfect MUI cancellation which was observed in standard GMAC [5, 8] is also present in the more practically relevant quasi-static fading model. So, we suspect that this effect is a characteristic of the many-user MAC.

The fact that orthogonalization is not optimal is one of the key practical implications of our work. It was observed before in the GMAC and here we again witness it in the more relevant QS-MAC. How to understand this suboptimality? First, in the fading case we have already seen this effect even in the classical regime (but under PUPE) – see (19). To give another intuition we consider a $K = \mu n$ user binary adder MAC

$$Y = \sum_{i=1}^{K} X_i \quad (130)$$

where $X_i \in \{0, 1\}$ and addition is over $\mathbb{Z}$. Now, using TDMA on this channel, each user can send at most $n/K = 1/\mu$ bits. Hence the message size is bounded by

$$\log M \leq \frac{1}{\mu} \quad (131)$$

Next, let us consider TIN. Assume $X_i \sim \text{Ber}(1/2)$. For user 1, we can treat $\sum_{i=2}^{\mu n} X_i$ as noise. By central limit theorem, this noise can be approximated as $\sqrt{\frac{1}{4} \mu n Z}$ where $Z \sim \mathcal{N}(0, 1)$. Thus we have a binary input AWGN (BIAWGN) channel

$$Y = X_1 + \sqrt{\frac{1}{4} \mu n} Z. \quad (132)$$

Therefore, the message size is bounded as

$$\log M \leq n C_{\text{BIAWGN}} \left(1 + \frac{4}{\mu n}\right)$$

$$\leq \frac{n}{2} \log \left(1 + \frac{4}{\mu n}\right) \to \frac{2}{\mu \ln 2} \quad (133)$$

where $C_{\text{BIAWGN}}$ is the capacity of the BIAWGN channel. Note that in both the above schemes the achievable message size is a constant as $n \to \infty$.

On the other hand, the true sum-capacity of the $K$-user adder MAC is given by

$$C_{\text{sum}} = \max_{X_1, \ldots, X_K} H(X_1 + \cdots + X_K).$$

As shown in [40] this maximum as achieved at $X_i \sim \text{Ber}(1/2)$. Since the the entropy of binomial distributions [41] can be computed easily, we obtain

$$C_{\text{sum}} = \frac{1}{2} \log K + o(\log K).$$

In particular, for our many-user MAC setting we obtain from the Fano inequality (and assuming PUPE is small)

$$\log M \leq \frac{\log(\mu n)}{2 \mu}.$$  

Surprisingly, there exist explicit codes that achieve this limit and with a very low-complexity (each message bit is sent separately)– a construction rediscovered several times [42, 44]. Hence the optimal achievable message size is

$$\log M \approx \frac{\log n}{2 \mu} \to \infty \quad (134)$$
as $n \to \infty$. And again, we see that TDMA and TIN are severely suboptimal for the many-user adder MAC as well.

We remark here that from figure 2 the no-CSI bound on $E^*$ (red curve) increases sharply in the neighborhood of $\mu = 0$. In fact, it can be seen from expressions in theorem IV.1 that $E^* = O(\sqrt{-\ln \mu})$ as $\mu \to 0$. Hence the bound is not optimal for small $\mu$.

There are a lot of interesting directions for future work. A natural extension would be to analyze the many-user massive MIMO fading channel with receiver having $N > 1$ antennas under both block and quasi static gains. Further, various asymptotics of $N$ can also be considered. Another direction is to come up with better achievability bounds using either a different decoding technique or performing better analysis, for example, using results on Gaussian processes (see [8] where it has been employed for the GMAC). From a practical standpoint, there is also a question of finding MAC architectures that would achieve $E^*$ for $\mu > 0$.

---

**Fig. 2:** $\mu$ vs $E_b/N_0$ for $\epsilon \leq 10^{-3}$, $k = 100$

**Fig. 3:** $\mu$ vs $E_b/N_0$ for $\epsilon \leq 10^{-1}$, $k = 100$
APPENDIX A
PROOFS OF SECTION III

A. Joint error

Proof of theorem III.I. Let \( R = (R_1, ..., R_K) \in C_{e,J} \). We need to show that there exists a sequence of \( (M_i^{(n)}, M_2^{(n)}, ..., M_K^{(n)}), n, \epsilon_n \) codes with projection decoding, such that

\[
\liminf_{n \to \infty} \frac{1}{n} \log \left( M_i^{(n)} \right) \geq R_i, \forall i \in [K] \\
\limsup_{n \to \infty} \epsilon_n \leq \epsilon.
\]  

(135)

(136)

Let \( \eta_i > 0, i \in [K] \). Choose \( M_i^{(n)} = [2^{n(R_i - \eta_i)}], \forall i \in [K] \). We use random coding: user \( j \), independently generates \( M_j^{(n)} \) vectors, each independently and uniformly distributed on the \( \sqrt{nP} \)-complex sphere. That is \( M_j \sim Unif \left( \sqrt{nP} (CS)^{n-1} \right) \).

Hence the channel inputs are given by \( X_i \). Let

\[
F \equiv \left\{ c \right\}
\]

such that \( \left( \hat{c}_1, \hat{c}_2, ..., \hat{c}_K \right) \) denote the decoded codewords.

Let \( c_{\mid S} = \{ c_i : i \in [S] \} \). Then, error occurs iff \( \exists S \subset [K] \) and \( S \neq \emptyset \), and \( \exists \{ c_i' : i \in [S], c_i' \neq c_i \} \) such that

\[
\| P_{c_{\mid S}, c_{\mid S}} \| Y_j^2 \geq \| P_{c_{\mid [K]}} \| Y_j^2.
\]

This can be equivalently written as follows. Let \( S \subset [K] \) be such that

\[
i \in [S] \iff \hat{c}_i \neq c_i
\]

(138)

where \( \{ c_i \}_{i=1}^K \) denote the decoded codewords.

Let \( c_{\mid S} = \{ c_i : i \in [S] \} \). Then, error occurs iff \( \exists S \subset [K] \) and \( S \neq \emptyset \), and \( \exists \{ c_i' : i \in [S], c_i' \neq c_i \} \) such that

\[
\| P_{c_{\mid S}, c_{\mid S}} \| Y_j^2 \geq \| P_{c_{\mid [K]}} \| Y_j^2.
\]

(139)

Let \( B_S = \left\{ P_{c_{\mid S}, c_{\mid S}} \right\} \geq \left\{ P_{c_{\mid [K]}} \right\} \) (here primes denote unsent codewords i.e., \( c_i' \) here means that it is independent of the channel inputs/output and distributed with the same law as \( c_i \)). Note that, for the sake of brevity, we are suppressing the dependence on \( c'_i \).

So, the average probability of error is given by

\[
\epsilon_n = \mathbb{P} \left[ \bigcup_{S \subset [K]} \bigcup_{S \neq \emptyset} \bigcup_{c, c'_i \neq c_i} B_S \right]
\]

\[
= \mathbb{P} \left[ \bigcup_{t \in [K]} \bigcup_{S \subset [K]} \bigcup_{|S| = t} \bigcup_{c, c'_i \neq c_i} B_S \right]
\]

(140)

Using ideas similar to the Random Coding Union (RCU) bound \( 45 \), we have

\[
\epsilon_n \leq \mathbb{E} \left[ \min \left\{ 1, \sum_{t \in [K]} \sum_{S \subset [K]: |S| = t} \left( \prod_{j \in S} (M_j - 1) \right) \mathbb{P} \left[ B_S | c_{[K]}, H_{[K]}, Z \right] \right\} \right]
\]

(141)

where \( H_{[K]} = \{ H_i : i \in [K] \} \).

From now on we denote \( \bigcup_{t \in [K]} \bigcup_{S \subset [K]} \bigcup_{|S| = t} \) by \( \bigcup_{i,t,S} \sum_{t \in [K]} \sum_{S \subset [K]: |S| = t} \) by \( \sum_{t,S} \) and \( \bigcup_{t \in [K]} \bigcup_{S \subset [K]} \bigcup_{|S| = t} \) by \( \bigcup_{t,S} \).

Claim 6. For \( t \in [K] \) and \( S \subset [K] \) with \( |S| = t \),

\[
P \left[ \left\| P_{c_{\mid S}, c_{\mid S}} \right\| Y_j^2 \geq \left\| P_{c_{\mid [K]}} \right\| Y_j^2 | c_{[K]}, H_{[K]}, Z \right] = F \left( \frac{\left\| Y \right\| - \left\| P_{c_{\mid [K]}} \right\| Y_j^2}{\left\| Y \right\|^2 - \left\| P_{c_{\mid [K]}} \right\| Y_j^2}; n - K, t \right)
\]

(142)

where \( F(x; a, b) \) is the cdf of beta distribution Beta(\( a, b \)). Further, from \( 18 \), we have

\[
F(x; n - K, t) \leq (n - K + t - 1)^{t-1}x^{n-K}
\]

(143)
Proof. See proof of claim 1. □

Letting \( G_S \equiv g(Y, c_{[K]}, S) = \frac{\|Y\|^2 - \|P_{c_{[K]}} Y\|^2}{\|P_{c_{[K]}} Y\|^2} \), \( M_S = \prod_{j \in S} (M_j - 1), s_t = (t - 1) \frac{\ln(n-K+t-1)}{(n-K)} \) and \( r_t = s_t + \frac{\ln M_S}{(n-K)} \), we have the following from (141), (142) and (143)

\[
\epsilon_n \leq \mathbb{E} \left[ \min \left\{ 1, \sum_{t,S} \exp (- (n-K) [-r_t - \ln G_S]) \right\} \right] \tag{144}
\]

Let \( \delta > 0 \) and let \( E_1 \) be the following event

\[
E_1 = \bigcap_{t,S} \{ - \ln G_S - r_t > \delta \} \tag{145}
\]

\[
= \bigcap_{t,S} \{ - \ln G_S > \tilde{V}_{n,S} \}
\]

\[
= \bigcap_{t,S} \{ G_S < V_{n,S} \} \tag{146}
\]

where \( \tilde{V}_{n,S} = r_t + \delta \) and \( V_{n,S} = e^{- \tilde{V}_{n,S}} \). Note that \( V_{n,S} \) depends on \( S \) and \( t \).

Then, from (144), we have the following

**Lemma A.1.** For the \( K \)-user MAC defined above, with the projection decoder, the average probability of error is upper bounded as

\[
\epsilon_n \leq \sum_{t,S} e^{- (n-K) \delta} + \mathbb{P} \left[ \bigcup_{t,S} G_S \geq V_{n,S} \right] \tag{147}
\]

Proof. By (144),

\[
\epsilon_n \leq \mathbb{E} \left[ \min \left\{ 1, \sum_{t,S} e^{-(n-K)[-r_t - \ln G_S]} \right\} (1\{E_1\} + 1\{E_1^c\}) \right]
\]

\[
\leq \sum_{t,S} e^{-(n-K)\delta} + \mathbb{P}[E_1^c] \tag{148}
\]

□

Hence, as \( n \to \infty \), it is the second term in the above expression that potentially dominates.

**Claim 7.** For \( t \in [K], S \subseteq [K] \) with \( |S| = t \), we have

\[
\mathbb{P} [ G_S \geq V_{n,S} ] = \mathbb{P} [ G_S \geq V_{n,S} ] \leq \mathbb{P} \left[ \left\| \left( 1 - V_{n,S} \right) P_{c_{[S^c]}} Z - V_{n,S} P_{c_{[S^c]}} \sum_{i \in S} H_i c_i \right\|^2 \geq V_{n,S} \left\| P_{c_{[S^c]}} \sum_{i \in S} H_i c_i \right\|^2 \right] \tag{149}
\]

where \( P_{c_{[S^c]}} \) represents the orthogonal projection onto the orthogonal complement of the space spanned by \( c_{[S^c]} \).

Proof. See proof of claim 2. □

To evaluate the above probability, we condition on \( c_{[K]} \) and \( H_{[K]} \). For ease of notation, we will not explicitly write the conditioning.

Let \( \chi^2_2(\lambda, d) \) denote the non-central chi-squared distributed random variable with non-centrality \( \lambda \) and degrees of freedom \( d \). That is, if \( Z_i \sim \mathcal{N}(\mu_i, 1), i \in [d] \) and \( \lambda = \sum_{i \in [d]} Z_i^2 \), then \( \chi^2_2(\lambda, d) \) has the same distribution as that of \( \sum_{i \in [d]} Z_i^2 \).

Since \( Z \sim \mathcal{CN}(0, I_n) \), we have \( Z - \frac{V_{n/2}}{V_{n/2}} \sum_{i \in S} H_i c_i \sim \mathcal{CN}(- \frac{V_{n/2}}{V_{n/2}} \sum_{i \in S} H_i c_i, I_n) \). Hence \( P_{c_{[S^c]}} \left( Z - \frac{V_{n/2}}{V_{n/2}} \sum_{i \in S} H_i c_i, P_{c_{[S^c]}} \right) \sim \mathcal{CN}(- \frac{V_{n/2}}{V_{n/2}} \sum_{i \in S} H_i c_i, P_{c_{[S^c]}}) \). Now using the fact that if \( W = P + iQ \sim \mathcal{CN}(\mu, \Gamma, 0) \) then

\[
\begin{bmatrix} P \\ Q \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} Re(\mu) \\ Im(\mu) \end{bmatrix}, \frac{1}{2} \begin{bmatrix} Re(\Gamma) & -Im(\Gamma) \\ Im(\Gamma) & Re(\Gamma) \end{bmatrix} \right) \tag{150}
\]

we can show the following
Lemma A.2. Let \( F = \left\| \frac{V_{n,S}}{1 - V_{n,S}} P_{c_{[\gamma]} \perp} \sum_{i \in S} H_i c_i \right\|^2 \) and \( n' = n - K + t \). Conditioned on \( H_{[K]} \) and \( c_{[K]} \), we have

\[
\left\| P_{c_{[\gamma]} \perp} \left( Z - \frac{V_{n,S}}{1 - V_{n,S}} \sum_{i \in S} H_i c_i \right) \right\|^2 \sim \frac{1}{2} \chi^2 (2F, 2n')
\]

(151)

Hence its conditional expectation is

\[
\mu = n' + F.
\]

(152)

Proof. See proof of claim 3.

Let \( U = \frac{V_{n,S}}{1 - V_{n,S}} \left\| \frac{P_{c_{[\gamma]} \perp}}{1 - V_{n,S}} \sum_{i \in S} H_i c_i \right\|^2 - n' \). Hence \( F = \frac{V_{n,S}}{1 - V_{n,S}} (U + n') \). Note that \( V_{n,S}, U, \lambda \) all depend on \( t \) and \( S \).

Letting \( T = \frac{1}{2} \chi^2 (2F, 2n') - (F + n') \), we have,

\[
\mathbb{P} \left[ \bigcup_{t,S} \left\| P_{c_{[\gamma]} \perp} \left( Z - \frac{V_{n,S}}{1 - V_{n,S}} \sum_{i \in S} H_i c_i \right) \right\|^2 \geq \frac{V_{n,S}}{1 - V_{n,S}} \left\| \frac{P_{c_{[\gamma]} \perp}}{1 - V_{n,S}} \sum_{i \in S} H_i c_i \right\|^2 \right]
\]

\[
= \mathbb{P} \left[ \bigcup_{t,S} \left\| P_{c_{[\gamma]} \perp} \left( Z - \frac{V_{n,S}}{1 - V_{n,S}} \sum_{i \in S} H_i c_i \right) \right\|^2 - \mu \geq U \right]
\]

\[
= \mathbb{E} \left[ \mathbb{P} \left[ \{ \bigcup_{t,S} \{ T \geq U \} \} \bigg| c_{[K]}, H_{[K]} \right] \right].
\]

(153)

Next we use lemma IV.2 to bound (153).

First, note that

\[
U = \frac{V_{n,S}}{1 - V_{n,S}} \left\| \frac{P_{c_{[\gamma]} \perp}}{1 - V_{n,S}} \sum_{i \in S} H_i c_i \right\|^2 - n'
\]

\[
= \frac{n'}{1 - V_{n,S}} \left( V_{n,S} \left( 1 + \frac{\left\| P_{c_{[\gamma]} \perp} \sum_{i \in S} H_i c_i \right\|^2}{n'} \right) - 1 \right)
\]

\[
= n' U^1
\]

(154)

where \( U^1 = \frac{1}{1 - V_{n,S}} (V_{n,S} W_S - 1) \) and \( W_S = \left( 1 + \frac{\left\| P_{c_{[\gamma]} \perp} \sum_{i \in S} H_i c_i \right\|^2}{n'} \right) \). Hence

\[
F = \frac{V_{n,S}}{1 - V_{n,S}} (U + n') = n' \frac{V_{n,S}}{1 - V_{n,S}} (U^1 + 1).
\]

(155)

Let \( \delta_1 > 0 \). Let \( E_{11} = \bigcap_{t,S} \{ U^1 > \delta_1 \} \). From (153) we have

\[
\mathbb{P} \left[ \bigcup_{t,S} \{ T \geq U \} \right] \leq \sum_{t,S} \mathbb{E} \left[ \mathbb{P} \left[ \{ T \geq U \} \bigg| c_{[K]}, H_{[K]} \right] \right] 1[E_{11}] + \mathbb{E} \left[ E_{11}^c \right]
\]

\[
\leq \sum_{t,S} \mathbb{E} \left[ \mathbb{P} \left[ \{ T \geq U \} \bigg| c_{[K]}, H_{[K]} \right] \right] 1[\{ U^1 > \delta_1 \}] + \mathbb{E} \left[ E_{11}^c \right]
\]

\[
\leq \sum_{t,S} \mathbb{E} \left[ e^{-n' f_n(U^1)} 1[\{ U^1 > \delta_1 \}] \right] + \mathbb{E} \left[ E_{11}^c \right]
\]

(156)

where the last inequality follows from (58), and

\[
f_n(x) = x + 1 + \frac{2V_{n,S}}{1 - V_{n,S}} (1 + x)
\]

\[
- \sqrt{1 + \frac{2V_{n,S}}{1 - V_{n,S}} (1 + x)} \sqrt{2x + 1 + \frac{2V_{n,S}}{1 - V_{n,S}} (1 + x)}
\]

(157)

Now, from claim 4 we have that for \( 0 < V_{n,S} < 1 \) and \( x > 0 \), \( f_n(x) \) is a monotonically increasing function of \( x \).
Hence we have

$$
\mathbb{P}\left[ \bigcup_{t,S} \{ T \geq U \} \right] \leq \sum_{t,S} e^{-n'f_n(\delta_1)} + \mathbb{P}[E_{11}].
$$

(158)

So, we have the following proposition

**Claim 8.** Let $A_S = \{ V_{n,S}W_S - 1 \leq \delta_1 \}$ and $E_{12} = \bigcup_{t,S} A_S$. If $0 < V_{n,S} < 1$ for all $t \in [K]$, $S \subset [K]$ with $|S| = t$ then we have

$$
\mathbb{P}\left[ \bigcup_{t,S} \{ G_S \geq V_{n,S} \} \right] \leq \sum_{t,S} e^{-n'f_n(\delta_1)} + \mathbb{P}[E_{12}].
$$

(159)

**Proof.**

$$
\mathbb{P}\left[ \bigcup_{t,S} \{ G_S \geq V_{n,S} \} \right] \leq \sum_{t,S} e^{-n'f_n(\delta_1)} + \mathbb{P}[E_{12}].
$$

(160)

Now, we need to upper bound $\mathbb{P}[E_{12}]$.

We have

$$
\left\| P_{\{e_{gS}\}}^\perp \sum_{i \in S} H_i e_i \right\|^2 = \sum_{i \in S} |H_i|^2 \left\| P_{\{e_{gS}\}}^\perp e_i \right\|^2 + 2 \sum_{i < j, i \in S} Re \left( \left\langle P_{\{e_{gS}\}}^\perp e_i, P_{\{e_{gS}\}}^\perp e_j \right\rangle H_i H_j \right).
$$

(161)

Further,

$$
\left\langle P_{\{e_{gS}\}}^\perp e_i, P_{\{e_{gS}\}}^\perp e_j \right\rangle = \langle e_i, e_j \rangle - \left\langle P_{\{e_{gS}\}} e_i, P_{\{e_{gS}\}} e_j \right\rangle
$$

(162)

Hence we have

$$
\left| Re \left( P_{\{e_{gS}\}}^\perp e_i, P_{\{e_{gS}\}}^\perp e_j \right) \right| \leq \left| \left\langle P_{\{e_{gS}\}}^\perp e_i, P_{\{e_{gS}\}}^\perp e_j \right\rangle \right| \leq |\langle e_i, e_j \rangle| + |\left\langle P_{\{e_{gS}\}} e_i, P_{\{e_{gS}\}} e_j \right\rangle| \leq |\langle e_i, e_j \rangle| + \left\| P_{\{e_{gS}\}} e_i \right\| \left\| P_{\{e_{gS}\}} e_j \right\| = n P \left( |\langle \hat{e}_i, \hat{e}_j \rangle| + \left\| P_{\{e_{gS}\}} \hat{e}_i \right\| \left\| P_{\{e_{gS}\}} \hat{e}_j \right\| \right)
$$

(163)

where hats denote corresponding normalized vectors. Since these unit vectors are high dimensional, their dot products and projection onto a smaller, fixed dimension surface is very small. Indeed, we have the following two lemmas.

**Lemma A.3.** If $e_1, e_2 \overset{iid}{\sim} Unif((\mathbb{C}S)^{n-1})$, then for any $\delta_2 > 0$, we have

$$
\mathbb{P}[|\langle e_1, e_2 \rangle| > \delta_2] \leq 4e^{-n\delta_2^2/2}
$$

(164)

**Proof.** First, lets take $e_1, e_2 \overset{iid}{\sim} S^{n-1}$. Let $x$ be a fixed unit vector in $\mathbb{R}^n$. Due to symmetry, we have $\mathbb{P}[\langle e_1, x \rangle \geq 0] = 1/2$. Hence, by Levy’s Isoperimetric inequality on the sphere [46], we have

$$
\mathbb{P}[\langle e_1, x \rangle > \delta_2] \leq e^{-n\delta_2^2/2}.
$$

(165)

Again by symmetry, and then taking $x$ as $e_2$, we have

$$
\mathbb{P}[|\langle e_1, e_2 \rangle| > \delta_2] \leq 2e^{-n\delta_2^2/2}.
$$

(166)

Now uniform distribution on $(\mathbb{C}S)^{n-1}$ is same as the uniform distribution on $S^{2n-1}$, and for complex vectors $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ we have $Re(\langle z_1, z_2 \rangle) = x_1^T x_2 + y_1^T y_2 = (x_1, y_1)^T (x_2, y_2)$. Hence if $e_1, e_2 \overset{iid}{\sim} (\mathbb{C}S)^{n-1}$, and $u_1, u_2 \overset{iid}{\sim} S^{2n-1}$ then $Re(\langle e_1, e_2 \rangle)$ has same law as $\langle u_1, u_2 \rangle$. Hence we have

$$
\mathbb{P}[|Re(\langle e_1, e_2 \rangle)| > \delta_2] \leq 2e^{-2n\delta_2^2/2}.
$$

(167)
Also, \( \text{Im} \langle z_1, z_2 \rangle = x_1^T y_2 - y_1^T x_2 \). Hence \( \text{Im} \langle e_1, e_2 \rangle \) has the same law as \( \text{Re} \langle e_1, e_2 \rangle \). Hence we have

\[
\begin{align*}
\mathbb{P} [ | \langle e_1, e_2 \rangle | > \delta_2 ] &= \mathbb{P} [ | \langle e_1, e_2 \rangle |^2 > \delta_2^2 ] \\
&= \mathbb{P} [ |\text{Re} \langle e_1, e_2 \rangle|^2 + |\text{Im} \langle e_1, e_2 \rangle|^2 > \delta_2^2 ] \\
&\leq \mathbb{P} [ |\text{Re} \langle e_1, e_2 \rangle| > \delta_2 \sqrt{2} ] + \mathbb{P} [ |\text{Im} \langle e_1, e_2 \rangle| > \delta_2 \sqrt{2} ] \\
&\leq 4e^{-n \delta_2^2 / 2} .
\end{align*}
\] (168)

Next we have a similar lemma for low dimensional projections from \([47], \text{Lemma 5.3.2}\]

**Lemma A.4**: Let \( x \sim \text{Unif}(S_{n-1}) \) and \( P \) be a projection to an \( m \) dimensional subspace of \( \mathbb{R}^n \). Then for any \( \delta_3 > 0 \), we have

\[
\mathbb{P} \left[ \left\| Px \right\| - \sqrt{\frac{m}{n}} > \delta_3 \right] \leq 2e^{-cn \delta_3^2}
\] (169)

where \( c \) is some absolute constant. Hence, by symmetry, the result remains true if \( P \) is a uniform random projection, independent of \( x \).

Now we need to prove that a similar result holds for the complex variable case as well. We have the following lemma

**Lemma A.5**: Let \( z \sim \text{Unif}(CS)^{n-1} \) and \( P \) be a projection to an \( m \) dimensional subspace \( V \) of \( \mathbb{C}^n \). Then for any \( \delta_3 > 0 \), we have

\[
\mathbb{P} \left[ \left\| Pz \right\| - \sqrt{\frac{m}{n}} > \delta_3 \right] \leq 2e^{-2cn \delta_3^2}
\] (170)

where \( c \) is some absolute constant. Hence, by symmetry, the result remains true if \( P \) is a uniform random projection, independent of \( z \).

**Proof**. Consider \( \left\| Pz \right\| \). Let \( U \) be the unitary change of basis matrix which converts \( V \) to first \( m \) coordinates. Hence \( \left\| Pz \right\| = \left\| UPz \right\| \). Therefore we can just consider the orthogonal projection onto first \( m \) coordinates. Hence the projection matrix \( P \) is real. Let \( e_1, \ldots, e_m \) be the standard basis corresponding to the first \( m \) coordinates. Let \( A \) be the \( n \times m \) matrix whose columns are \( e_1, \ldots, e_m \). Then \( P = A A^T \) (\( A^T \) denotes conjugate transpose). Since \( A \) is real, we have \( \text{Re}(Pz) = AA^T \text{Re}(z) \) and \( \text{Im}(Pz) = AA^T \text{Im}(z) \). Now, if \( z \sim \text{Unif}(CS)^{n-1} \) then \( \text{Re}(z) \) has same law as \( \text{Im}(z) \). Hence \( \text{Re}(Pz) \) has same law as \( \text{Im}(Pz) \). Further \( A^T = A^T \). Also note that, if \( z = x + iy \) then \( \left\| Pz \right\|^2 = z^* AA^T x + y^T AA^T y = \left[ x^T \ y^T \right] \begin{bmatrix} AA^T & 0 \\ 0 & AA^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \left\| P \begin{bmatrix} x \\ y \end{bmatrix} \right\| \) where \( P \) denotes the orthogonal projection from \( \mathbb{R}^{2n} \) to a \( 2m \) dimensional subspace. Hence \( \left\| Pz \right\|^2 \) has the same law as that of the projection of a uniform random vector on \( S^{2n-1} \) to a \( 2m \) dimensional subspace. Hence using lemma [A.3] we have

\[
\mathbb{P} \left[ \left\| Pz \right\| - \sqrt{\frac{m}{n}} > \delta_3 \right] \leq 2e^{-2cn \delta_3^2}
\] (171)

Since \( H_i \sim \text{CN}(0,1) \), we have \( |H_i|^2 \sim \frac{1}{2} \chi_2^2(2) = \text{exp}(1) \) where \( \chi_2(d) \) denotes the chi-squared distribution with \( d \) degrees of freedom and \( \text{exp}(1) \) represents an exponentially distributed random variable with rate 1. Therefore, for \( \nu \geq 0 \),

\[
\mathbb{P} [ |H_i|^2 \geq \nu ] = e^{-\nu}
\] (172)

Now, we are in a position to bound \( \mathbb{P} [ E_{12} ] \). For \( S \subset [K] \) with \( |S| = t \), define the events \( E_2, E_3 \) and \( E_4 \) as follows:

\[
E_2 = \bigcap_{i<j: i,j \in [K]} \left\{ |\langle \hat{c}_i, \hat{c}_j \rangle| \leq \delta_2 \right\}
\] (172a)

\[
E_3(S,t) = \bigcap_{i \in S} \left\{ \left\| P_{c_{B_i}} \hat{c}_i \right\| - \sqrt{\frac{|K-t|}{n}} \right\} \leq \delta_3
\] (173a)

\[
E_4 = \bigcap_{i \in [K]} \left\{ |H_i|^2 \leq \nu \right\}
\] (173b)
where we choose \( \delta_2 = n^{-\frac{1}{4}} = \delta_3 \) and \( \nu = n^{\frac{1}{4}} \). Hence we have

\[
\mathbb{P} [ E_{12} ] \leq \mathbb{P} \left[ \bigcup_{t,S} (A_S \cap E_2 \cap E_3 \cap E_4) \right] + \mathbb{P} [ E_2^r ] + \mathbb{P} [ E_4^r ] + \sum_{t,S} \mathbb{P} [ E_3^r(S,t) ].
\]  

(174)

Using lemmas \[A.3\] and \[A.5\] and eq. (172), we have

\[
\mathbb{P} [ E_2^r ] + \mathbb{P} [ E_4^r ] + \sum_{t,S} \mathbb{P} [ E_3^r(S,t) ] \leq 2K(K-1) e^{-\frac{n_2^2}{2}} + Ke^{-\nu} + \sum_{t,S} 2te^{-\text{cn}\delta_3^2}.
\]  

(175)

Note that the above quantity goes to 0 as \( n \to \infty \) due to the choice of \( \delta_2, \delta_3 \) and \( \nu \). Also, the choice of parameters is not the optimum. Nevertheless, this is enough to prove the result.

Let \( \delta_{1,t} = \left( \delta_2 + \left( \delta_3 + \sqrt{\frac{K-t}{n}} \right) \right)^n \). \( \delta_{2,t} = \left( \delta_3 + \sqrt{\frac{K-t}{n}} \right)^n \). Observe that on the sets \( E_2, E_3 \) and \( E_4 \), we have from (163)

\[
\left| \text{Re} \left( \frac{P_{c_1|c_2}}{P_{c_1|c_2}} c_i, P_{c_1|c_2} c_j \right) \right| H_i H_j \leq \nu \delta_{1,t} = O(n^{-\frac{\alpha}{2}})
\]

(176a)

\[
|H_i|^2 \left| \frac{P_{c_1|e_S}}{P_{c_1|e_S}} \right| c_i \|^2 \leq \nu \delta_{2,t} = O(n^{-\frac{\alpha}{2}})
\]  

(176b)

So we have

\[
\mathbb{P} \left[ \bigcup_{t,S} (A_S \cap E_2 \cap E_3 \cap E_4) \right] \leq \mathbb{P} \left[ \bigcup_{t,S} \left( V_{n,S} \left| 1 + \frac{nP}{n'} \sum_{i \in S} |H_i|^2 \|\hat{c}_i\|^2 - n\delta_{2,t} - t(t-1)\delta_{1,t} \right| - 1 \leq \delta_1 \right) \right]
\]

\[
\leq \mathbb{P} \left[ \bigcup_{t,S} \left( V_{n,S} \left| 1 + \frac{nP}{n'} \sum_{i \in S} |H_i|^2 \right| \leq 1 + \delta_1 + O(n^{-\frac{\alpha}{2}}) \right) \right]
\]

\[
\leq \mathbb{P} \left[ \bigcup_{t,S} \left( \ln \left| 1 + P \sum_{i \in S} |H_i|^2 \right| \leq V_{n,S}' \right) \right]
\]  

(177)

where \( V_{n,S}' = V_{n,S} + \ln(1 + \delta_1 + O(n^{-1/2})) \), and \( O \) depends on \( K \) and \( t \).

Let \( \delta_n = \ln(1 + \delta_1 + O(n^{-1/2})) \). We have \( \frac{\log M_s}{n-K} = \left( \sum_{i \in S}(R_i - \eta_i) \right) (1 + o(1)) \) and \( s_t = O \left( \frac{\log n}{n} \right) \).

By the choice of \( M_t^{(n)} \), for sufficiently large \( n \), sufficiently small \( \delta \) and \( \delta_1 \), we have

\[
\mathbb{P} \left[ \bigcup_{t,S} \left( \ln \left| 1 + P \sum_{i \in S} |H_i|^2 \right| \leq V_{n,S}' \right) \right] = \mathbb{P} \left[ \bigcup_{t,S} \ln \left| 1 + P \sum_{i \in S} |H_i|^2 \right| \leq s_t + \frac{\log M_s}{n-K} + \delta + \delta_n \right]
\]

\[
= \mathbb{P} \left[ \bigcup_{t,S} \log \left| 1 + P \sum_{i \in S} |H_i|^2 \right| \right]
\]

\[
\leq s_t \log_2(e) + \left( \sum_{i \in S}(R_i - \eta_i) \right) (1 + o(1)) + (\delta + \delta_n) \log_2(e)
\]

\[
\leq \mathbb{P} \left[ \bigcup_{t,S} \left( \log \left| 1 + P \sum_{i \in S} |H_i|^2 \right| \leq \left( \sum_{i \in S} R_i \right) \right) \right] \]

(178)

Finally combining everything, we have

\[
\epsilon_n \leq \mathbb{P} \left[ \bigcup_{t,S} \left( \log \left| 1 + P \sum_{i \in S} |H_i|^2 \right| \leq s_t \log_2(e) + \left( \sum_{i \in S}(R_i - \eta_i) \right) (1 + o(1)) + (\delta + \delta_n) \log_2(e) \right] \right]
\]

\[
+ 2K(K-1) e^{-\frac{n_2^2}{2}} + Ke^{-\nu} + \sum_{t,S} 2te^{-\text{cn}\delta_3^2} + \epsilon^{-\delta(n-K)} + e^{-nf_n(\delta_1)} \]

(179)

Therefore for this choice of \( M_t^{(n)} \), from (178) we have

\[
\lim_{n \to \infty} \epsilon_n
\]
\[
\mathbb{P} \left[ \bigcup_{i,S} \left\{ \log \left( 1 + P \sum_{i \in S} |H_i|^2 \right) \leq \left( \sum_{i \in S} R_i \right) \right\} \right] \leq \epsilon
\]

Since \( \eta_i > 0 \) were arbitrary, we are done. That is (136) is also satisfied.

\( \Box \)

\section*{B. Per-user error}

\textbf{Proof of theorem III.2.} We need to show that there exists a sequence of \( \left( M_1^{(n)}, M_2^{(n)}, \ldots, M_K^{(n)} \right) \) codes with the decoder given by (15) and (16) such that

\[
\lim_{n \to \infty} \frac{1}{n} \log \left( M_i^{(n)} \right) \geq R_i, \forall i \in [K] \quad (181)
\]

\[
\lim_{n \to \infty} \sup_{\epsilon_n} \epsilon_n \leq \epsilon. \quad (182)
\]

Let \( P_{\epsilon}^S(R) < \epsilon \) and \( \eta_i > 0, i \in [K] \). Choose \( M_i^{(n)} = \left[ e^{n(R_i - \eta_i)} \right], \forall i \in [K] \). We use random coding with Gaussian codebooks: user \( i \) generates \( M_i^{(n)} \) codewords \( \{ c_j^{i} : j \in [M_i^{(n)}] \} \) distributed as \( \mathcal{CN}(0, P_n^{i} I_n) \) independent of other users, where \( P_n^{i} = \frac{P}{1 + \eta_i} \). Here \( \mathcal{CN}(\mu, \Sigma) \) denotes the complex normal distribution with mean \( \mu \), covariance \( \Sigma \) and pseudo-covariance 0. For the (random) message \( W_i \in [M_i] \), user \( i \) transmits \( X_i = c_j^{W_i} 1 \{ \| c_j^{W_i} \|^2 > nP \} \). The channel model is given in (5) and the decoder is given by (15) and (16). The per-user probability of error is given by

\[
P_e = \mathbb{E} \left[ \frac{1}{K} \sum_{j=1}^{K} 1 \left\{ W_j \neq (g_D(Y))_j \right\} \right]. \quad (183)
\]

Similar to the proof of Theorem 1, we change the measure over which \( \mathbb{E} \) is taken in (183) to the one where \( X_i = c_j^{W_i} \) at the cost of adding a total variation distance. Hence the probability of error under this change of measure becomes

\[
P_e \leq p_1 + p_0
\]

with

\[
p_0 = K \mathbb{P} \left[ \| w \|^2 > n \frac{P}{P_n^{i}} \right] \quad (184)
\]

\[
p_1 = \mathbb{E} \left[ \frac{1}{K} \sum_{j=1}^{K} 1 \left\{ W_j \neq (g_D(Y))_j \right\} \right] \quad (185)
\]

where \( w \sim \mathcal{CN}(0, I_n) \) and, with abuse of notation, \( \mathbb{E} \) in \( p_1 \) is taken over the new measure. It can be easily seen that by the choice of \( P_n^{i} \) and lemma IV.2, \( p_0 \to 0 \) as \( n \to \infty \). From now on, we exclusively focus on bounding \( p_1 \).

\( p_1 \) can also be written as

\[
p_1 = \frac{1}{K} \mathbb{E} \left[ \sum_{j \in D} 1 \left\{ W_j \neq (g_D(Y))_j \right\} + |D^c| \right] = 1 - \frac{\mathbb{E} |D|}{K} + \frac{1}{K} \mathbb{E} \left[ \sum_{j \in D} 1 \left\{ W_j \neq (g_D(Y))_j \right\} \right] \quad (186)
\]

because, for \( i \in D^c, 1 \left\{ W_j \neq (g_D(Y))_j \right\} = 1, \ a.s. \) Define \( p_2 \) as

\[
p_2 = \mathbb{P} \left[ \sum_{j \in D} 1 \left\{ W_j \neq (g_D(Y))_j \right\} > 0 \right]. \quad (187)
\]

So, it’s enough to show that \( p_2 \to 0 \) as \( n \to \infty \). This is because, if \( p_2 \to 0 \), then the non-negative random variables \( A_n = \sum_{j \in D} 1 \left\{ W_j \neq (g_D(Y))_j \right\} \) converge to 0 in probability. Since \( A_n \leq K, \ a.s. \), we have, by dominated convergence,

\[
\mathbb{E} [A_n] = \mathbb{E} \left[ \sum_{j \in D} 1 \left\{ W_j \neq (g_D(Y))_j \right\} \right] \to 0. \quad (189)
\]

To this end, we upper bound \( p_2 \).
Let \( c = (c_1, \ldots, c_K) \) be the tuple of sent codewords. Let \( K_1 = |D| \). Let \( c_{(D)} \) denote the ordered tuple corresponding to indices in \( D \). That is, if \( i_1 < i_2 < \ldots < i_{K_1} \) are the elements of \( D \), then \( (c_{(D)})_j = c_{i_j}, \forall j \in [K_1] \). Let

\[
P_S = \left\{ \left\| P_{c_{(D)}} Y \right\| > \left\| P_{c_{(D)}} Y \right\| \right\}
\]

Then \( p_2 \) can also be written as

\[
p_2 = \mathbb{P} \left[ \sum_{i \in D} 1 \left\{ W_j \neq (g_D(Y))_j \right\} > 0 \right]
\]

\[
= \mathbb{P} \left[ \exists S \subset D, S \neq \emptyset : \forall i \in S, (g_D(Y))_i \neq W_i \right]
\]

\[
= \mathbb{P} \left[ \exists c_{(D)} \neq c_{(D)} : \left\| P_{c_{(D)}} Y \right\| > \left\| P_{c_{(D)}} Y \right\| \right]
\]

\[
= \mathbb{P} \left[ \bigcup_{t \in [K_1]} \bigcup_{|S| = t} \bigcup_{i \in S} B_S \right]
\]

Let \( \delta > 0, G_S = g(Y, c_{[K]}, S, D) = \frac{\|Y\|^2 - \|P_{c_{(D)}} Y\|^2}{\|P_{c_{(D)}} Y\|^2}, M_S = \prod_{i \in S} (M_j - 1), s_t = (t - 1)\ln(\frac{n-K_t-1}{n-1}), r_t = s_t + \frac{\ln M_S}{n-K_t}, \)

\( \tilde{V}_{n,S} = r_t + \delta \) and \( V_{n,S} = e^{-\tilde{V}_{n,S}}. \) Denote \( \bigcup_{t \in [K_1]} \bigcup_{|S| = t} \bigcup_{i \in S} B_S \) as \( \bigcup_{t \in [K_1]} \bigcup_{|S| = t} \bigcup_{i \in S} B_S \), similarly for \( \bigcap \) and \( \Sigma \). Further, denote \( \bigcup_{t \in [K_1]} \bigcup_{|S| = t} \) as \( \bigcup_{t \in [K_1]} \), again similarly for \( \bigcap \) and \( \Sigma \).

Note that, since \( D \) is random, both \( M_S \) and \( V_{n,S} \) are random. But in the symmetric case only \( M_S \) is not random. Now, following steps similar to (141), (142), (144) and (147), we have

\[
p_2 \leq \mathbb{E} \left[ \sum_{t,S,K_1} e^{-(n-K_t)\delta} \right] + \mathbb{P} \left[ \bigcup_{t,S,K_1} G_S \geq V_{n,S} \right]
\]

\[
\leq \sum_{t,K} e^{-(n-K_t)\delta} + \mathbb{P} \left[ \bigcup_{t,S,K_1} G_S \geq V_{n,S} \right].
\]

So, the first term goes to 0 as \( n \to \infty \).

Let \( Z_D = Z + \sum_{i \in D^e} H_i c_i \). It can be easily seen that, similar to (149), we have

\[
\mathbb{P} \left[ G_S \geq V_{n,S} \right] \leq \mathbb{P} \left[ \left( 1 - V_{n,S} \right) P_{c_{[K]}}^\perp Z_D - V_{n,S} P_{c_{[K]}}^\perp \sum_{i \in S} H_i c_i \right] \geq V_{n,S} \left( \left\| P_{c_{[K]}}^\perp \sum_{i \in S} H_i c_i \right\| \right) \leq V_{n,S} \left( \left\| P_{c_{[K]}}^\perp \sum_{i \in S} H_i c_i \right\| \right).
\]

Now, conditional of \( H_{[K]} \) and \( c_{[D]} \), \( Z_D \sim \mathcal{CN}(0, 1 + P_n \sum_{i \in D^e} |H_i|^2) \). Hence \( P_{c_{[K]}}^\perp \left( Z_D - \frac{V_{n,S}}{1 - V_{n,S}} \sum_{i \in S} H_i c_i \right) \sim \mathcal{CN}(-\frac{V_{n,S}}{1 - V_{n,S}} \sum_{i \in S} H_i c_i, (1 + P_n \sum_{i \in D^e} |H_i|^2) P_{c_{[K]}}^\perp). \) Therefore

\[
\left\| P_{c_{[K]}}^\perp \left( Z_D - \frac{V_{n,S}}{1 - V_{n,S}} \sum_{i \in S} H_i c_i \right) \right\|^2 \sim \left( 1 + P_n \sum_{i \in D^e} |H_i|^2 \right) \frac{1}{2} \chi^2(2F, 2n')
\]

where

\[
F = \left( \frac{V_{n,S}}{1 - V_{n,S}} \right) \left( \frac{\sum_{i \in S} H_i c_i}{1 + P_n \sum_{i \in D^e} |H_i|^2} \right)
\]

\[
n' = n - K_1 + t.
\]

Let

\[
U = \frac{V_{n,S}}{1 - V_{n,S}} \left( P_{c_{[K]}}^\perp \sum_{i \in S} H_i c_i \right)^2 - n'
\]

\[
U' = \frac{1}{1 - V_{n,S}} (V_{n,S} W_S - 1)
\]

where \( W_S = \left( 1 + \frac{\sum_{i \in S} H_i c_i}{n'(1 + P_n \sum_{i \in D^e} |H_i|^2)} \right) \)
Hence $U = n'U^1$ and $F = \frac{V_{n',s}}{1 - V_{n',s}} n'(1 + U^1)$. So, similar to (153), we have

$$P \left[ \bigcup_{t,S,K_1} \{ G_s \geq V_{n,s} \} \right] \leq P \left[ \bigcup_{t,S,K_1} \{ T \geq U \} \right]$$

(200)

where $T = \frac{1}{2}
\chi'^2(2F; 2n') - (F + n')$.

Let $\delta_1 > 0$ and $E_{11} = \bigcap_{t,S,K_1} \{ U^1 > \delta_1 \} \in \sigma(H_{[K]}, c[D])$.

Now, similar to (158), we have

Let $\hat{A}$ and choose

where $T = \delta$.

So, by the chose of $\delta_1$, $\hat{A}$ and dominated convergence, the first term in (201) converges to 0 as $n \to \infty$. Next, we upper bound the second term $P \left[ E_{11}^c \right]$.

Let $A_S = \{ V_{n,s}W_s - 1 \leq \delta_1 \}$ and $E_{12} = \bigcup_{t,S,K_1} A_S$. Similar to (160), we have

$$P \left[ E_{11}^c \right] = P \left[ \bigcup_{t,S,K_1} \{ U^1 \leq \delta_1 \} \right] \leq P \left[ E_{12} \right].$$

(202)

Let $\hat{c}_i = c_i / \| c_i \|$. Let $\delta_2 > 0$, $\delta_3 > 0$, $\delta_4 > 0$ and $\nu > 1$. Define the events

$$E_2 = \bigcap_{i < j, i, j \in [K]} \{ | \langle \hat{c}_i, \hat{c}_j \rangle | \leq \delta_2 \}$$

(203a)

$$E_3(S, t) = \bigcap_{i \in S} \left\{ \| P_{e^{[i]}\hat{c}_i} \| - \sqrt{\frac{K_1 - t}{n}} \leq \delta_3 \right\}$$

(203b)

$$E_4 = \bigcap_{i \in [K]} \{ | H_i |^2 \leq \nu \}$$

(203c)

$$E_5 = \bigcap_{i \in [K]} \{ | \| c_i \| - \sqrt{nP_n^{i}} \| \leq \delta_4 \sqrt{nP_n^{i}} \}$$

(203d)

and choose $\delta_2 = O(n^{-\frac{1}{4}}) = \delta_3 = \delta_4$ and $\nu = O(n^{1/4})$.

Using these events we can bound $P \left[ E_{11}^c \right]$ as

$$P \left[ E_{11}^c \right] \leq \sum_{t,S,K_1} P \left[ E_{12}^c \right] + P \left[ E_2^c \right] + P \left[ E_3^c \right] + P \left[ E_4^c \right] + P \left[ E_5^c \right] +$$

$$E \left[ \sum_{t,S,K_1} P \left[ E_{3}^c(S, t) | H_{[K]} \right] \right].$$

(204)

From [47] Theorem 3.1.1], we have

$$P \left[ E_5^c \right] \leq 2Ke^{-c_1n\delta_2^2}$$

(205)

for some constant $c_1 > 0$. So, from lemma A.3, lemma A.5, (172) and (205), we have

$$P \left[ E_{11}^c \right] \leq \sum_{t,S,K_1} P \left[ A_S \cap E_2 \cap E_3(S, t) \cap E_4 \cap E_5 \right] +$$

$$2K(K - 1)e^{-n\delta_2^2} + K e^{-\nu} + 2K e^{-c_1n\delta_4^2} + \sum_{t,S} 2te^{-cn\delta_2^2}$$

(206)

So, by the chose of $\delta_i, i \in \{2, 3, 4\}$ and $\nu$, the exponential terms in the last expression go to 0 as $n \to \infty$. 

33
Let \( N = (1 + P_n \sum_{i \in D'} |H_i|^2) \), \( \delta_{1,t} = \left( \delta_2 + \left( \delta_3 + \sqrt{\frac{K_1 - 1}{n}} \right)^2 \right) \), \( \delta_{2,t} = \left( \delta_3 + \sqrt{\frac{K_1 - 1}{n}} \right)^2 \). Let \( \text{SINR}_n = \frac{P_n \sum_{i \in S} |H_i|^2}{N} \).

Now, arguing similar to (177), we get

\[
P \left[ \bigcup_{t,S,K_1} \left( A_S \cap E_2 \cap E_3(S,t) \cap E_4 \cap E_5 \right) \right] \\
\leq P \left[ \bigcup_{t,S,K_1} \left\{ V_{n,S} \left[ 1 + \left\{ \frac{n(1 - \delta_4)^2}{n'} \right\} \text{SINR}_n - (1 + \delta_4)^2 \left( \frac{nP_n}{n-K} t \nu \delta_2,t + \frac{nP_n'(t-1)}{n-K} \delta_1,t \right) \right] - 1 \leq \delta_1 \right\} \right] \\
\leq P \left[ \bigcup_{t,S,K_1} \{ \ln [1 + \text{SINR}_n] \leq V_{n,S}' \} \right] \\
\tag{207}
\]

where \( V_{n,S}' = \hat{V}_{n,S} + \ln(1 + \delta_1 + O(n^{-1/2})) \).

Let \( \delta_n = \ln(1 + \delta_1 + O(n^{-1/2})) \). We have \( \frac{\log M_n}{n-K} = \left( \sum_{i \in S} (R_i - \eta_i) \right) \left( 1 + O(1) \right) \). There for sufficiently large \( n \) and sufficiently small \( \delta \) and \( \delta_1 \), we have \( V_{n,S}' \leq \sum_{i \in S} R_i \) a.s. Hence

\[
P \left[ \bigcup_{t,S,K_1} \ln [1 + \text{SINR}_n] \leq \log_2(\epsilon) V_{n,S}' \right] \leq P \left[ \bigcup_{t,S,K_1} \ln [1 + \text{SINR}_n] \leq \sum_{i \in S} R_i \right] \\
\tag{209}
\]

But we know that \( P_n' \to P \), and on \( D \), from (15) we have

\[
\sum_{i \in S} R_i < \log \left( 1 + \frac{P \sum_{i \in S} |H_i|^2}{1 + P \sum_{i \in D'} |H_i|^2} \right) \quad a.s. \\
\tag{210}
\]

Hence the probability in (209) goes to 0 as \( n \to \infty \).

So combining everything from (193), (202), (206), (207), (208), (209) and (210), we get \( p_2 \to 0 \) as \( n \to \infty \). Therefor \( p_1 \to 1 - \frac{E[D]}{K} \) as \( n \to \infty \). Hence we have

\[
\epsilon_n = P_\epsilon \to 1 - \frac{E[D]}{K} < \epsilon. \\
\tag{211}
\]

Hence \( \limsup_{n \to \infty} \epsilon_n \leq \epsilon \). Further, since \( \eta_i > 0 \) were arbitrary, we can ensure \( \liminf_{n \to \infty} \frac{1}{n} \log M_i^{(n)} \geq R_i, \forall i \in [K] \).

\( \square \)

C. Proof of proposition 7

Proof. We prove the second upper bound in (17). This is based on a single-user converse using the genie argument. Formally, since we consider per-user error, it is enough to look at the event that a particular user is not decoded. Let \( W_i \overset{\text{iid}}{\sim} \text{unif}[M] \) be the message of user \( i \). The channel (5) can be written as \( Y = H_i X_1 + \tilde{Z} + Z \) where \( \tilde{Z} = \sum_{i=2}^{K} H_i X_i \) denotes the interference. Let \( L(Y) \) be the decoder output. Also, let \( L(Y, \tilde{Z}) \) be the decoder output when it has knowledge of \( \tilde{Z} \). Hence a converse bound \( \Pr[W_1 \neq (L(Y))] \geq \epsilon \) is implied by \( \Pr[W_1 \neq (L(Y, \tilde{Z}))] \geq \epsilon \) for all \( L(\cdot, \cdot) \). Since \( Y - \tilde{Z} \) is a sufficient statistic of \( (Y, \tilde{Z}) \) for \( W_1 \), we have, equivalently, \( \Pr[W_1 \neq (L(Y - \tilde{Z}))] \geq \epsilon \) for all \( L(\cdot) \). Letting \( \hat{Y} = Y - \tilde{Z} \), this is equivalent to a converse for the channel \( \hat{Y} = H_1 X_1 + Z \): \( \Pr[W_1 \neq (L(\hat{Y}))] \geq \epsilon \) for all \( L(\cdot) \). This is just the usual single user converse, and hence the bound is given by \( R \leq C_\epsilon = \sup \{ \epsilon : \Pr[\log_2(1 + P |H_1|^2) \leq \xi] \leq \epsilon \} = \log_2(1 - P \ln(1 - \epsilon)) \) (18).

\( \square \)
Proof of claim. We have \( \|Y\|^2 - \|P_{A_0}Y\|^2 = \|P_{A_0}^\perp \hat{Z}\|^2 \leq \|\hat{Z}\|^2 - \|P_{A_1} \hat{Z}\|^2 = \|P_{A_1}^\perp \hat{Z}\|^2 \).

Also, \( \|P_{A_1}^\perp Y\|^2 = \|P_{A_1}^\perp \sum_{i \in S_2^c} H_{ci} + P_{A_1}^\perp \hat{Z}\|^2 \). Hence we have

\[
p_2 = P \left\{ \sum_{i \in S_2^c} |H_{ci}| \geq V_{n,t} \right\}
= P \left\{ \sum_{i \in S_2^c} |H_{ci}| \geq V_{n,t} \sum_{i \in S_2^c} |H_{ci}| \right\}
= P \left\{ \sum_{i \in S_2^c} |H_{ci}| \geq V_{n,t} \right\}
= P \left\{ \sum_{i \in S_2^c} |H_{ci}| \geq V_{n,t} \sum_{i \in S_2^c} |H_{ci}| \right\}
= \left\{ \sum_{i \in S_2^c} |H_{ci}| \geq V_{n,t} \right\}
\]

(212)

Proof of claim. Conditional of \( H_{[k]} \) and \( A_0, \hat{Z} \sim \mathcal{CN} \left( 0, \left( 1 + P^* \sum_{i \in S \setminus S_2^c} |H_i|^2 \right) \right) \). Hence

\[
P_{A_1}^\perp \left( \hat{Z} - \frac{V_{n,t}}{1 - V_{n,t}} \sum_{i \in S_2^c} H_{ci} \right) \sim \mathcal{CN} \left( - \frac{V_{n,t}}{1 - V_{n,t}} P_{A_1}^\perp \sum_{i \in S_2^c} H_{ci}, \left( 1 + P^* \sum_{i \in S \setminus S_2^c} |H_i|^2 \right) P_{A_1}^\perp \right).
\]

Now, the rank of \( P_{A_1}^\perp \) is \( n - K_1 + t \) because the vectors in \( A_1 \) are linearly independent almost surely. Let \( U \) be a unitary change of basis matrix that rotates the range space of \( P_{A_1}^\perp \) to the space corresponding to first \( n - K_1 + t \) coordinates. Then

\[
\| \mathcal{CN} \left( - \frac{V_{n,t}}{1 - V_{n,t}} P_{A_1}^\perp \sum_{i \in S_2^c} H_{ci}, \left( 1 + P^* \sum_{i \in S \setminus S_2^c} |H_i|^2 \right) P_{A_1}^\perp \right) \|^2
= \| U \mathcal{CN} \left( - \frac{V_{n,t}}{1 - V_{n,t}} P_{A_1}^\perp \sum_{i \in S_2^c} H_{ci}, \left( 1 + P^* \sum_{i \in S \setminus S_2^c} |H_i|^2 \right) P_{A_1}^\perp \right) U^* \|^2
= \mathcal{CN} \left( - \frac{V_{n,t}}{1 - V_{n,t}} U P_{A_1}^\perp \sum_{i \in S_2^c} H_{ci}, \left( 1 + P^* \sum_{i \in S \setminus S_2^c} |H_i|^2 \right) U P_{A_1}^\perp U^* \right).
\]

(214)

Observe that \( UP_{A_1}^\perp U^* \) is a diagonal matrix with first \( n - K_1 + t \) diagonal entries being ones and rest all 0. Also, if \( W = \hat{P} + i\hat{Q} \sim \mathcal{CN}(\mu, \Gamma) \) (with pseudo-covariance being 0) then

\[
\left[ \begin{array}{c} \hat{P} \\ \hat{Q} \end{array} \right] \sim \mathcal{N} \left( \left[ \begin{array}{cc} \text{Re}(\mu) & 1/2 \\ \text{Im}(\mu) & -\text{Re}(\mu) \end{array} \right] \right).
\]

(215)

Using this and the definition of non-central chi-squared distribution the claim follows.
Proof of Claim \[4\] We have

\[
f_n(x) = x + 1 + \frac{2V_{n,t}}{1 - V_{n,t}}(1 + x) - \sqrt{1 + \frac{2V_{n,t}}{1 - V_{n,t}}(1 + x)} \sqrt{2x + 1 + \frac{2V_{n,t}}{1 - V_{n,t}}(1 + x)}
\]

\[
= \frac{1}{1 - V_{n,t}} \left[ (1 + V_{n,t})(x + 1)) - 2\sqrt{V_{n,t}}\left(\frac{(1 + V_{n,t})^2}{4V_{n,t}} \right)^2 - \frac{(1 - V_{n,t}^2)^2}{16V_{n,t}^2} \right] \tag{216}
\]

Hence

\[
f'(x) = \frac{1}{1 - V_{n,t}} \left[ 1 + V_{n,t} - 2\sqrt{V_{n,t}} \frac{a}{\sqrt{a^2 - b^2}} \right]
\]

\[
= \frac{1}{1 - V_{n,t}} \left( \sqrt{V_{n,t}} - \sqrt{\frac{a + b}{a - b}} \left( \sqrt{V_{n,t}} - \sqrt{\frac{a - b}{a + b}} \right) \right) \tag{217}
\]

where \(a = \left( x + \frac{(1 + V_{n,t})^2}{4V_{n,t}} \right)\) and \(b = \frac{1 - V_{n,t}^2}{4V_{n,t}}\). Also \(a > 0\) and \(b > 0\). Further \(a + b > a - b\) and

\[
\sqrt{V_{n,t}} < \sqrt{\frac{a - b}{a + b}} = \sqrt{\frac{V_{n,t}(1 + V_{n,t} + 2x)}{1 + V_{n,t} + 2V_{n,t}x}}
\]

\[
\iff 2V_{n,t}x + 1 + V_{n,t} < 2x + 1 + V_{n,t}
\]

\[
\iff 0 < V_{n,t} < 1
\]

which is true. Hence both the factors in (217) are negative. Therefore \(f'(x) > 0\). \(\square\)
Appendix C
Maximal Per-User Error

In this section we briefly describe relations between maximal per-user error (PUPE-max) defined in (9) and PUPE. First, we represent our system as in (120)

\[ Y = AHW + Z. \]  

(218)

Let \( P_{e,i}(A) = \mathbb{P}[W_i \neq \hat{W}_i] \). We are interested in bounding the variance of \( P_{e,i}(A) \) so that

\[ \mathbb{E} \left[ P_{e,u}^{\text{max}}(A) \right] = \mathbb{E} \left[ \max P_{e,i}(A) : i \in [K] \right] \]

can be related to \( \mathbb{E} \left[ P_{e,i}(A) \right] = \mathbb{E} \left[ P_{e,u} \right] \) due to symmetry on users by random codebook generation. Consider two coupled systems

\[ Y = AHW + Z \]
\[ Y' = A'HW + Z \]

(219)

(220)

where \( A \) and \( A' \) are fixed so that the channels are dependent on these.

Now we have

\[ |P_{e,i}(A) - P_{e,i}(A')| \leq d_{TV}(P_{Y,H,W}, P_{Y',H,W}) \leq \sqrt{\frac{1}{2} D(P_{Y,H,W} || P_{Y',H,W})} \]  

(221)

where \( d_{TV}(P, Q) = \sup \{ |P(A) - Q(A)| : A \text{ is measurable} \} \) is the total variation distance between measures \( P \) and \( Q \), \( D(P || Q) = \mathbb{E}_P \left[ \ln \frac{dP}{dQ} \right] \) is the Kullback-Leibler divergence (in nats) and the last inequality is the Pinsker’s inequality (see [48]). Now using properties of \( D \) (see [49] Theorem 2.2)

\[ D(P_{Y,H,W} || P_{Y',H,W}) = D(P_{Y|H,W} || P_{Y'|H,W} | P_{H,W}) \]
\[ = \int_{H,W} D(P_{Y|H=W=w} || P_{Y'|H=W=w}) dP_{H,W}(h, w) \]  

(222)

Now note that conditioned on \( H = h, W = w \), we have \( Y \sim \mathcal{C}\mathcal{N}(Ahw, I_n) \) and \( Y' \sim \mathcal{C}\mathcal{N}(A'hw, I_n) \). Hence a simple computation shows that \( D(P_{Y|H=h,W=w} || P_{Y'|H=h,W=w}) = \| Ahw - A'hw \|^2 \). Therefore we have

\[ D(P_{Y,H,W} || P_{Y',H,W}) = \mathbb{E} \left[ \| (A - A')HW \|^2 \right]. \]  

(223)

Now let \( B = A - A' \) and \( X = HW \). Then

\[ \mathbb{E} \left[ \| BX \|^2 \right] = \sum_{i \in [n]} \mathbb{E} \left[ \sum_{j,k \in [KM]} B_{ij} \bar{B}_{ik} X_i \bar{X}_k \right] \]  

(224)

Note that \( \mathbb{E} \left[ X_j \bar{X}_k \right] \) is zero if \( j \neq k \) and it is \( 1/M \) otherwise. Hence

\[ \mathbb{E} \left[ \| BX \|^2 \right] = \frac{1}{M} \sum_{i \in [n]} \sum_{j \in [KM]} B_{ij} \bar{B}_{ij} = \frac{1}{M} \| B \|^2_F. \]  

(225)

Therefore

\[ D(P_{Y,H,W} || P_{Y',H,W}) = \frac{1}{M} \| A - A' \|^2_F. \]  

(226)

So combining this with (221), we obtain

\[ |P_{e,i}(A) - P_{e,i}(A')| \leq \frac{1}{2M} \| A - A' \|^2_F. \]  

(227)

Now let each entry of \( A \) and \( A' \) to be distributed iid as \( \mathcal{C}\mathcal{N}(0, P) \) where \( P = P_{tot}/K \). Further, let \( \tilde{A} = \sqrt{\frac{K}{P_{tot}}} A \) and \( \tilde{A}' = \sqrt{\frac{K}{P_{tot}}} A' \). So the entries of \( \tilde{A} \) and \( \tilde{A}' \) are iid \( \mathcal{C}\mathcal{N}(0, 1) \). Therefore, with slight abuse of notation, we can rewrite (227) as

\[ |P_{e,i}(\tilde{A}) - P_{e,i}(\tilde{A}')| \leq \frac{P_{tot}}{2MK} \| \tilde{A} - \tilde{A}' \|^2_F. \]  

(228)
Hence the function $P_{e,i}$ is Lipschitz with Lipschitz constant $L = \sqrt{\frac{P_{\text{tot}}}{2M}}$. By concentration of Lipschitz functions of Gaussian random vectors [30, Theorems 5.5, 5.6], we have that $P_{e,i}(\hat{A})$ is sub-Gaussian with

$$\text{Var}(P_{e,i}(A)) \leq 4L^2 = \frac{2P_{\text{tot}}}{KM}. \quad (229)$$

Hence, using bounds on expected maximum of sub-Gaussian random variables (see [30, Section 2.5]), we obtain

$$\mathbb{E}\left[\max_{i \in [K]} P_{e,i}(A)\right] \leq \mathbb{E}[P_{e,u}] + \sqrt{\text{Var}(P_{e,i}(A)) \ln K}$$

$$= \mathbb{E}[P_{e,u}] + \sqrt{\frac{2P_{\text{tot}} \ln K}{M}} \xrightarrow{K \to \infty} \mathbb{E}[P_{e,u}]. \quad (230)$$

Therefore, a random coding argument along with (230) shows that PUPE-max has same asymptotics as PUPE in the linear scaling regime. For FBL performance, if each user sends $k = 100$ bits then $M = 2^k$ and hence $\mathbb{E}[P_{e,u}^{\text{max}}] \approx \mathbb{E}[P_{e,u}]$.