Minimum energy to send k bits through the Gaussian channel with and without feedback

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Abstract

The minimum achievable energy per bit over memoryless Gaussian channels has been previously addressed in the limit when the number of information bits goes to infinity, in which case it is known that the availability of noiseless feedback does not lower the minimum energy per bit, which is -1.59 dB below the noise level. This paper analyzes the behavior of the minimum energy per bit for memoryless Gaussian channels as a function of k, the number of information bits. It is demonstrated that in this non-asymptotic regime, noiseless feedback leads to significantly better energy efficiency. In particular, without feedback achieving energy per bit of -1.57 dB requires coding over at least $k = 10^6$ information bits, while we construct a feedback scheme that transmits a single information bit with energy -1.59 dB and zero error. We also show that unless k is very small, approaching the minimal energy per bit does not require using the feedback link except to signal that transmission should stop.

Index Terms

Shannon theory, channel capacity, minimum energy per bit, feedback, non-asymptotic analysis, Gaussian channels, Brownian motion, stop-feedback.

I. Introduction

A problem of broad practical interest is to transmit a message with minimum energy. For the additive white Gaussian noise (AWGN) channel, the key parameters of the code are:

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- n: number of degrees of freedom,
- k: number of information bits,
- ϵ: probability of block error and
- E: total energy budget.

Of course, it is not possible to construct a code with arbitrary values of n, k, ϵ and E. Determining the region of feasible (n, k, ϵ, E) has received considerable attention in information theory, primarily in various asymptotic regimes:

1) The first asymptotic result dates back to [1], where Shannon demonstrates that in the limit of $\epsilon \to 0$, $k \to \infty$, $n \to \infty$ and $\frac{k}{n} \to 0$ the smallest achievable energy per bit $E_{\mathsf{b}} \triangleq \frac{E}{k}$ converges to
$$\left(\frac{E_{\mathsf{b}}}{N_0}\right)_{min} = \log_e 2 = -1.59 \text{ dB}\,, \tag{1}$$
where $\frac{N_0}{2}$ is the noise power per degree of freedom. The limit does not change if ϵ is fixed, if noiseless causal feedback is available at the encoder, if the channel is subject to fading, or even if the modulation is suitably restricted.
2) Alternatively, if one fixes $\epsilon > 0$ and the rate $\frac{k}{n} = R$ then as $k \to \infty$ and $n \to \infty$ we have (e.g., [2])
$$\frac{E_{\mathsf{b}}}{N_0} \to \frac{4^R - 1}{2R}\,. \tag{2}$$
Thus in this case the minimum energy per bit becomes a function of R, but not ϵ. In contrast to (1), (2) only holds with coherent demodulation and is sensitive to both modulation and fading; see [3].
3) Non-asymptotically, in the regime of fixed rate R and ϵ, bounds on the minimum E_{b} for finite k have been proposed [4], [5], studied numerically [6]–[10] and tightly approximated [5], [11].

In this paper we investigate the minimal energy E required to transmit k bits allowing error probability $\epsilon \geq 0$ and $n \to \infty$. Equivalently, we determine the maximal number of bits of information that can be transmitted with a fixed (non-asymptotic) energy budget and an error probability constraint, but without any limitation on the number of degrees of freedom (time-bandwidth product). This is different from [1] in that we do not take $k \to \infty$, and from [4]–[11] in that we do not fix a non-zero rate $\frac{k}{n}$. By doing so, we obtain a *bona fide* energy-information tradeoff in the simplest possible setting of the AWGN channel not subject to fading. Even though

the asymptotic value (1) can be obtained from (2) (i.e. from the regime of restricted rate) by taking $R \to 0$, the minimum energy for finite k cannot be obtained from the asymptotic limit in (2).

The paper is organized as follows. In Section II we state the problem formally for both cases of communication with and without feedback. In Section III we present the main results of the paper and compare the two cases numerically. In particular, we demonstrate that without feedback achieving -1.57 dB energy per bit necessarily requires coding over $k = 10^6$ information bits while with feedback we construct a code that transmits $k = 1$ bit at the optimal -1.59 dB. This is the discrete-time counterpart of Turin's result [13] on infinite bandwidth continuous-time communication in the presence of white noise and noiseless feedback. Moreover, we show that as long as k is not too small (say, more than 100) a stop-feedback code (which uses the feedback link only to signal that the receiver does not need further transmissions) also closely approaches the fundamental limit, thereby eliminating the need for an instantaneous noiseless feedback link. In general, for values of k ranging from 1 to 2000 feedback results in about 10 to 0.5 dB improvement in energy efficiency, respectively.

II. Problem statement

Without constraints on the number of degrees of freedom, the AWGN channel acts between input space $\mathsf{A} = \mathbb{R}^\infty$ and output space $\mathsf{B} = \mathbb{R}^\infty$ by addition:

$$\mathbf{y} = \mathbf{x} + \mathbf{z}\,, \tag{3}$$

where $\mathbb{R}^\infty$ is the vector space of real valued sequences[1] $(x_1, x_2, \ldots, x_n, \ldots)$, $\mathbf{x} \in \mathsf{A}$, $\mathbf{y} \in \mathsf{B}$ and $\mathbf{z}$ is a random vector with independent and identically distributed (i.i.d.) Gaussian components $Z_k \sim \mathcal{N}(0, N_0/2)$ independent of $\mathbf{x}$.

Definition 1: An (E, M, ϵ) code (without feedback) is a list of codewords $(\mathbf{c}_1, \ldots, \mathbf{c}_M) \in \mathsf{A}^M$, satisfying

$$||\mathbf{c}_j||^2 \le E\,, j = 1, \ldots, M\,, \tag{4}$$

and a decoder $g : \mathsf{B} \to \{1, \ldots, M\}$ satisfying

$$\mathbb{P}[g(\mathbf{y}) \neq W] \le \epsilon\,, \tag{5}$$

[1]In this paper, boldface letters $\mathbf{x}$, $\mathbf{y}$ etc. denote the infinite dimensional vectors with coordinates x_k, y_k etc., correspondingly.

where $\mathbf{y}$ is the response to $\mathbf{x} = \mathbf{c}_W$, and W is the message which is equiprobable on $\{1, \ldots, M\}$. The fundamental energy-information tradeoff is given by

$$M^*(E, \epsilon) = \max\{M : \exists (E, M, \epsilon)\text{-code}\} \,. \tag{6}$$

Equivalently, we define the minimum energy per bit:

$$E_{\mathsf{b}}^*(k, \epsilon) = \frac{1}{k} \inf\{E : \exists (E, 2^k, \epsilon)\text{-code}\} \,. \tag{7}$$

Although we are interested in (7), $M^*(E, \epsilon)$ is more suitable for expressing our results and (7) is the solution to

$$k = \log_2 M^*(kE_{\mathsf{b}}^*(k, \epsilon), \epsilon) \,. \tag{8}$$

Note that (3) also models an infinite-bandwidth continuous-time Gaussian channel without feedback observed over an interval $[0, T]$, in which each component corresponds to a different tone in an orthogonal frequency division representation. In that setup, E corresponds to the allowed power P times T, and $\frac{N_0}{2}$ is the power spectral density of the white Gaussian noise.

Definition 2: An (E, M, ϵ) code with feedback is a sequence of encoder functions $\{f_k\}_{k=1}^{\infty}$ determining the channel input as a function of the message W and the past channel outputs,

$$X_k = f_k(W, Y_1^{k-1}) \,, \tag{9}$$

satisfying

$$\mathbb{E}\left[||\mathbf{x}||^2 | W = j\right] \leq E \,, \qquad j = 1, \ldots, M \,, \tag{10}$$

and a decoder $g : \mathsf{B} \to \{1, \ldots, M\}$ satisfying (5). The fundamental energy-information tradeoff with feedback is given by

$$M_{\mathsf{f}}^*(E, \epsilon) = \max\{M : \exists (E, M, \epsilon)\text{-code with feedback}\} \tag{11}$$

and the minimum energy per bit by

$$E_{\mathsf{f}}^*(k, \epsilon) = \frac{1}{k} \inf\{E : \exists (E, 2^k, \epsilon)\text{-code with feedback}\} \,. \tag{12}$$

We also define a special subclass of feedback codes:

Definition 3: An (E, M, ϵ) code with feedback is a *stop-feedback code* if its encoder functions satisfy

$$f_k(W, Y^{k-1}) = \tilde{f}_k(W) 1\{\tau \geq k\} \,, \tag{13}$$

for some sequence of functions $\tilde{f}_k : \{1, \ldots, M\} \to \mathbb{R}$ and a stopping time τ of the filtration $\sigma\{Y_1, \ldots, Y_j\}$. Therefore, the stop-feedback code uses the feedback link only once to send a "ready-to-decode" signal, which terminates the transmission.

Notice that instead of (10) we could have defined a weaker energy constraint by averaging over the codebook as follows:

$$\frac{1}{M} \sum_{j=1}^{M} \mathbb{E}\left[||\mathbf{x}||^2 | W = j\right] \leq E\,. \tag{14}$$

However, in the context of feedback codes constraints (10) and (14) are equivalent:

Lemma 1: Any (E, M, ϵ) feedback code satisfying energy constraint (14) can be modified to satisfy a stronger energy constraint (10).

The proof is given in Appendix A.

Similarly, one can show that for feedback codes, allowing random transformations in place of deterministic functions f_n does not lead to any improvements of fundamental limits M_{f}^* and E_{f}^*. Such claims, however are not true for either the non-feedback codes (Definition 1) or stop-feedback codes (Definition 3). In fact, for the former allowing either a randomized encoder $\{1, \ldots, M\} \to \mathbb{R}^\infty$ or imposing an average-over-the-codebook energy constraint (14) affects the asymptotic behavior of $\log M^*(E, \epsilon)$ considerably; see [14, Section 4.3.3].

III. Main results

In the context of finite-blocklength codes without feedback, we showed in [5] that the maximum rate compatible with a given error probability ϵ for finite blocklength n admits a tight analytical approximation which can be obtained by proving an asymptotic expansion under fixed ϵ and $n \to \infty$. We follow a similar approach in this paper obtaining upper and lower bounds on $\log M^*(E, \epsilon)$ and $\log M_{\mathsf{f}}^*(E, \epsilon)$ and corresponding asymptotics for fixed ϵ and $E \to \infty$.

A. No feedback

Theorem 2: For every $M > 0$ there exists an (E, M, ϵ) code for channel (3) with[2]

$$\epsilon \;=\; \mathbb{E}\left[\min\left\{MQ\left(\sqrt{\frac{2E}{N_0}} + Z\right), 1\right\}\right]\,, \tag{15}$$

[2]As usual, $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$ is defined for $-\infty < x < \infty$ and satisfies $Q^{-1}(1-x) = -Q^{-1}(x)$.

and $Z \sim \mathcal{N}(0,1)$. Conversely, any (E, M, ϵ) code without feedback satisfies

$$\frac{1}{M} \geq Q\left(\sqrt{\frac{2E}{N_0}} + Q^{-1}(1-\epsilon)\right) . \tag{16}$$

Proof: To prove (15), consider a codebook with M orthogonal codewords

$$\mathbf{c}_j = \sqrt{E}\,\mathbf{e}_j\,, \quad j = 1, \ldots, M \tag{17}$$

where $\{\mathbf{e}_j, j = 1, \ldots\}$ is a an orthonormal basis of $L_2(\mathbb{R}^\infty)$. Such a codebook under a maximum likelihood decoder has probability of error equal to

$$P_e = 1 - \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} \left[1 - Q\left(\sqrt{\frac{2}{N_0}} z\right)\right]^{M-1} e^{-\frac{(z-\sqrt{E})^2}{N_0}} dz\,, \tag{18}$$

which is obtained by observing that conditioned on $(W = j, Z_j)$ the events $\{||\mathbf{c}_j + \mathbf{z}||^2 \leq ||\mathbf{c}_j + \mathbf{z} - \mathbf{c}_i||^2\},\ i \neq j$ are independent. A change of variables $x = \sqrt{\frac{2}{N_0}} z$ and application of the bound $1 - (1-y)^{M-1} \leq \min\{My, 1\}$ weakens (18) to (15).

To prove (16) fix an arbitrary codebook $(\mathbf{c}_1, \ldots, \mathbf{c}_M)$ and a decoder $g : \mathsf{B} \to \{1, \ldots, M\}$. We denote the measure $P^j = P_{\mathbf{y}|\mathbf{x}=\mathbf{c}_j}$ on $\mathsf{B} = \mathbb{R}^\infty$ as the infinite dimensional Gaussian distribution with mean $\mathbf{c}_j$ and independent components with individual variances equal to $\frac{N_0}{2}$; i.e.,

$$P^j = \prod_{k=1}^{\infty} \mathcal{N}\left(c_{j,k}, \frac{N_0}{2}\right)\,, \quad n = 1, 2, \ldots \tag{19}$$

where $c_{j,k}$ is the k-th coordinate of the vector $\mathbf{c}_j$. We also define an auxiliary measure

$$\Phi = \prod_{k=1}^{\infty} \mathcal{N}\left(0, \frac{N_0}{2}\right)\,, \quad n = 1, 2, \ldots \tag{20}$$

Assume for now that the following holds for each j and event $F \in \mathcal{B}^\infty$:

$$P^j(F) \geq \alpha \Longrightarrow \Phi(F) \geq \beta_\alpha(E)\,, \tag{21}$$

where the right-hand side of (16) is denoted by

$$\beta_\alpha(E) = Q\left(\sqrt{\frac{2E}{N_0}} + Q^{-1}(\alpha)\right) . \tag{22}$$

From (21) we complete the proof of (16):

$$\begin{aligned} \frac{1}{M} &= \frac{1}{M} \sum_{j=1}^{M} \Phi(g^{-1}(j)) && (23) \\ &\geq \frac{1}{M} \sum_{j=1}^{M} \beta_{P^j(g^{-1}(j))}(E) && (24) \\ &\geq \beta_{1-\epsilon}(E)\,, && (25) \end{aligned}$$

where (23) follows because $g^{-1}(j)$ partitions the space B, (24) follows from (21), and (25) follows since the function $\alpha \to \beta_\alpha(E)$ is non-decreasing convex (e.g., [5, Section III.D-3]) for any E and

$$\frac{1}{M}\sum_{j=1}^{M} P^j(g^{-1}(j)) \geq 1-\epsilon \tag{26}$$

is equivalent to (5), which holds for every (E, M, ϵ) code.

To prove (21) we compute the Radon-Nikodym derivative

$$\log_e \frac{dP^j}{d\Phi}(\mathbf{y}) = \sum_{k=1}^{\infty}\left(-\tfrac{1}{2}c_{j,k}^2 + c_{j,k}Y_k\right) , \tag{27}$$

and hence $\log_e \frac{dP^j}{d\Phi}$ is distributed as

$$\log_e \frac{dP^j}{d\Phi}(\mathbf{y}) \sim \mathcal{N}\left(\frac{||\mathbf{c}_j||^2}{2}, N_0\frac{||\mathbf{c}_j||^2}{2}\right) \tag{28}$$

if $\mathbf{y} \sim P^j$ and as

$$\log_e \frac{dP^j}{d\Phi}(\mathbf{y}) \sim \mathcal{N}\left(-\frac{||\mathbf{c}_j||^2}{2}, N_0\frac{||\mathbf{c}_j||^2}{2}\right) \tag{29}$$

if $\mathbf{y} \sim \Phi$. Then, (21) follows by the Neyman-Pearson lemma since $||\mathbf{c}_j||^2 \leq E$ for all $j \in \{1, \ldots, M\}$. This method of proving a converse result is in the spirit of the meta-converse in [5, Theorem 26]. ■

For $M = 2$ bound (16) is equivalent to

$$\epsilon \geq Q\left(\sqrt{\frac{2E}{N_0}}\right) , \tag{30}$$

which coincides with the upper bound obtained via antipodal signalling. It is not immediately obvious, however, that the bounds on $\log M^*(E, \epsilon)$ (and, equivalently, on $E_{\mathsf{b}}^*(k, \epsilon)$) obtained in Theorem 2 are tight in general. The next result, however, shows that they do agree up to the first three terms in the asymptotic expansion. Naturally, these bounds are expected to be very sharp non-asymptotically, which is validated by the numerical evaluation in Section IV.

Theorem 3: In the absence of feedback, the number of bits that can be transmitted with energy E and error probability $0 < \epsilon < 1$ behaves as[3]

$$\log M^*(E, \epsilon) = \frac{E}{N_0}\log e - \sqrt{\frac{2E}{N_0}}Q^{-1}(\epsilon)\log e + \frac{1}{2}\log\frac{E}{N_0} + O(1) \tag{31}$$

[3]All logarithms, $\log$, and exponents, $\exp$, in this paper are taken with respect to an arbitrary fixed base, which also determines the information units.

as $E \to \infty$.

Proof: To obtain (31) fix $0 < \epsilon < 1$ and denote

$$x^* = \sqrt{\frac{2E}{N_0}} + Q^{-1}\left(1 - \epsilon + \sqrt{\frac{2N_0}{E}}\right). \tag{32}$$

We now choose $M = \frac{1}{Q(x^*)}$ and observe that we have

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \min(MQ(x), 1) e^{-\frac{1}{2}\left(x - \sqrt{\frac{2E}{N_0}}\right)^2} dx && (33)\\
&= 1 - Q\left(x^* - \sqrt{\frac{2E}{N_0}}\right) + \frac{M}{\sqrt{2\pi}} \int_{x^*}^{+\infty} Q(x) e^{-\frac{1}{2}\left(x - \sqrt{\frac{2E}{N_0}}\right)^2} dx && (34)\\
&= \epsilon - \sqrt{\frac{2N_0}{E}} + \frac{M}{\sqrt{2\pi}} \int_{x^*}^{+\infty} Q(x) e^{-\frac{1}{2}\left(x - \sqrt{\frac{2E}{N_0}}\right)^2} dx && (35)\\
&\leq \epsilon - \sqrt{\frac{2N_0}{E}} + \frac{M}{2\pi x^*} \int_{x^*}^{+\infty} e^{-\frac{1}{2}\left(x - \sqrt{\frac{2E}{N_0}}\right)^2 - \frac{x^2}{2}} dx && (36)\\
&= \epsilon - \sqrt{\frac{2N_0}{E}} + \frac{e^{-\frac{E}{2N_0}} Q\left(\sqrt{2}x^* - \sqrt{\frac{E}{N_0}}\right)}{2\sqrt{\pi} x^* Q(x^*)} && (37)\\
&= \epsilon - \sqrt{\frac{2N_0}{E}} + \frac{e^{-\frac{E}{2N_0} + \frac{(x^*)^2}{2} - \frac{1}{2}\left(\sqrt{2}x^* - \sqrt{\frac{E}{N_0}}\right)^2 + o(1)}}{2\sqrt{\pi}\left(\sqrt{2}x^* - \sqrt{\frac{E}{N_0}}\right)} && (38)\\
&\leq \epsilon - \sqrt{\frac{2N_0}{E}} + \frac{1 + o(1)}{2\sqrt{\pi}\left(\sqrt{2}x^* - \sqrt{\frac{E}{N_0}}\right)} && (39)\\
&\leq \epsilon - \sqrt{\frac{2N_0}{E}} + \sqrt{\frac{N_0}{E}}(1 + o(1)), && (40)
\end{aligned}$$

as $E \to \infty$, where (36) is by [15, (3.35)]

$$Q(x) \leq \frac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}}, \tag{41}$$

while in (38) we used [15, (3.53)]

$$\log Q(x) = -\frac{x^2 \log e}{2} - \log x - \frac{1}{2}\log 2\pi + o(1), \quad x \to \infty \tag{42}$$

(39) is by

$$\begin{aligned}
-\frac{E}{N_0} + (x^*)^2 - \left(\sqrt{2}x^* - \sqrt{\frac{E}{N_0}}\right)^2 &= -\left(Q^{-1}\left(1 - \epsilon + \sqrt{\frac{2N_0}{E}}\right)\right)^2 && (43)\\
&< 0, && (44)
\end{aligned}$$

which follows from (32), and (40) is because

$$2\sqrt{\pi}\left(\sqrt{2}x^* - \sqrt{\frac{E}{N_0}}\right) \geq \sqrt{\frac{E}{N_0}}\,, \tag{45}$$

is equivalent to (according to (32))

$$\epsilon \geq \sqrt{\frac{2N_0}{E}} + Q\left(\sqrt{\frac{E}{2N_0}}\left(1 - \frac{1}{2\sqrt{\pi}}\right)\right)\,, \tag{46}$$

which holds for all sufficiently large E.

Therefore, by (15) with $M = \left\lfloor \frac{1}{Q(x^*)} \right\rfloor$ we have demonstrated that for all sufficiently large E

$$\begin{aligned}
\log M^*(E,\epsilon) &\geq -\log Q\left(\sqrt{\frac{2E}{N_0}} + Q^{-1}\left(1 - \epsilon + \sqrt{\frac{2N_0}{E}}\right)\right) && (47)\\
&= -\log Q\left(\sqrt{\frac{2E}{N_0}} + Q^{-1}(1-\epsilon) + O\left(\sqrt{\frac{N_0}{E}}\right)\right) && (48)\\
&= \frac{E}{N_0}\log e + \sqrt{\frac{2E}{N_0}}Q^{-1}(1-\epsilon)\log e + \frac{1}{2}\log\frac{E}{N_0} + O(1)\,, && (49)
\end{aligned}$$

where (48) is by applying Taylor expansion to $Q^{-1}(x)$ for $x = 1-\epsilon$ and (49) is by using (42) and Taylor expansion of $\log x$. Finally, application of (42) to (16) results in a lower bound matching (49) up to $O(1)$ terms. ■

As discussed in Section II, Theorems 2 and 3 may be interpreted in the context of the infinite-bandwidth continuous-time Gaussian channel with noise spectral density $\frac{N_0}{2}$. Indeed, denote by $M_c^*(T,\epsilon)$ the maximum number of messages that is possible to communicate over such a channel over the time interval $[0,T]$ with probability of error ϵ and power-constraint P. According to Shannon [1] we have

$$\lim_{T\to\infty}\frac{1}{T}\log M_c^*(T,\epsilon) = \frac{P}{N_0}\log e\,. \tag{50}$$

Theorem 3 sharpens (50) to

$$\log M_c^*(T,\epsilon) = \frac{PT}{N_0}\log e - \sqrt{\frac{2PT}{N_0}}Q^{-1}(\epsilon)\log e + \frac{1}{2}\log\frac{PT}{N_0} + O(1) \tag{51}$$

as $T\to\infty$. Furthermore, Theorem 2 provides tight non-asymptotic bounds on $\log M_c^*(T,\epsilon)$.

B. Communication with feedback

We start by stating a non-asymptotic converse bound.

Theorem 4: Let $0 \le \epsilon < 1$. Any (E, M, ϵ) code with feedback for channel (3) must satisfy

$$d\left(1-\epsilon || \tfrac{1}{M}\right) \le \frac{E}{N_0} \log e\,, \tag{52}$$

where $d(x||y) = x \log \frac{x}{y} + (1-x) \log \frac{1-x}{1-y}$ is the binary relative entropy.

Note that in the special case $\epsilon = 0$ (52) reduces to

$$\log M \le \frac{E}{N_0} \log e\,. \tag{53}$$

Proof: Consider an arbitrary (E, M, ϵ) code with feedback, namely a sequence of encoder functions $\{f_n\}_{n=1}^{\infty}$ and a decoder map $g : \mathsf{B} \to \{1, \ldots, M\}$. The "meta-converse" part of the proof proceeds step by step as in the non-feedback case (19)-(26), with the exception that measures $P^j = P_{\mathbf{y}|W=j}$ on B are defined as

$$P^j = \prod_{k=1}^{\infty} \mathcal{N}(f_k(j, Y_1^{k-1}), \tfrac{1}{2} N_0) \tag{54}$$

for $j = 1, \ldots, M$ and β_α is replaced by $\tilde{\beta}_\alpha$, which is the unique solution $\tilde{\beta} < \alpha$ to

$$\tilde{\beta}_\alpha : \quad d(\alpha || \tilde{\beta}) = \frac{E}{N_0} \log e\,. \tag{55}$$

We need only to show that (21) holds with these modifications, i.e. for any $\alpha \in [0, 1]$

$$\inf_{F \subset \mathsf{B}: P^j(F) \ge \alpha} \Phi(F) \ge \tilde{\beta}_\alpha\,. \tag{56}$$

Once $W = j$ is fixed, the channel inputs X_k become functions $\mathsf{B} \to \mathbb{R}$:

$$X_k = f_k(j, Y_1^{k-1})\,. \tag{57}$$

To find the critical set F achieving the infimum in the hypothesis testing problem (56) we compute the Radon-Nikodym derivative:

$$\log_e \frac{dP^j}{d\Phi} = \sum_{k=1}^{\infty} X_k Y_k - \tfrac{1}{2} X_k^2\,. \tag{58}$$

Denote the total energy spent by the code

$$\tau = \sum_{k=1}^{\infty} X_k^2\,. \tag{59}$$

The key part of the proof is to show that (58) is equal to a Brownian motion with drift $\pm\frac{1}{2}$ (where the sign depends on the hypothesis P^j or Φ), evaluated at time τ, which in fact can be interpreted as a stopping time of the Brownian motion. Assuming this, the proof is completed by applying the following result of Shiryaev [30, Theorem 3, Section IV.2]:

Lemma 5 (Shiryaev): Consider a space

$$\Omega = C(\mathbb{R}_+, \mathbb{R}) \tag{60}$$

of continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}$ with the standard filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on Ω such that $\phi_t \sim B_t$ (under $\mathbb{P}$) and $\phi_t \sim \bar{B}_t$ (under $\mathbb{Q}$), where B_t and $\bar{B}_t$ denote Brownian motions

$$B_t = \frac{t}{2} + \sqrt{\tfrac{N_0}{2}} W_t \,, \tag{61}$$

$$\bar{B}_t = -\frac{t}{2} + \sqrt{\tfrac{N_0}{2}} W_t \,, \tag{62}$$

and W_t is a standard Wiener process for $t \in [0, \infty)$. Then

$$\min_{\tau_1, F_1} \mathbb{Q}(F_1) = \tilde{\beta}_\alpha \,, \tag{63}$$

where $\tilde{\beta}_\alpha$ is defined in (55) and the minimization is over all stopping times τ_1 and sets $F_1 \in \mathcal{F}_{\tau_1}$ such that

$$\int_\Omega \tau_1 d\mathbb{P} \leq E \,, \tag{64}$$

$$\mathbb{P}(F_1) \geq \alpha \,. \tag{65}$$

The application of Lemma 5 to our setting is the following. The left side of (56) can be lower bounded as

$$\inf_{F \subset \mathsf{B}: P^j(F) \geq \alpha} \Phi(F) = \inf_{F_1 \in \mathcal{F}_\tau : \mathbb{P}(F_1) \geq \alpha} \mathbb{Q}(F_1) \tag{66}$$

$$\geq \min_{F_1, \tau_1} \mathbb{Q}(F_1) \tag{67}$$

$$= \tilde{\beta}_\alpha \,, \tag{68}$$

where (66) is by the assumed equivalence $\log_e \frac{dP^j}{d\Phi} \sim B_\tau$ (under P^j) and $\log_e \frac{dP^j}{d\Phi} \sim \bar{B}_\tau$ (under Φ), (67) follows by minimizing over all stopping times τ_1 satisfying (64) which is valid since the expectation of τ (under P^j) satisfies (64) by energy constraint (10), and (68) is by Lemma 5.

We proceed to show that under P^j we have $\log_e \frac{dP^j}{d\Phi} \sim B_\tau$. To do this, we will redefine random variables $(Y_1, Y_2, \ldots)$ in terms of the Brownian motion B_t. First, note that without loss of generality we can choose nonvanishing f_k in (57), since having $X_k = 0$ does not help in discriminating P^j vs. Φ[4]. Then each Y_k is a one-to-one function of

$$L_k = X_k Y_k - \frac{1}{2} X_k^2 \,, \quad k = 1, \ldots \tag{69}$$

According to (57) we can rewrite then

$$X_k = \hat{f}_k(L_1^{k-1}) \,, \tag{70}$$

where $\hat{f}_k$ depends on the original encoder function f_k as well as the message $j \in \{1, \ldots, M\}$.

Given an element $\phi \in \Omega$ (see (60)) we define the following sequences:

$$\hat{\tau}_0 = 0 \,, \tag{71}$$

$$\hat{X}_k = \hat{f}_k(\hat{L}_1^{k-1}) \,, \tag{72}$$

$$\hat{\tau}_k = \hat{\tau}_{k-1} + \hat{X}_k^2 \,, \tag{73}$$

$$\hat{L}_k = \phi_{\hat{\tau}_k} - \phi_{\hat{\tau}_{k-1}} \,, \qquad k = 1, \ldots \tag{74}$$

We now show that each $\hat{\tau}_k$ is a stopping time of the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ on Ω. The proof is by induction. Clearly the statement holds for $\hat{\tau}_0$. Assume $\hat{\tau}_{k-1}$ is a stopping time. Then by (73) the time $\hat{\tau}_k$ is a positive increment of $\hat{\tau}_{k-1}$ by a $\mathcal{F}_{\hat{\tau}_{k-1}}$-measurable value. Thus $\hat{\tau}_k$ is also a stopping time. Consequently, the increasing limit

$$\hat{\tau} \triangleq \lim_{k \to \infty} \hat{\tau}_k \tag{75}$$

$$= \sum_{k=1}^{\infty} \hat{X}_k^2 \tag{76}$$

is also a stopping time of $\{\mathcal{F}_t\}_{t \geq 0}$.

Now, since $\mathbb{P}$ is such that (Lemma 5)

$$\phi_t \sim B_t \,, \tag{77}$$

[4]Note that a good coding scheme will always allow $X_k = 0$ for the purpose of conserving energy. However, we are free to make modifications to the encoding maps f_k provided that they do not increase the left-hand side of (56).

under $\mathbb{P}$ the distribution of $\hat{L}_n$ given $\hat{L}_1^{n-1}$ is $\mathcal{N}\left(\frac{1}{2}\hat{X}_n^2, \frac{N_0}{2}\hat{X}_n^2\right)$. On the other hand, under P^j the distribution of L_n given L_1^{n-1} is $\mathcal{N}\left(\frac{1}{2}X_n^2, \frac{N_0}{2}X_n^2\right)$. Since by (70) and (72), $\hat{X}_n$ and X_n are identical functions of $\hat{L}_1^{n-1}$ and L_1^n, respectively, we conclude that

$$(L_1^\infty, X_1^\infty) \sim (\hat{L}_1^\infty, \hat{X}_1^\infty) \tag{78}$$

Then, comparing (59) and (76) we obtain

$$\tau \sim \hat{\tau} \tag{79}$$

and, in particular,

$$\int_\Omega \hat{\tau} d\mathbb{P} \leq E \tag{80}$$

by (10).

Finally, we have

$$\begin{aligned} \log_e \frac{dP^j}{d\Phi} &= \sum_{k=1}^\infty L_k && (81)\\ &\sim \sum_{k=1}^\infty \hat{L}_k && (82)\\ &= \phi_{\hat{\tau}} && (83)\\ &\sim B_\tau\,, && (84) \end{aligned}$$

where (82) is by (78), (83) is by (74) and (76) and (84) is by (79) and (77).

Similarly, one shows that under Φ the distribution of $\log_e \frac{dP^j}{d\Phi}$ is equal to that of $\bar{B}_\tau$. Indeed, relations (78), (79) and (83) remain true if Y_1^∞ is given distribution Φ and ϕ is given a distribution $\mathbb{Q}$ (as in Lemma 5). ■

In [12] we have shown the following result:

Theorem 6: For any $E > N_0$, there exists an $(E, 2, 0)$-code with feedback. Consequently, for all positive integers k we have

$$E_{\mathsf{f}}^*(k, 0) \leq N_0\,. \tag{85}$$

Furthermore, the ternary constellation $\{-1, 0, +1\}$ suffices for the $(E, 2, 0)$ code.

At the expense of allowing constellations of unbounded cardinality Theorem 6 can be considerably sharpened. In fact, the next result shows that the availability of noiseless feedback allows the transmission of a single information bit ($k = 1$) at the optimal value of -1.59 dB. As in

the continuous-time AWGN channel with feedback [13], the proof of this result turns out to be rather non-trivial.

Theorem 7: For any $E > N_0 \log_e 2$ there exists an $(E, 2, 0)$-code with feedback. Consequently, for all positive integers k we have

$$E_{\mathsf{f}}^*(k, 0) = N_0 \log_e 2 \,. \tag{86}$$

Proof: We first show that the second claim follows from the first. Indeed, an $(E_1, M_1, 0)$ code and an $(E_2, M_2, 0)$ code can be combined into an $(E_1 + E_2, M_1 M_2, 0)$ code by using the first code on odd numbered channel inputs and the second code on even inputs. Thus, function $E_{\mathsf{f}}^*(\cdot, 0)$ is non-increasing and according to the first claim we have

$$E_{\mathsf{f}}^*(k, 0) \leq N_0 \log_e 2 \tag{87}$$

for all $k > 0$. Then (86) follows from (53) with $M = 2^k$.

To prove the first claim, it is convenient to assume that the message set is $\{-1, +1\}$ (instead of $\{1, 2\}$). We use the following encoding functions:

$$f_n(W, Y^{n-1}) = \frac{Wd}{1 + \exp\{W \cdot S_{n-1}\}} \,. \tag{88}$$

To motivate this choice assume that the sequence of encoder functions f_k is already fixed for $k = 1, \ldots, n-1$. Then the joint distribution of $(W, X_1^{n-1}, Y_1^{n-1})$ is completely specified once we specify that $W = \pm 1$ is equiprobable. Consequently, we can define information densities

$$\imath(w; y_1^k) = \sum_{j=1}^{k} \log \frac{P_{Y_j|X_j}(y_j | f_j(w; y_1^{j-1}))}{P_{Y_j|Y_1^{j-1}}(y_j | y_1^{j-1})} \,, \quad k = 1, \ldots, n-1 \tag{89}$$

and the log-likelihood process

$$\begin{aligned} S_k &= \log \frac{\mathbb{P}[W = +1 | Y^k]}{\mathbb{P}[W = -1 | Y^k]} && (90) \\ &= \imath(+1; Y_1^k) - \imath(-1; Y_1^k) \,, \quad k = 1, \ldots, n-1 \,. && (91) \end{aligned}$$

Notice now that the choice of f_n contributes $\mathbb{E}\left[|f_n(+1, Y^{n-1})|^2 | W = +1\right]$ to the energy $\mathbb{E}\left[||\mathbf{x}||^2 | W = +1\right]$ and $\mathbb{E}\left[|f_n(-1, Y^{n-1})|^2 \exp\{-S_{n-1}\} | W = +1\right]$ to the energy $\mathbb{E}\left[||\mathbf{x}||^2 | W = -1\right]$. Thus the contribution to the unconditional $\mathbb{E}\left[|\mathbf{x}|^2\right]$ is given by the expectation of

$$|f_n(+1, Y_1^{n-1})|^2 + |f_n(-1, Y_1^{n-1})|^2 \exp\{-S_{n-1}\} \,. \tag{92}$$

If we now fix an arbitrary $d > 0$ and impose an additional constraint

$$f_n(+1, Y_1^{n-1}) - f_n(-1, Y_1^{n-1}) = d\,, \tag{93}$$

then the minimum of (92) is achieved with the encoder function (88).

Having specified the full sequence of the encoder functions $f_j, j = 1, \ldots$, we have also determined the probability distribution of (W, X^∞, Y^∞). We now need to show that measures $P_{Y^\infty|W=+1}$ and $P_{Y^\infty|W=-1}$ are mutually singular and also to estimate the total energy spent by the scheme, that is the expectation of

$$E_d \stackrel{\triangle}{=} ||\mathbf{x}||^2 = \sum_{j=1}^{\infty} ||X_j||^2\,. \tag{94}$$

Note that by symmetry it is sufficient to analyze the case of $W = +1$, and so in all arguments below we assume that the distribution on (W, X^∞, Y^∞) is in fact normalized by conditioning on $W = +1$. For example, we now have $X_1 = \frac{d}{2}$ almost surely.

Notice that according to the definition in (89), we have

$$\begin{aligned}
&\imath(+1; y^n) - \imath(-1; y^n) && (95)\\
&= \imath(+1; y^{n-1}) - \imath(-1; y^{n-1}) + \log \frac{P_{Y_n|X_n}(y_n|f_n(+1, y^{n-1}))}{P_{Y_n|X_n}(y_n|f_n(-1, y^{n-1}))} && (96)\\
&= \imath(+1; y^{n-1}) - \imath(-1; y^{n-1}) + \frac{\log e}{N_0}\left[(y_n - f_n(-1, y^{n-1}))^2 - (y_n - f_n(+1, y^{n-1}))^2\right] && (97)\\
&= \imath(+1; y^{n-1}) - \imath(-1; y^{n-1}) + \frac{\log e}{N_0}\left(2y_n d - \frac{1-\exp\{S_{n-1}\}}{1+\exp\{S_{n-1}\}} d^2\right)\,, && (98)
\end{aligned}$$

where in the last step we have used definition of the encoder (88). If we now replace y^n with random variable Y^n in (98), then (under $W = +1$) we have $Y_n \sim \frac{d}{1+\exp\{S_{n-1}\}} + Z_n$, where $Z_n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. Therefore, almost surely we have for each n:

$$S_n = S_{n-1} + \frac{2\log e}{N_0}\left[\tfrac{1}{2}d^2 + d\,Z_n\right]\,, \tag{99}$$

where Z_n are i.i.d. with common distribution $\mathcal{N}\left(0, \frac{N_0}{2}\right)$.

From (99) we see that under $W = +1$, S_n is a submartingale drifting towards $+\infty$, which implies that the measures $P_{Y^\infty|W=+1}$ and $P_{Y^\infty|W=-1}$ are mutually singular and therefore W can be recovered from Y^∞ with zero error. To complete the proof we need to show

$$\lim_{d\to 0} \mathbb{E}\left[E_d\right] = N_0 \log_e 2\,. \tag{100}$$

First, notice that conditioned on $W = +1$ we have

$$E_d = \sum_{j=1}^{\infty} \left(\frac{d}{1 + \exp\{S_j\}} \right)^2 . \tag{101}$$

To simplify the computation of $\mathbb{E}\left[E_d\right]$, from now on replace $d\, Z_n$ in (99) with $W_{nd^2} - W_{(n-1)d^2}$, where W_t is a standard Wiener process. For convenience we also define the Brownian motion B_t as in (61). In this way, we can write

$$S_n = \frac{2 \log e}{N_0} B_{nd^2} \,, \tag{102}$$

i.e. S_n is just a sampling of B_t on a d^2-spaced grid. Consequently, the conditional energy in (101) is then given by

$$E_d = \sum_{j=1}^{\infty} \left(\frac{d}{1 + e^{\frac{2}{N_0} B_{jd^2}}} \right)^2 . \tag{103}$$

We now show that the collection of random variables $\{E_d, d \in (0, \sqrt{N_0})\}$ is uniformly integrable. Notice that for all

$$0 < d \leq \sqrt{N_0} \tag{104}$$

we have

$$\left\{ 4B_{jd^2} > jd^2 \text{ for all } j \geq \frac{\mathcal{E}}{d^2} \right\} \subseteq \left\{ E_d \leq \mathcal{E} + c \right\} , \tag{105}$$

where

$$c = \frac{1}{1 - e^{-1}} \frac{1}{N_0} > 0 \,. \tag{106}$$

Indeed, for any realization belonging to the set in the left-hand side of (105) we have

$$\begin{aligned}
E_d &= \sum_{j=0}^{\infty} \left(\frac{d}{1 + e^{\frac{2}{N_0} B_{jd^2}}} \right)^2 1\left\{ 4B_{jd^2} > jd^2 \right\} && (107)\\
&\quad + \sum_{j=0}^{\infty} \left(\frac{d}{1 + e^{\frac{2}{N_0} B_{jd^2}}} \right)^2 1\left\{ 4B_{jd^2} \leq jd^2 \right\} && (108)\\
&\leq \sum_{j=0}^{\infty} d^2 e^{-\frac{4}{N_0} B_{jd^2}} 1\left\{ 4B_{jd^2} > jd^2 \right\} + \sum_{j=0}^{\infty} d^2 1\left\{ 4B_{jd^2} \leq jd^2 \right\} && (109)\\
&\leq d^2 \sum_{j=0}^{\infty} e^{-\frac{1}{N_0} jd^2} + \sum_{j=0}^{\infty} d^2 1\left\{ 4B_{jd^2} \leq jd^2 \right\} && (110)\\
&\leq c + \sum_{j=0}^{\infty} d^2 1\left\{ 4B_{jd^2} \leq jd^2 \right\} && (111)\\
&\leq c + \mathcal{E} \,, && (112)
\end{aligned}$$

where (109) follows from the inequalities $(1+e^x)^{-1} \le e^{-x}$ and $(1+e^x)^{-1} \le 1$ applied to the first and second sum, respectively; (110) is because $4B_{jd^2} > jd^2$ in the first sum, (111) is by the inequality

$$\sum_{j=0}^{\infty} e^{-\lambda j} = \frac{1}{1-e^{-\lambda}} \le \frac{1}{1-e^{-1}}\frac{1}{\lambda}, \quad \forall 0 < \lambda \le 1, \tag{113}$$

applicability of which is assured by (104); and finally (112) follows since by assumption the realization satisfies

$$4B_{jd^2} > jd^2 \text{ for all } jd^2 \ge \mathcal{E}. \tag{114}$$

This establishes (105).

Assume the following identity (to be shown below):

$$\mathbb{P}[\tilde{B}_t > 0 \text{ for all } t > \mathcal{E}] = 1 - 2Q\left(\sqrt{\frac{\mathcal{E}}{8N_0}}\right), \tag{115}$$

where

$$\tilde{B}_t = \frac{1}{4}t + \sqrt{\frac{N_0}{2}}W_t. \tag{116}$$

Then consider the following chain

$$\begin{aligned}
\mathbb{P}[E_d > \mathcal{E} + c] &\le \mathbb{P}\left[\exists j \ge \frac{\mathcal{E}}{d^2} : \tilde{B}_{jd^2} \le 0\right] && (117)\\
&\le \mathbb{P}\left[\exists t > \mathcal{E} : \tilde{B}_t \le 0\right] && (118)\\
&= 2Q\left(\sqrt{\frac{\mathcal{E}}{8N_0}}\right) && (119)\\
&\le \sqrt{\frac{16N_0}{\pi\mathcal{E}}}e^{-\frac{\mathcal{E}}{16N_0}}, && (120)
\end{aligned}$$

where (117) is by (105), (119) is by (115), and (120) follows by the inequality (41). Clearly, a uniform (in d) exponential upper bound on the tail of the distribution E_d implies that random variables $\{E_d, 0 < d \le \sqrt{N_0}\}$ are uniformly integrable.

To show (115), define a collection of stopping times for $b > 0 > a$:

$$\tau_{a,b} = \inf\left\{t > 0 : \tilde{B}_t \notin (a,b)\right\}. \tag{121}$$

Applying Doob's optional stopping theorem to the stopping moment $\tau_{a,b}$ and martingale $e^{-\sqrt{\frac{1}{2N_0}}W_t - \frac{1}{4N_0}t}$, which is bounded (and hence uniformly integrable) on $[0, \tau_{a,b}]$, we obtain

$$\mathbb{P}[\tilde{B}_{\tau_{a,b}} = b] = \frac{1 - e^{-\frac{a}{N_0}}}{e^{-\frac{b}{N_0}} - e^{-\frac{a}{N_0}}}. \tag{122}$$

Moreover, as $b \to \infty$ we have $\{\tilde{B}_{\tau_{a,b}} = b\} \searrow \{\tilde{B}_t > a \text{ for all } t > 0\}$. Therefore, from (122) we get

$$\mathbb{P}[\tilde{B}_t > a \text{ for all } t > 0] = 1 - e^{\frac{a}{N_0}} \,. \tag{123}$$

Expression (115) now follows by the following calculation:

$$\begin{aligned}
\mathbb{P}[\tilde{B}_t > 0 \text{ for all } t > \mathcal{E}] &= \mathbb{E}\left[\mathbb{P}[\tilde{B}_t - \tilde{B}_{\mathcal{E}} > -\tilde{B}_{\mathcal{E}} \text{ for all } t > \mathcal{E} | \tilde{B}_{\mathcal{E}}]\right] && (124)\\
&= \int_0^\infty \frac{1}{\sqrt{\pi N_0 \mathcal{E}}} e^{-\frac{\left(x-\frac{\mathcal{E}}{4}\right)^2}{N_0 \mathcal{E}}} \mathbb{P}[\tilde{B}_t > -x \text{ for all } t > 0] dx && (125)\\
&= \int_0^\infty \frac{1}{\sqrt{\pi N_0 \mathcal{E}}} e^{-\frac{\left(x-\frac{\mathcal{E}}{4}\right)^2}{N_0 \mathcal{E}}} \left(1 - e^{-\frac{x}{N_0}}\right) dx && (126)\\
&= \int_0^\infty \frac{1}{\sqrt{\pi N_0 \mathcal{E}}} \left(e^{-\frac{\left(x-\frac{\mathcal{E}}{4}\right)^2}{N_0 \mathcal{E}}} - e^{-\frac{\left(x+\frac{\mathcal{E}}{4}\right)^2}{N_0 \mathcal{E}}}\right) dx && (127)\\
&= 1 - 2Q\left(\sqrt{\frac{\mathcal{E}}{8N_0}}\right) , && (128)
\end{aligned}$$

where (124) is by conditioning on $\tilde{B}_{\mathcal{E}}$, (125) is by the Markov property of Brownian motion and integrating over the distribution of $\tilde{B}_{\mathcal{E}} \sim \mathcal{N}\left(\frac{\mathcal{E}}{4}, \frac{N_0 \mathcal{E}}{2}\right)$, (126) is by (123), and (128) is just an elementary calculation.

According to (103) and the continuity of sample paths of B_t, we have

$$\lim_{d \to 0} E_d = \int_0^\infty \left(\frac{1}{1 + e^{\frac{2}{N_0} B_t}}\right)^2 dt \,. \tag{129}$$

Now taking the expectation we get

$$\begin{aligned}
\lim_{d \to 0} \mathbb{E}\left[E_d\right] &= \mathbb{E}\left[\lim_{d \to 0} E_d\right] && (130)\\
&= \mathbb{E}\left[\int_0^\infty \left(\frac{1}{1 + e^{\frac{2}{N_0} B_t}}\right)^2 dt\right] && (131)\\
&= \int_0^\infty \mathbb{E}\left[\left(\frac{1}{1 + e^{\frac{2}{N_0} B_t}}\right)^2\right] dt && (132)\\
&= \int_0^\infty \int_{-\infty}^\infty \left(\frac{1}{1 + e^{\frac{2}{N_0} x}}\right)^2 \frac{1}{\sqrt{\pi N_0 t}} e^{-\frac{\left(x-\frac{1}{2}t\right)^2}{N_0 t}} dx dt && (133)\\
&= N_0 \log_e 2 \,, && (134)
\end{aligned}$$

where (130) is by uniform integrability, (131) is by (129), and (132) is by Fubini's theorem, (133) is by using the fact that $B_t \sim \mathcal{N}\left(\frac{t}{2}, \frac{N_0 t}{2}\right)$, and (134) is obtained by the following argument.[5] If we define

$$u(x) \triangleq \int_0^\infty \frac{1}{\sqrt{\pi N_0 t}} e^{-\frac{\left(x-\frac{1}{2}t\right)^2}{N_0 t}} dt\,, \tag{135}$$

then its two-sided Laplace transform is given by

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-vx} u(x) dx &= \int_0^\infty \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0 t}} e^{-\frac{\left(x-\frac{1}{2}t\right)^2}{N_0 t} - vx} dx dt && (136)\\
&= \int_0^\infty e^{-t\frac{v}{2} + \frac{tN_0}{4}v^2} dt && (137)\\
&= \frac{4}{2v - N_0 v^2} && (138)\\
&= \frac{2}{v} + \frac{2N_0}{2 - N_0 v}\,, && (139)
\end{aligned}$$

provided that $0 < v < \frac{2}{N_0}$. It is straightforward to check that (139) is a Laplace transform of the function $2\min\{e^{\frac{2}{N_0}x}, 1\}$. By the uniqueness of the Laplace transform we conclude that

$$u(x) = 2\min\{e^{\frac{2}{N_0}x}, 1\}\,. \tag{140}$$

Now substituting this expression into (133) we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{1}{1+e^{\frac{2}{N_0}x}}\right)^2 u(x) dx &= 2\int_{-\infty}^{0} \left(\frac{1}{1+e^{\frac{2}{N_0}x}}\right)^2 \min\left\{e^{\frac{2}{N_0}x}, 1\right\} dx && (141)\\
&= N_0 \int_{-\infty}^{\infty} \left(\frac{1}{1+e^x}\right)^2 e^x dx + N_0 \int_0^\infty \left(\frac{1}{1+e^x}\right)^2 dx && (142)\\
&= \frac{N_0}{2} + N_0\left(\log_e 2 - \frac{1}{2}\right) && (143)\\
&= N_0 \log_e 2\,, && (144)
\end{aligned}$$

which completes the proof of (134). ■

We proceed to give a tight analysis of the large-energy behavior, based on Theorem 7.

Theorem 8: In the presence of feedback, the number of bits that can be transmitted with energy E and error probability $0 \le \epsilon < 1$ behaves as

$$\log M_{\mathsf{f}}^*(E, \epsilon) = \frac{E}{N_0}\frac{\log e}{1-\epsilon} + O(1) \tag{145}$$

[5]This elegant method was suggested by Yihong Wu.

as $E \to \infty$. More precisely, we have

$$\left\lceil \frac{E}{N_0} \frac{1}{1-\epsilon} \frac{1}{\log_e 2} - 1 \right\rceil \log 2 \;\leq\; \log M_{\mathsf{f}}^*(E,\epsilon) \tag{146}$$

$$\leq\; \frac{E}{N_0} \frac{\log e}{1-\epsilon} + \frac{h(\epsilon)}{1-\epsilon}\,, \tag{147}$$

where $h(x) = -x \log x - (1-x)\log(1-x)$ is the binary entropy function.

Proof: Fix $\epsilon \geq 0$ and $E > 0$. Then by Theorem 7 there exists an $\left(\frac{E}{1-\epsilon}, M, 0\right)$ feedback code with

$$\log M \;=\; \left\lceil \frac{E}{N_0} \frac{1}{1-\epsilon} \frac{1}{\log_e 2} - 1 \right\rceil \log 2 \tag{148}$$

Then, we can randomize between this code and a trivial $(0, M, 1)$ code (which sends an all-zero codeword for all messages) by using the former with probability $(1-\epsilon)$.

We now describe this randomization procedure formally, by constructing a code satisfying Definition 2. Let (f_n, g) be the sequence of encoders and a decoder corresponding to the code in (148). We construct a new code as follows:

$$f_n'(W, Y^{n-1}) \;=\; \begin{cases} 0\,, & n = 1\,, \\ 1\left\{Y_1 \leq \sqrt{\frac{N_0}{2}} Q^{-1}(\epsilon)\right\} f_{n-1}(W, Y_2^{n-1})\,, & n \geq 2\,, \end{cases} \tag{149}$$

$$g'(Y^\infty) \;=\; g(Y_2^\infty)\,. \tag{150}$$

Denote the event

$$S = \left\{ Y_1 \leq \sqrt{\frac{N_0}{2}} Q^{-1}(\epsilon) \right\}\,, \tag{151}$$

which has probability

$$\mathbb{P}[S] = 1 - \epsilon\,. \tag{152}$$

The probability of error of the new code is estimated as

$$\mathbb{P}[g'(Y^\infty) \neq W] \;=\; \mathbb{P}[g'(Y^\infty) \neq W | S]\mathbb{P}[S] + \mathbb{P}[g'(Y^\infty) \neq W | S^c]\mathbb{P}[S^c] \tag{153}$$

$$\leq\; 0 \cdot \mathbb{P}[S] + 1 \cdot \mathbb{P}[S^c] \tag{154}$$

$$\leq\; \epsilon\,, \tag{155}$$

since conditioned on S the transmission is governed by the code $\{f_n, n = 1, \ldots\}$ whose decoder g recovers the message with zero-error by construction. Similarly, the average energy of the new

code is

$$\sum_{n=1}^{\infty}\mathbb{E}\left[\left|f'_n(W,Y^{n-1})\right|^2\right] = \sum_{n=2}^{\infty}\mathbb{E}\left[\left|f_{n-1}(W,Y_2^{n-1})\right|^2\,\middle|\,S\right]\mathbb{P}\left[S\right] \tag{156}$$

$$\leq (1-\epsilon)\frac{E}{1-\epsilon} \tag{157}$$

$$= E\,, \tag{158}$$

where (156) is because on S^c the energy expenditure is zero, and (157) is because conditioned on S the transmission is governed by the original code $\{f_n, n=1,\ldots\}$, which by construction has an average energy not exceeding $\frac{E}{1-\epsilon}$. Therefore (f'_n, g') is indeed an (E, M, ϵ) feedback code with M satisfying (148). Existence of such a code implies (146). Bound (147) follows from (52) and

$$d(\alpha||\beta) \geq \alpha\log\frac{1}{\beta} - h(\alpha)\,. \tag{159}$$

■

A similar argument shows that in terms of $E_{\mathsf{f}}^*(k,\epsilon)$ the converse (Theorem 4) and the achievability (Theorem 7 plus randomization) translate into

$$\left(1-\epsilon-\frac{h(\epsilon)}{k}\right)\log_e 2 \leq \frac{E_{\mathsf{f}}^*(k,\epsilon)}{N_0} \leq (1-\epsilon)\log_e 2\,. \tag{160}$$

C. Stop feedback

The codes constructed in the previous section achieve the optimal value of energy per bit already for $k=1$. However, they require the availability of full instantaneous noiseless feedback. From a practical point of view, this may not always be attractive. In contrast, stop-feedback codes only exploit the feedback link to terminate the transmission, which makes such codes robust to noise in the feedback link. In this section we show that such codes also achieve the optimal value of energy per bit as long as the value of k is not too small.

Theorem 9: For any $E>0$ and positive integer M there exists an (E, M, ϵ) stop-feedback code whose probability of error is bounded by

$$\epsilon \leq (M-1)e^{-\frac{E}{N_0}}\,. \tag{161}$$

The proof of this result is given in Appendix B. Asymptotically, Theorem 9 implies the following lower bound on $\log M_{\mathsf{f}}^*$:

Theorem 10: For any error probability $0 < \epsilon < 1$, stop-feedback codes achieve

$$\log M_{\mathsf{f}}^*(E,\epsilon) \geq \frac{E}{N_0}\frac{\log e}{1-\epsilon} - \log\frac{E}{N_0} + O(1) \tag{162}$$

as $E \to \infty$.

Proof: Fix $\epsilon > 0$ and $E > 1$. By Theorem 9 there exists an $\left(\frac{E-1}{1-\epsilon}, M, \frac{1}{E}\right)$ stop-feedback code with

$$M \geq \frac{1}{E} e^{\frac{E-1}{N_0(1-\epsilon)}} . \tag{163}$$

Then, we can randomize between this code and a trivial $\left(0, M, 1-\frac{1}{M}\right)$ code (which sends an all-zero codeword for all messages) by using the latter with probability $\epsilon - \frac{1-\epsilon}{E-1}$.

We now describe this randomization procedure formally. The stop-feedback code with size lower-bounded by (163) is defined by the following three functions (see Definition 3):

1) a sequence of non-feedback encoder maps $\tilde{f}_n : \{1,\ldots,M\} \to \mathbb{R}$, $n = 1,\ldots$,
2) a decoder map $g : \mathbb{R}^\infty \to \{1,\ldots,M\}$, and
3) a stopping time: $\tau : \mathbb{R}^\infty \to \mathbb{Z}_+$, which is a measurable function satisfying an additional requirement that for any $n \geq 0$ the set $\{\tau(y^\infty) \leq n\}$ is a function of only $y^n = (y_1,\ldots,y_n)$.

From $(\tilde{f}, g, \tau)$ we construct a new code $(\tilde{f}', g', \tau')$ as follows:

$$\tilde{f}'_n(W) = \begin{cases} 0\,, & n = 1\,, \\ \tilde{f}_{n-1}(W)\,, & n \geq 2 \end{cases} \tag{164}$$

$$g'(Y^\infty) = g(Y_2^\infty) \tag{165}$$

$$\tau'(Y^\infty) = 1 + \tau(Y_2^\infty) 1\left\{ Y_1 \leq \sqrt{\frac{N_0}{2}} Q^{-1}\left(\epsilon - \frac{1-\epsilon}{E-1}\right)\right\} \tag{166}$$

One easily verifies that $\{\tau'(Y^\infty) \leq n\}$ depends only on Y^n for any $n \geq 1$, i.e. τ' is indeed a stopping time of the filtration $\{\sigma(Y^n), n \geq 0\}$.

The overall probability of error is then upper-bounded by

$$\begin{aligned}
&\mathbb{P}[g'(Y^\infty) \neq W] \\
&= \mathbb{P}[g(Y_2^\infty) \neq W | \tau' = 1]\mathbb{P}[\tau' = 1] + \mathbb{P}[g(Y_2^\infty) \neq W | \tau' > 1]\mathbb{P}[\tau' > 1] && (167)\\
&= \left(1 - \frac{1}{M}\right)\mathbb{P}[\tau' = 1] + \mathbb{P}[g(Y_2^\infty) \neq W | \tau' > 1]\mathbb{P}[\tau' > 1] && (168)\\
&\leq \mathbb{P}[\tau' = 1] + \frac{1}{E}\mathbb{P}[\tau' > 1] && (169)\\
&\leq \left(\epsilon - \frac{(1-\epsilon)}{E-1}\right) + \frac{1}{E} \cdot \frac{(1-\epsilon)E}{E-1} && (170)\\
&= \epsilon\,, && (171)
\end{aligned}$$

where (168) is because conditioned on $\tau' = 1$, random variables W and Y_2^∞ are independent, (169) is because conditioned on $\tau' > 1$ transmission is governed by the original code (f_n, g) which has probability of error $\frac{1}{E}$ by construction, and (170) is because

$$\begin{aligned}
\mathbb{P}[\tau' = 1] &= \mathbb{P}\left[Y_1 > \sqrt{\frac{N_0}{2}} Q^{-1}\left(\epsilon - \frac{1-\epsilon}{E-1}\right)\right] && (172)\\
&= \epsilon - \frac{1-\epsilon}{E-1}\,. && (173)
\end{aligned}$$

Similarly, the average energy of the encoder $\{f'_n, n = 1, \ldots\}$ is upper-bounded by

$$0 \cdot \left(\epsilon - \frac{(1-\epsilon)}{E-1}\right) + \frac{E-1}{1-\epsilon} \cdot \frac{(1-\epsilon)E}{E-1} = E\,. \qquad (174)$$

Thus, we have constructed an (E, M, ϵ) stop-feedback code with M satisfying (163) as required.

■

D. Schalkwijk-Kailath codes

It is instructive to compare our results with the various constructions based on the Schalkwijk-Kailath method [16]. Although none of such constructions can beat the codes of Theorem 7 (which essentially match the converse bound; see (146)-(147)), we discuss them here for completeness. Detailed proofs can be found in Appendix C.

There are several different non-asymptotic bounds that can be obtained from the Schalkwijk-Kailath method. Here are some of the results:

1) The original result of Schalkwijk-Kailath [16, (6)-(12)] proves that for any $E > \frac{N_0}{2}$ and positive integers L and M there exists an (E, M, ϵ) code with[6]

$$\epsilon = 2Q\left(\frac{\sqrt{3L}\sqrt{\frac{2E}{N_0} - \log_e L - 1}}{M}\right) . \tag{175}$$

Notice that when $e^{\frac{2E}{N_0}-2}$ is a positive integer the value of L minimizing the right-hand side of (175) is given by that integer. For such values of E we get from (175) the following lower bound on $\log M_{\mathsf{f}}^*(E, \epsilon)$:

$$\log M_{\mathsf{f}}^*(E, \epsilon) \geq \frac{E}{N_0}\log e + \frac{1}{2}\log\frac{3}{e} - \log Q^{-1}\left(\frac{\epsilon}{2}\right) . \tag{176}$$

2) Elias [17] proposed a method for transmitting a Gaussian random variable over the AWGN channel with feedback (see also [18] and [19]). Such a method leads to another variation of Schalkwijk-Kailath, whose precise analysis is reported in [20, Section III] (see also [21, p. 18-6]). Taking the infimum in [20, (21)] over all $nS = \frac{E}{N_0}$ proves that (176) holds for all values of energy $E > 0$.

3) Zigangirov [23, (20)] optimized the locations of a uniform pulse amplitude modulation (PAM) constellation in [16] to better approximate the normal distribution obtaining

$$\log M_{\mathsf{f}}^*(E, \epsilon) \geq \frac{E}{N_0}\log e + \frac{1}{2}\log\frac{\pi}{2} - \log Q^{-1}\left(\frac{\epsilon}{2}\right) , \tag{177}$$

for all $E > 0$ and $0 < \epsilon < 1$, which improves (176).

Pinsker [22] claimed that there exist coding schemes for the AWGN channel with noiseless feedback achieving m-fold exponential decrease of probability of error (in blocklength). For the formal proof of this result, Zigangirov [23] proposed to supplement the Schalkwijk-Kailath method by a second phase which significantly reduces average energy by adaptively modifying the constellation so that the most likely message (as estimated by the receiver) is mapped to zero. A similar idea has been proposed by Kramer [24] for communication with orthogonal waveforms and was shown to achieve an m-fold exponential probability of error. In the context of fixed-energy, Zigangirov's method results in the following zero-error bound, whose proof is found in Appendix C:

[6]We used an upper-bound $\sum_{i=1}^{L-1}\frac{1}{i} < 1 + \log_e L$ in [16, (12)].

Theorem 11: For any $M \in \{2, 3, \ldots\}$ and

$$\frac{E}{N_0} > \frac{1}{2} + \log_e \frac{87(M-1)}{32} \tag{178}$$

there exists an $(E, M, 0)$ code. Equivalently, we have

$$\frac{E_{\mathsf{f}}^*(k,0)}{N_0} \quad \leq \quad \log_e 2 + \frac{1}{k}\left(\frac{1}{2} + \log_e \frac{87(1-2^{-k})}{32}\right) . \tag{179}$$

Gallager and Nakiboğlu [20] devised a modification of Zigangirov's second phase in order to obtain a better bound on the optimal behavior of the probability of error in the regime of fixed-rate feedback communication over the AWGN channel. In the present zero-error context, which is not the main focus of [20], the analysis in [20, Section V.B] can be shown to imply the following zero-error feedback achievability bound:

$$\log M_{\mathsf{f}}^*(E,0) \geq \frac{E}{N_0} \log e - \log \frac{2e^3}{\sqrt{3}} , \tag{180}$$

or, equivalently,

$$\frac{E_{\mathsf{f}}^*(k,0)}{N_0} \leq \log_e 2 + \frac{1}{k} \log_e \frac{2e^3}{\sqrt{3}} . \tag{181}$$

Numerical comparison of the bounds (146), (161), (176) and (177) for $\epsilon = 10^{-3}$ is shown on Fig. 1. Each bound is computed by fixing a number of information bits k and finding the smallest E for which a $(2^k, E, 10^{-3})$ code is guaranteed to exist; the plot shows $\frac{E_b}{N_0} = \frac{E}{kN_0}$ (dB). The converse bound (Theorem 4) is not shown since it is indistinguishable, see (160), from the bound achieved by the codes of Theorem 8 (hence the name, "optimal"). It can be seen that for $k \gtrsim 300$ the difference between the bounds becomes negligible so that even the stop-feedback bound (the weakest on the plot) achieves energies below -1.5 dB, while for smaller values of k the advantage of 1-bit method of Theorem 7 becomes more significant.

Fig. 2 compares the *zero-error* feedback achievability bounds (181), (179) and the optimal code as given by Theorem 7. As expected the optimal code yields a significantly better energy per bit for smaller values k. Further discussion and comparison with the non-feedback case is given in Section IV.

E. Discussion

At first sight it may be plausible that, when zero-error is required, infinite bandwidth may allow finite energy per bit even in the absence of feedback. However, by taking $\epsilon \to 0$ in (16)

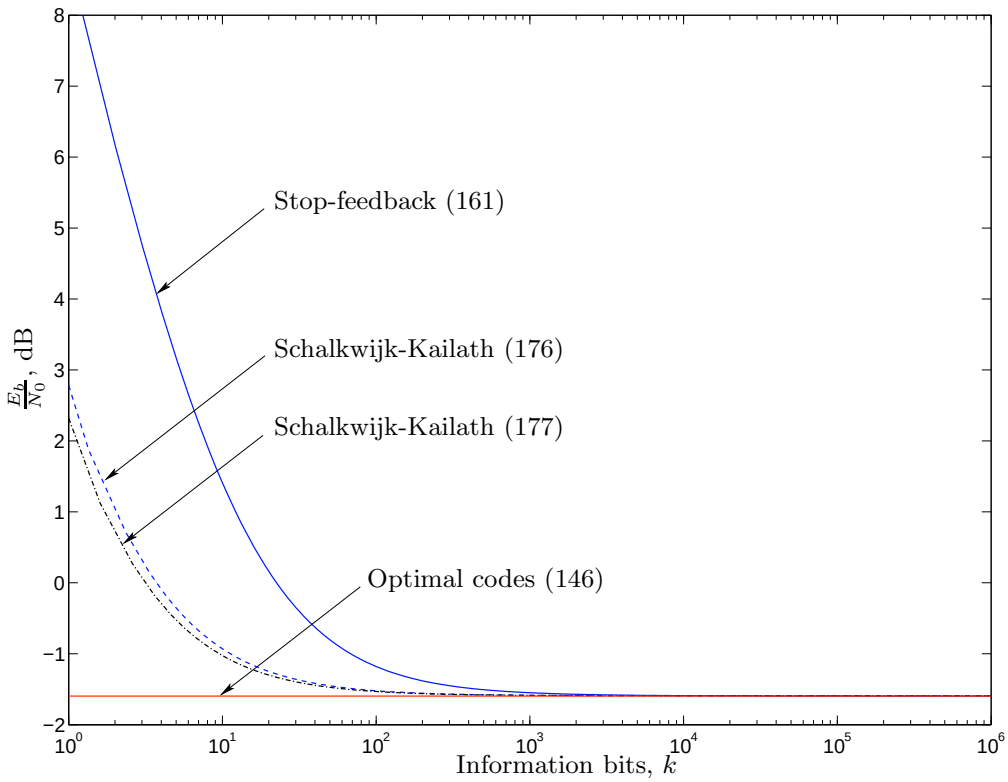

Fig. 1. Comparison of various feedback achievability bounds on the minimum energy per bit as a function of the number of information bits; block error rate $\epsilon = 10^{-3}$.

we obtain

$$M^*(E, 0) = 1 \tag{182}$$

for all $E > 0$. Equivalently, this can be seen as a consequence of [25]. At the same time, for $\epsilon = 0$ with feedback we have (Theorem 8)

$$\log M_{\mathsf{f}}^*(E, 0) = \frac{E}{N_0} \log e + O(1)\,, \tag{183}$$

in stark contrast with the non-feedback case (182).

Note also that as $\epsilon \to 0$, the leading term in (145) coincides with the leading term in (31). As we know, in the regime of arbitrarily reliable communication (and therefore $k \to \infty$) feedback does not help.

Theorems 6, 7, 11 and (180) demonstrate that noiseless feedback (along with infinite bandwidth) allows for zero-error communication with finite average energy. This phenomenon is not unique to the AWGN as the following simple argument demonstrates.

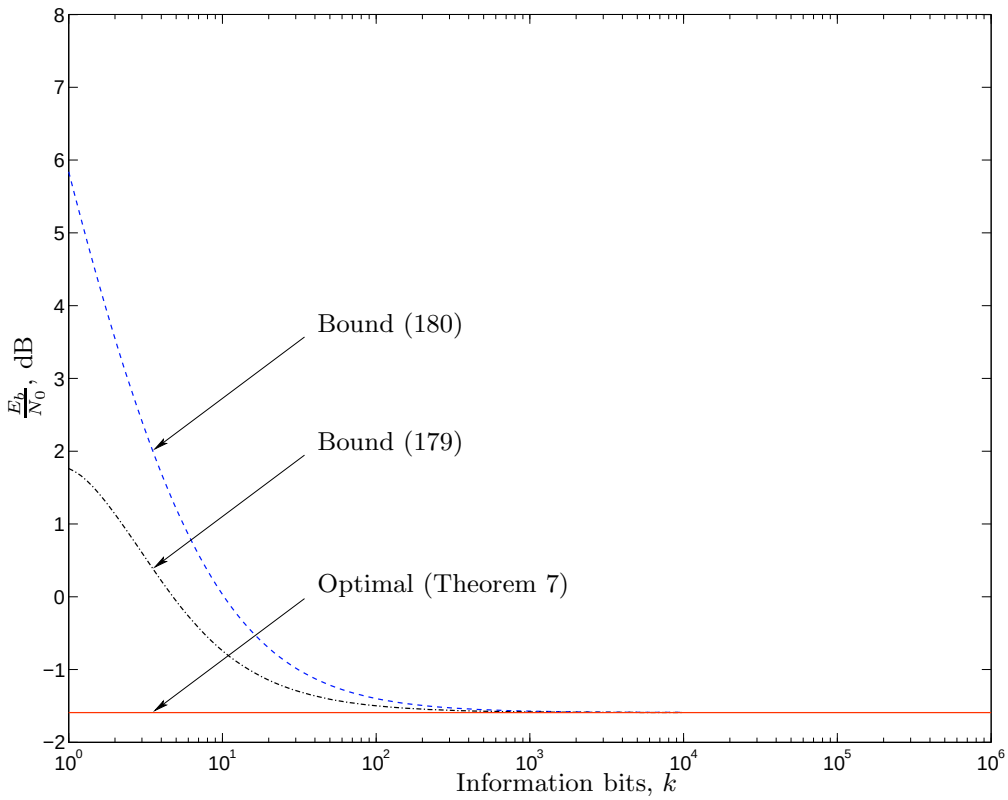

Fig. 2. Comparison of the minimum energy per bit achievable by the *zero-error* feedback codes obtained from Gallager-Nakiboğlu [20] (upper bound (180)), Zigangirov [23] (upper bound (179)) and Theorem 7.

Consider an arbitrary memoryless channel $P_{Y|X}$ with cost function $\mathsf{c}(x)$ and a zero-cost symbol x_0; see [31] for details. Pick an arbitrary symbol x_1 such that $\mathsf{c}(x_1) > 0$ and

$$D(P_{Y|X=x_1}||P_{Y|X=x_0}) > 0\,. \tag{184}$$

First, consider a non-feedback code with $M = 2$ mapping message $W = 1$ to an infinite string of x_0's and message $W = 2$ to an infinite string of x_1's. Due to the memorylessness of the channel and (184), the maximum likelihood message estimate $\hat{W}$ based on an infinite string of observations $(Y_1, \ldots)$ is exact:

$$\mathbb{P}[W \neq \hat{W}] = 0\,. \tag{185}$$

Moreover the maximum likelihood estimate $\hat{W}_n$ based on the first n observations $(Y_1, \ldots, Y_n)$ satisfies

$$\mathbb{P}[W \neq \hat{W}_n | W = m] \leq \exp\{-n\theta\}\,, \quad m = 1, 2 \tag{186}$$

for some positive θ. The total cost for such a two-codeword code is infinite because

$$\mathbb{E}\left[\sum_{j=1}^{\infty}\mathsf{c}(X_j)\,\middle|\,W=2\right]=\infty\,. \tag{187}$$

To work around this problem we employ the feedback link as follows. After the n-th channel use the transmitter computes the estimate $\hat{W}_n$ and relabels the messages before issuing X_{n+1} so that the most likely message $\hat{W}_n$ is mapped to a zero-cost symbol x_0. This relabeling can clearly be undone at the receiver side due to the knowledge of $\hat{W}_n$. Therefore, (185) and (186) continue to hold. The average total cost for this modified scheme, however, becomes

$$\begin{aligned}
\mathbb{E}\left[\sum_{n=1}^{\infty}\mathsf{c}(X_n)\right] &= \sum_{n=1}^{\infty}\mathsf{c}(x_1)\mathbb{P}[W\neq\hat{W}_n] && (188)\\
&\leq \sum_{n=1}^{\infty}\mathsf{c}(x_1)\exp\{-n\theta\} && (189)\\
&\leq \frac{\mathsf{c}(x_1)}{\exp\{\theta\}-1} && (190)\\
&< \infty\,, && (191)
\end{aligned}$$

where (188) is because our scheme spends a non-zero cost $\mathsf{c}(x_1)$ only in the case $\hat{W}_n\neq W$, (189) is by (186), and (190) is because $\theta>0$. As required, we have obtained a zero-error feedback code transmitting one bit of information with finite average cost.

This illustrates that achieving zero-error relies essentially on the infinite bandwidth assumption (see [20, Section VI] for a lower bound on the probability of error with finite number of degrees of freedom). At the same time, the main code constructions presented here, Theorems 7 and 9, can be restated for the case of a finite number of degrees of freedom, L, that satisfies $L\gg k$. For example, in Theorem 7, instead of taking the limit $d\to 0$ (see the proof of Theorem 7) we can consider the code obtained with a small fixed $d>0$. Then application of Lévy's modulus of continuity theorem [32] implies that the energy per bit increases to approximately

$$N_0\log_e 2+N_0\cdot O\left(\sqrt{\frac{d^2}{N_0}\log\frac{N_0}{d^2}}\right)\,,\quad d\to 0\,. \tag{192}$$

Regarding the probability of error, we know from (102) that after L channel uses, the log-likelihood is distributed as $\mathcal{N}\left(\frac{Ld^2\log e}{N_0},\frac{2Ld^2\log^2 e}{N_0}\right)$. Thus, the probability of error increases from 0 to approximately

$$\epsilon\approx e^{-\frac{3Ld^2}{N_0}}\,. \tag{193}$$

Hence, if a finite probability of error ϵ needs to be achieved with a finite number of degrees of freedom L, then Theorem 7 can be modified to achieve an energy per bit

$$E \geq N_0 \log_e 2 + N_0 \cdot O\left(\sqrt{\frac{\log\frac{1}{\epsilon}}{L}\log\frac{L}{\log\frac{1}{\epsilon}}}\right), \quad L \to \infty, \tag{194}$$

which follows from taking $d^2 = \frac{N_0}{L}\log\frac{1}{\epsilon}$ in (192). A similar argument shows that the stop-feedback construction of Theorem 9 can also be modified to allow for $L \gg \log M$.

Note that in the case when L is small, i.e. $L \sim \log M$, the problem changes completely and falls in the category of the finite blocklength analysis for the AWGN channel undertaken in [5, Section III.J].

Finally, a natural question is whether the same improvements can be achieved by feedback codes satisfying a stronger energy constraint, namely, if (10) is replaced by the requirement

$$\mathbb{P}[||\mathbf{x}||^2 \leq E | W = j] = 1\,, \qquad j = 1, \ldots, M\,. \tag{195}$$

The answer to this question is negative, as follows from the following result:

Theorem 12: Let $0 \leq \epsilon < 1$. Any (E, M, ϵ) code with feedback satisfying energy constraint (195) must satisfy the non-feedback converse bound in (16).

Proof: We follow the proof of Theorem 4 with the only change being that instead of (80) and (64) we have a stronger condition

$$\hat{\tau} \leq E\,, \qquad \mathbb{P}\text{-a.s.}\,. \tag{196}$$

Then, the minimizing set F in (56) necessarily belongs to the σ-algebra $\mathcal{F}_E$, where we recall that $\{\mathcal{F}_t, t \geq 0\}$ is a standard filtration on Ω in (60). Thus F becomes a conventional, fixed observation time (or "fixed-sample-size") binary hypothesis test for the drift of the Brownian motion, or in other words, between $\mathbb{P}$ and $\mathbb{Q}$ restricted to $\mathcal{F}_E$. A simple computation shows

$$\frac{dP^j}{d\Phi} \sim \left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_E} = \phi_E\,, \tag{197}$$

and by the Neyman-Pearson lemma (since $\phi_E \sim B_E$ under $\mathbb{P}$ and $\phi_E \sim \bar{B}_E$ under $\mathbb{Q}$, see Lemma 5), we have

$$\inf_{F \in \mathcal{F}_E : P'(F) \geq \alpha} \Phi'(F) = \beta_\alpha\,, \tag{198}$$

where β_α is defined in (22). This completes the proof of (56) with $\tilde{\beta}_\alpha$ replaced by β_α and results in the bound (16) as shown in the proof of Theorem 2. ■

Theorem 12 parallels the result of Pinsker [22, Theorem 2] on block coding for the AWGN channel with fixed rate. We discuss the relationship to his results below.

In the converse part of [22, Theorem 2] Pinsker demonstrated that Shannon's cone-packing lower bound on the probability of error [4] holds in the presence of noiseless feedback provided that the power-constraint is in the almost sure sense, such as in (195). (Wyner [26] has also demonstrated explicitly that enforcing constraint (195) for the Schalkwijk-Kailath scheme results in probability of error decaying only exponentially.)

In particular, Pinsker's result implies that for rates above critical the error exponent for the AWGN channel is not improved by the availability of feedback. At the other extreme, for $M = 2$ feedback is again useless [22, (12)] and [28]. For $M \geq 3$ and up to the critical rate, however, feedback does indeed improve the error exponent. In fact, in the achievability part of [22, Theorem 2] Pinsker derived a simple scheme achieving Shannon's cone-packing error exponent for all rates. His scheme consisted of an encoder employing a random spherical code, which constantly monitors the decoding progress over the feedback link and switches to the Schalkwijk-Kailath mode once the true message is found among the L most likely (the Schalkwijk-Kailath encoder is then used to select the actual message out of the list of L).

Theorem 12 shows that a lower bound of Theorem 2 for the fixed-energy context serves the same role as Shannon's cone-packing lower bound does for the fixed-rate one. In particular, if we fix M and let $E \to \infty$ the converse (16) becomes

$$\epsilon \geq \exp\left\{-\frac{E}{N_0}\log e + o(E)\right\} . \tag{199}$$

This bound matches Pinsker's feedback achievability bound [22, Theorem 1 and (33)]. The non-feedback achievability bound in Theorem 2, only yields

$$\epsilon \leq \exp\left\{-\frac{E}{2N_0}\log e + o(E)\right\} \tag{200}$$

for the regime of $M = \mathrm{const}$ and $E \to \infty$ (for the regime $M = \exp\{O(E)\}$ see [27, p.345]). Thus, although codes in Theorem 2 are optimal up to $O(1)$ terms in the fixed-ϵ regime (according to (31)), in the regime of exponentially decaying probability of error they become quite suboptimal. This example illustrates that conclusions in the fixed-ϵ regime (which loosely corresponds to working "close to capacity") and the fixed-rate (or fixed M) regime may not coincide.

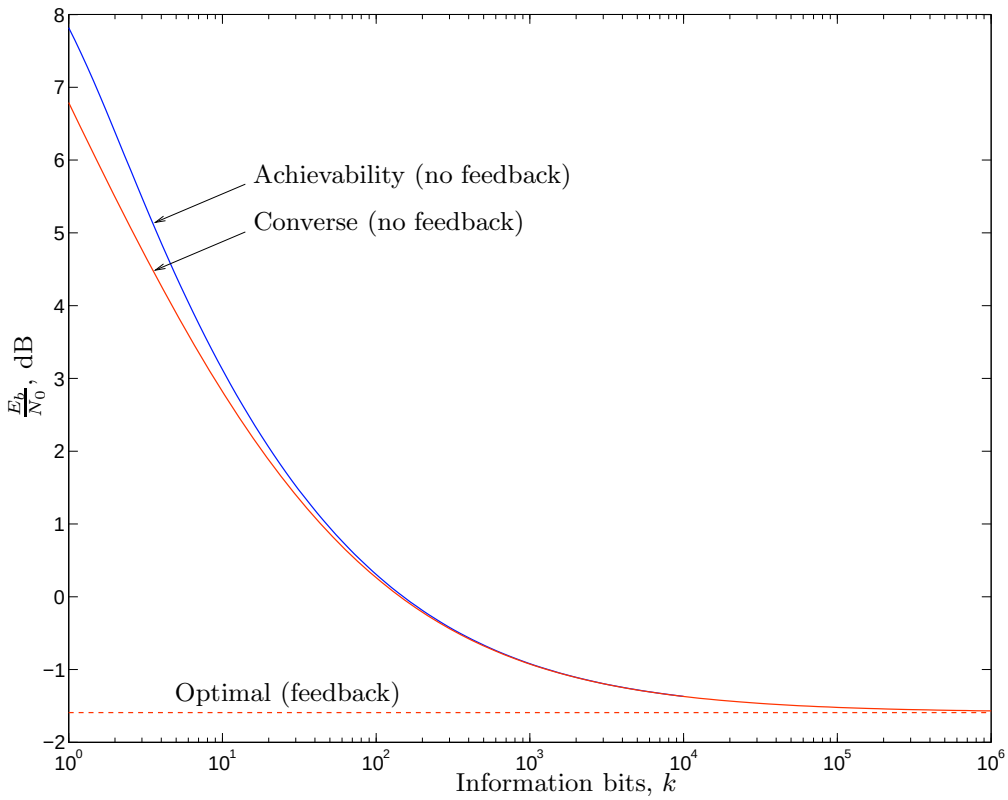

Fig. 3. Bounds on the minimum energy per bit as a function of the number of information bits with and without feedback; block error rate $\epsilon = 10^{-3}$.

We have shown that the lower-bound of Theorem 12 is tight for regimes $M = \text{const}$ and $\epsilon = \text{const}$. It is natural, therefore, to expect that similarly to [22, Theorem 2] one can show that Theorem 12 is also exponentially tight when M scales with $E \to \infty$ according to $M = 2^{\frac{E}{E_b}}$ where $E_b > N_0 \log_e 2$ is a fixed energy-per-bit. Likely, the same two-phase strategy of Pinsker will succeed.

IV. Conclusion

This paper finds new non-asymptotic bounds for the minimum achievable energy per bit and uses those bounds to refine the current understanding of the asymptotic behavior. The main new bounds are:

- Theorem 2: tight upper and lower bounds without feedback;
- Theorem 4: a converse bound with feedback;

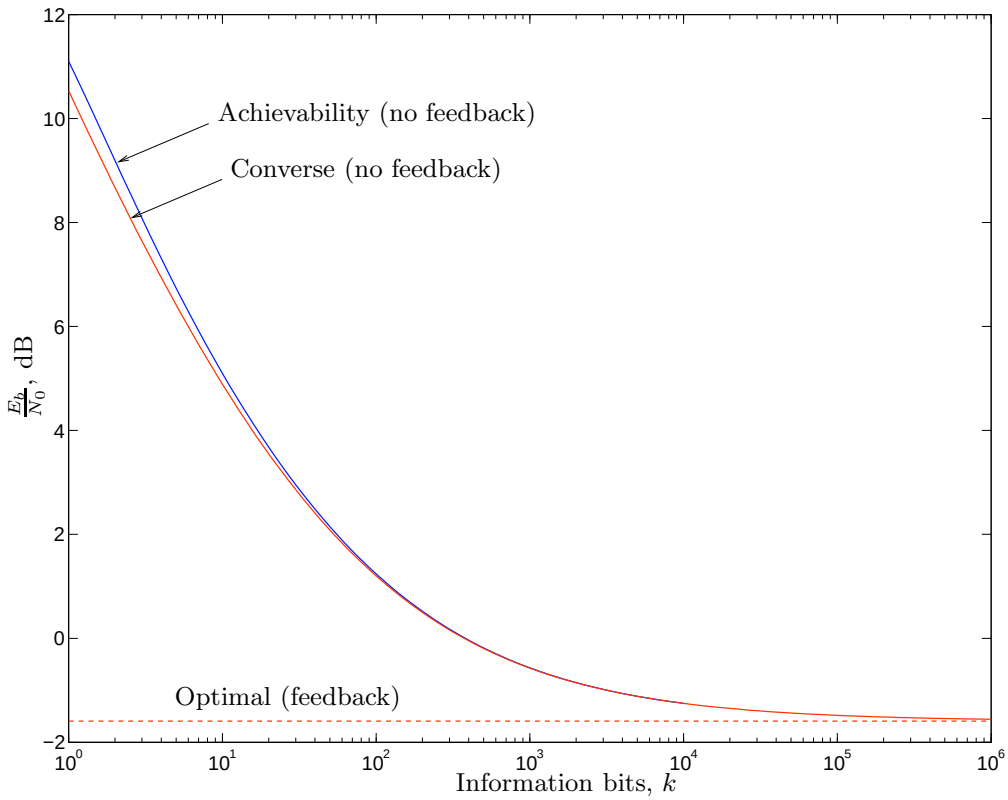

Fig. 4. Bounds on the minimum energy per bit as a function of the number of information bits with and without feedback; block error rate $\epsilon = 10^{-6}$.

- Theorem 7: a 1-bit zero-error feedback scheme achieving the optimal -1.59 dB energy per bit;
- Theorem 9: a stop-feedback achievability bound.

In addition we have analyzed variations of the schemes of Schalkwijk-Kailath [16] and Zigangirov [23] adapted for the purpose of minimizing the energy per bit (Section III-D and Theorem 11).

Regarding the asymptotic expansions with $E \to \infty$, our main results are given by Theorems 3, 8 and 10 and can be compared as follows:

$$\log M^*(E,\epsilon) = \frac{E}{N_0}\log e + O(\sqrt{E}) \qquad \text{(no feedback)}\,, \tag{201}$$

$$\log M_{\mathsf{f}}^*(E,\epsilon) = \frac{E}{N_0}\frac{\log e}{1-\epsilon} + O(\log E) \qquad \text{(stop-feedback)}\,, \tag{202}$$

$$\log M_{\mathsf{f}}^*(E,\epsilon) = \frac{E}{N_0}\frac{\log e}{1-\epsilon} + O(1) \qquad \text{(full feedback)} \tag{203}$$

as $E \to \infty$.

As the number of information bits, k, goes to infinity, the minimum energy per bit required for arbitrarily reliable communication is equal to -1.59 dB with or without feedback. However, in the non-asymptotic regime, in which the block error probability is set to ϵ, the minimum energy per bit may substantially reduced thanks to the availability of feedback. Comparing Theorems 3 and 8, we observe a double benefit: feedback reduces the leading term in the minimum energy by a factor of $1-\epsilon$, and the penalty due to the second-order term in (31) disappears.

Theorem 7 shows that the optimal energy per bit of -1.59 dB is achievable already at $k=1$ bit. This remarkable fact was observed by Turin [13] in the context of a continuous-time AWGN channel with feedback. The Poisson channel counterpart has been investigated recently in [29], which shows that the minimum average energy per bit with feedback[7] satisfies

$$E_f^*(k,\epsilon) = \frac{1}{k}(1-\epsilon)\,, \qquad 0 < \epsilon \le 1. \tag{204}$$

The result also holds for $\epsilon = 0$ in the special case when a) the dark current is absent and b) signals of infinite duration are allowed.

The bounds developed above enable a quantitative analysis of the dependence of the required energy on the number of information bits. In Fig. 3 we take $\epsilon = 10^{-3}$ and compare the bounds on $E_{\mathsf{b}}^*(k,\epsilon)$ and $E_{\mathsf{f}}^*(k,\epsilon)$ developed in Section III. Non-feedback upper (15) and lower (16) bounds are tight enough to conclude that for messages of size $k \sim 100$ bits the minimum $\frac{E_{\mathsf{b}}}{N_0}$ is 0.20 dB, whereas the Shannon limit is only approachable within 0.02 dB at $k \gtrsim 10^6$ bits. With feedback, the gap between the achievability and converse bounds is negligible enough, see (160), to determine the value of the minimal energy per bit (denoted "Feedback (optimal)" on the Fig. 3) for all practical purposes. Compared to the non-feedback case, Fig. 3 demonstrates the significant advantages of using feedback with practical values of k. In Fig. 4 we compute the bounds for $\epsilon = 10^{-6}$, in which case the advantages of the feedback codes become even more pronounced.

Another way to interpret Figs. 3 and 4 is to note that for moderate values of k an improvement of up to 10 dB is achievable with feedback codes. As discussed, this effect is analytically

[7]The result in [29] differs from (204) by a factor of $\frac{2^{k-1}-1}{2^k}$ due to the fact that [29] uses an average over the codebook energy constraint (14) instead of the per-codeword energy constraint in (10). The factor reflects that under the optimal scheme one message has energy zero and all (2^k-1) others have energy $1-\epsilon$.

expressed by the absence of the $O(\sqrt{E})$ penalty term in expansion (145). Notice that under the maximal energy constraint (195), feedback is unable to improve upon the non-feedback converse bound and thus becomes useless even non-asymptotically (Theorem 12).

Surprisingly, our results demonstrate that the benefits of feedback are largely realized by stop-feedback codes that use the feedback link only to send a single "stop transmission" signal (as opposed to requiring a full noiseless feedback available at the transmitter). Indeed, Theorem 10 demonstrates that the asymptotic expansion for stop-feedback codes remains free from the $\sqrt{E}$ penalty term. Moreover, as seen from the comparison in Fig. 1, for practically interesting values of k, the suboptimality of our stop-feedback bound is insignificant compared to the gain with respect to the non-feedback codes. Consequently, we conclude that for such values of k the dominant benefit of feedback on the energy per bit is already brought about by the stop-feedback scheme of Theorem 9. In this way, the results of Section III-B (in particular (202)) easily extend to noisy and/or finite capacity feedback links. Where the noiselessness of feedback plays the crucial role, however, is in offering the possibility of achieving zero error with finite energy.

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APPENDIX A
PROOF OF LEMMA 1

Proof: Given a sequence of encoder maps $f_n, n = 1, \ldots$ we construct a different sequence f'_n as follows:

$$f'_1(W) = 0\,, \tag{205}$$

$$f'_n(W, Y^{n-1}) = f_{n-1}(\sigma(Y_1, W), Y_2^{n-1})\,, \quad n \geq 2\,, \tag{206}$$

where $\sigma : \mathbb{R} \times \{1, \ldots, M\} \to \{1, \ldots, M\}$ is a measurable map with two properties: 1) for any $y \in \mathbb{R}$ the map $m \mapsto \sigma(y, m)$ is a bijection of $\{1, \ldots, M\}$; 2) for any m the distribution of $\sigma(Z, m)$ is equiprobable on $\{1, \ldots, M\}$ whenever Z is Gaussian with variance $\frac{N_0}{2}$. The existence of such a map is obvious. We define the decoder g' to satisfy

$$\sigma(Y_1, g'(Y_1^\infty)) = g(Y_2^\infty)\,, \tag{207}$$

which is consistent since $m \mapsto \sigma(y, m)$ is a bijection. Clearly, the probability of error of (f'_n, g') is the same as that of (f_n, g). By assumption the original code satisfies (14) and therefore

$$\frac{1}{M} \sum_{j=1}^{M} \mathbb{E}\left[\sum_{n=2}^{\infty} \left| f_{n-1}(j, Y_2^{n-1}) \right|^2 \middle| \sigma(Y_1, W) = j \right] \leq E\,. \tag{208}$$

Now for any $j \in \{1, \ldots, M\}$ per-codeword energy is:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{n=1}^{\infty} \left| f'_n(W, Y^{n-1}) \right|^2 \middle| W = j \right] \\
&= \mathbb{E}\left[\sum_{n=2}^{\infty} \left| f_{n-1}(\sigma(Y_1, W), Y_2^{n-1}) \right|^2 \middle| W = j \right] && (209)\\
&= \mathbb{E}\left[\mathbb{E}\left[\sum_{n=2}^{\infty} \left| f_{n-1}(\sigma(Y_1, W), Y_2^{n-1}) \right|^2 \middle| \sigma(Y_1, W) \right] \middle| W = j \right] && (210)\\
&= \frac{1}{M} \sum_{i=1}^{M} \mathbb{E}\left[\sum_{n=2}^{\infty} \left| f_{n-1}(\sigma(Y_1, j), Y_2^{n-1}) \right|^2 \middle| \sigma(Y_1, j) = i \right] && (211)\\
&\leq E\,, && (212)
\end{aligned}$$

where (209) is by (205) and (206), (211) is because $\mathbb{P}[\sigma(Y_1, W) = i | W = j] = \frac{1}{M}$, and (212) is by (208). Thus by (212) the encoder sequence $f'_n, n = 1, \ldots$ satisfies a per-codeword constraint (10). ■

APPENDIX B
STOP-FEEDBACK CODES

The stop-feedback bound in Theorem 9 is just a representative of the following family of bounds.

Theorem 13: For any $E > 0$ and positive integer M there exists an (E, M, ϵ) code with feedback for channel (3) satisfying

$$\epsilon \leq \inf \{1 - \alpha + (M-1)\beta\} \ , \tag{213}$$

where the infimum is over all $0 < \beta < \alpha \leq 1$ satisfying

$$d(\alpha||\beta) = \frac{E}{N_0} \log e \, . \tag{214}$$

Moreover, there exists an (E, M, ϵ) stop-feedback code; its probability of error is bounded by (213) with $\alpha = 1$, namely,

$$\epsilon \leq (M-1) e^{-\frac{E}{N_0}} \, . \tag{215}$$

Proof: Fix a list of elements $(\mathbf{c}_1, \ldots, \mathbf{c}_M) \in \mathsf{A}^M$ to be chosen later; $||\mathbf{c}_j||^2$ need not be finite. Upon receiving channel outputs $Y_1, \ldots, Y_n$ the decoder computes the likelihood $S_{j,n}$ for each codeword $j = 1, \ldots, M$, cf. (27) and (58):

$$S_{j,n} = \sum_{k=1}^{n} C_{j,k} Y_k - \tfrac{1}{2} C_{j,k}^2 \, , \quad j = 1, \ldots, M \, . \tag{216}$$

Fix two scalars $\gamma_0 < 0 < \gamma_1$ and define M stopping times

$$\tau_j = \inf\{n > 0 : S_{j,n} \notin (\gamma_0, \gamma_1)\} \, . \tag{217}$$

Among those processes $\{S_{i,n}\}$ that upcross γ_1 without having previously downcrossed γ_0, we choose the process $\{S_{j,n}\}$ for which the γ_1 upcrossing occurs earliest. Then decoder outputs $\hat{W} = j$. The encoder conserves energy by transmitting only up until time τ_j (when the true message $W = j$):

$$X_n \stackrel{\triangle}{=} f_n(j, Y_1^{n-1}) = C_{j,n} 1\{\tau_j \geq n\} \, . \tag{218}$$

At first, it might seem that we could further reduce the energy spent by replacing τ_j in (218) with the actual decoding moment $\tilde{\tau}$. This however, is problematic for two reasons. First, whenever $\gamma_0 > -\infty$, $\tilde{\tau}$ equals ∞ with some non-zero probability since it is possible for all M processes $\{S_{i,n}\}$ to downcross γ_0 without first upcrossing γ_1. Second, even if $\gamma_0 = \infty$ the expectation

of $\mathbb{E}\left[\tilde{\tau}|W=j\right]$ becomes unmanageable unless one upper-bounds $\tilde{\tau}$ with τ_j, which is simply equivalent to (218). Similarly, the possibility of downcrossings precludes the interpretation of our scheme as stop-feedback unless γ_0 is taken to be $-\infty$.

To complete the construction of the encoder-decoder pair we need to choose $(\mathbf{c}_1,\ldots,\mathbf{c}_M)$. This is done by a random-coding argument. Fix $d>0$ and generate each $\mathbf{c}_j$ independently with equiprobable antipodal coordinates:

$$\mathbb{P}[C_{j,k}=+d]=\mathbb{P}[C_{j,k}=-d]=\frac{1}{2}\,,\ j=1,\ldots,M. \tag{219}$$

We now upper-bound the probability of error P_e averaged over the choice of the codebook. By symmetry it is sufficient to analyze the probability $\mathbb{P}[\hat{W}\neq 1|W=1]$. We then have

$$\mathbb{P}[\hat{W}\neq 1|W=1]\leq \mathbb{P}[S_{1,\tau_1}\leq\gamma_0|W=1]+\sum_{j=2}^{M}\mathbb{P}[S_{j,\tau_j}\geq\gamma_1,\tau_j\leq\tau_1|W=1]\,, \tag{220}$$

because there are only two error mechanisms: S_1 downcrosses γ_0 before upcrossing γ_1, or some other S_j upcrosses γ_1 before S_1. Notice that in computing probabilities $\mathbb{P}[S_{1,\tau_1}\leq\gamma_0|W=1]$ and $\mathbb{P}[S_{2,\tau_2}\geq\gamma_1,\tau_2\leq\tau_1|W=1]$ on the right-hand side of (220) we are interested only in time instants $0\leq n\leq\tau_1$. For all such moments $X_n=C_{1,n}$. Therefore, below for simplicity of notation we will assume that $X_n=C_{1,n}$ for all n (whereas in reality $X_n=0$ for all $n>\tau_1$, which becomes relevant only for calculating the total energy spent).

We define B_t and $\bar{B}_t$ as in (61) and (62); then conditioned on $W=1$ the process S_1 can be rewritten as

$$S_{1,n}=B_{nd^2}\,, \tag{221}$$

because according to (220) we are interested only in $0\leq n\leq\tau_1$ and thus $X_k=C_{1,k}$. The stopping time τ_1 then becomes

$$d^2\tau_1=\inf\{t>0:B_t\notin(\gamma_0,\gamma_1)\,,t=nd^2,n\in\mathbb{Z}\}\,. \tag{222}$$

If we now define

$$\tau = \inf\{t>0:B_t\notin(\gamma_0,\gamma_1)\}\,, \tag{223}$$

$$\bar{\tau} = \inf\{t>0:\bar{B}_t\notin(\gamma_0,\gamma_1)\}\,, \tag{224}$$

then the path-continuity of B_t implies that

$$d^2\tau_1\searrow\tau\ \text{ as } d\to 0\,. \tag{225}$$

Similarly, still under the condition $W = 1$ we can rewrite (216) in the case of the second codeword as

$$S_{2,n} = d^2 \sum_{k=1}^{n} L_k + \bar{B}_{nd^2} \,, \tag{226}$$

where L_k are i.i.d., independent of $\bar{B}_t$ and

$$\mathbb{P}[L_k = +1] = \mathbb{P}[L_k = -1] = \frac{1}{2} \,. \tag{227}$$

Note that one should not infer from (226) that the processes $S_{1,n}$ and $S_{2,n}$ have dependence as B_t and $\bar{B}_t$ which determine each other; see (61) and (62). The equality in (226) makes sense as long as the process $S_{2,n}$ is considered separately from $S_{1,n}$.

Extending (225), we will show below that as $d \to 0$ we have

$$\mathbb{P}[S_{1,\tau_1} \leq \gamma_0 | W = 1] \;\; \to \;\; 1 - \alpha(\gamma_0, \gamma_1) \,, \tag{228}$$

$$\mathbb{P}[S_{2,\tau_2} \geq \gamma_1, \tau_2 < \infty | W = 1] \;\; \to \;\; \beta(\gamma_0, \gamma_1) \,, \tag{229}$$

where $\alpha(\gamma_0, \gamma_1)$ and $\beta(\gamma_0, \gamma_1)$ are

$$\alpha(\gamma_0, \gamma_1) \;\; = \;\; \mathbb{P}[B_\tau = \gamma_1] \,, \tag{230}$$

$$\beta(\gamma_0, \gamma_1) \;\; = \;\; \mathbb{P}[\bar{B}_{\bar{\tau}} = \gamma_1, \bar{\tau} < \infty] \,, \tag{231}$$

i.e. the probabilities of hitting the upper threshold γ_1, without having gone below γ_0 by B_t and $\bar{B}_t$, respectively[8]. Thus, the interval (γ_0, γ_1) determines the boundaries of the sequential probability ratio test. As shown by Shiryaev [30, Section 4.2], α and β satisfy

$$d(\alpha(\gamma_0, \gamma_1) || \beta(\gamma_0, \gamma_1)) = \frac{\log e}{N_0} \mathbb{E}\left[\tau\right] . \tag{232}$$

Assuming (228) and (229) as $d \to 0$ the probability of error is upper-bounded by (220):

$$\mathbb{P}[\hat{W} \neq 1 | W = 1] \leq 1 - \alpha(\gamma_0, \gamma_1) + (M-1)\beta(\gamma_0, \gamma_1) \,. \tag{233}$$

At the same time, the average energy spent by our scheme is

$$\lim_{d \to 0} \mathbb{E}\left[||\mathbf{x}||^2\right] = \lim_{d \to 0} \mathbb{E}\left[d^2 \tau_1\right] = \mathbb{E}\left[\tau\right] , \tag{234}$$

because of (225).

[8]The condition $\bar{\tau} < \infty$ is required for handling the special case $\gamma_0 = -\infty$.

Finally, comparing (214) and (232) it follows that optimizing (233) over all $\gamma_0 < 0 < \gamma_1$ satisfying $\mathbb{E}[\tau] = E$ we obtain (213). To prove (215) simply notice that when $\alpha = 1$ we have $\gamma_0 = -\infty$, and hence the decision is taken by the decoder the first time any S_j upcrosses γ_1. Therefore, the time τ_j (whose computation requires the full knowledge of Y_k) can be replaced in (218) with the time of decoding decision, which requires sending only a single signal. Obviously, this modification will not change the probability of error and will conserve energy even more (since under $\gamma_0 = -\infty$, τ_j cannot occur before the decision time).

We now prove (228) and (229). By (221) and (225) we have

$$S_{1,\tau_1} = B_{d^2\tau_1} \to B_\tau \,, \tag{235}$$

because of the continuity of B_t. From (235) we obtain (228) after noticing that again due to continuity

$$\mathbb{P}[B_\tau \leq \gamma_0] = 1 - \mathbb{P}[B_\tau \geq \gamma_1] = 1 - \mathbb{P}[B_\tau = \gamma_1] \,. \tag{236}$$

The proof of (229) requires a slightly more intricate argument for which it is convenient to introduce a probability space denoted by $(\Omega, \mathcal{H}, \mathbb{P})$ which is the completion of the probability space generated by $\{\bar{B}_t\}_{t=0}^{\infty}$ and $\{L_k\}_{k=1}^{\infty}$ defined in (62) and (227), respectively. For each $0 < d \leq 1$ we define the following random variables, where their explicit dependence on d is omitted for brevity:

$$D_t = d^2 \sum_{k \leq \lfloor t/d^2 \rfloor} L_k \,, \tag{237}$$

$$\Sigma_t = D_t + \bar{B}_{d^2 \lfloor \frac{t}{d^2} \rfloor} \,, \tag{238}$$

$$\tau_2 = \inf\{t > 0 : \Sigma_t \notin (\gamma_0, \gamma_1)\} \,, \tag{239}$$

$$\bar{\tau} = \inf\{t > 0 : B_t \notin (\gamma_0, \gamma_1)\} \,. \tag{240}$$

In comparison with the random variables appearing in (229) Σ_{nd^2} and τ_2 take the role of $S_{2,n}$ and $d^2\tau_2$, respectively; and also $\mathbb{P}$ henceforth is already normalized by the conditioning on $W = 1$. Thus in the new notation we need to prove

$$\lim_{d \to 0} \mathbb{P}[\Sigma_{\tau_2} \geq \gamma_1, \tau_2 < \infty] = \mathbb{P}[\bar{B}_{\bar{\tau}} = \gamma_1, \bar{\tau} < \infty] \,. \tag{241}$$

We define the following subsets of Ω:

$$E_0 = \{\omega \in \Omega : \exists T < \infty \ \forall t > T : \sup_{0<d\leq 1} \Sigma_t < 0\}\,, \tag{242}$$

$$E_1 = \{\bar{\tau} = \infty\} \cup \{\bar{\tau} < \infty, \forall \epsilon > 0 \ \exists t_1, t_2 \in (0, \epsilon) \text{ s.t. } \bar{B}_{\bar{\tau}+t_1} > \bar{B}_{\bar{\tau}}, \bar{B}_{\bar{\tau}+t_2} < \bar{B}_{\bar{\tau}}\}\,, \tag{243}$$

$$E_2 = \{\omega \in \Omega : \lim_{d\to 0} D_t = 0 \text{ uniformly on compacts}\}\,, \tag{244}$$

$$E = E_0 \cap E_1 \cap E_2\,. \tag{245}$$

According to Lemma 14 the sets in (242)-(245) belong to $\mathcal{H}$ and have probability 1.

The next step is to show

$$\{\bar{B}_{\bar{\tau}} = \gamma_1, \bar{\tau} < \infty\} \cap E \subset \liminf_{d\to 0} \{\Sigma_{\tau_2} \geq \gamma_1, \tau_2 < \infty\} \cap E\,. \tag{246}$$

To that end select an arbitrary element $\omega \in \{\bar{B}_{\bar{\tau}} = \gamma_1, \bar{\tau} < \infty\} \cap E$. Since $\bar{B}_t$ is continuous it must attain its minimum b_0 on $[0; \bar{\tau}]$; of course, $b_0 > \gamma_0$. Again, due to continuity of $\bar{B}_t$ at $t = \bar{\tau}$ there must exist an $\epsilon_1 > 0$ such that

$$b_0' \stackrel{\triangle}{=} \min_{0\leq t\leq \bar{\tau}+\epsilon_1} \bar{B}_t > \gamma_0\,. \tag{247}$$

On the other hand, because $\omega \in E_1$ we have

$$b_1 \stackrel{\triangle}{=} \max_{0\leq t\leq \bar{\tau}+\epsilon_1} \bar{B}_t > \gamma_1\,. \tag{248}$$

Moreover, since $\omega \in E_2$ we have $D_t \to 0$ uniformly on $[0; \bar{\tau} + \epsilon_1]$; therefore, there exists a $d_1 > 0$ such that for all $d \leq d_1$ we have

$$\sup_{t\in[0;\bar{\tau}+\epsilon_1]} |D_t| \leq \epsilon_2\,, \tag{249}$$

where

$$\epsilon_2 = \frac{1}{3} \min(b_1 - \gamma_1, b_0' - \gamma_0) > 0\,. \tag{250}$$

If we denote by t_1 the point at which $B_{t_1} = b_1$, then by continuity of B_t at t_1 there exists a $\delta > 0$ such that

$$\forall t \in (t_1 - \delta; t_1 + \delta) : B_t > b_1 - \epsilon_2\,. \tag{251}$$

Then for every $d < \sqrt{\delta}$ we have

$$\max_{t\in[0,\bar{\tau}+\epsilon_1]} \bar{B}_{d^2\lfloor \frac{t}{d^2} \rfloor} > b_1 - \epsilon_2\,. \tag{252}$$

Finally, for every $d \leq \min(\sqrt{\delta}, d_1)$ we have

$$\sup_{t\in[0,\bar{\tau}+\epsilon_1]} \Sigma_t \geq b_1 - 2\epsilon_2 > \gamma_1 \tag{253}$$

and

$$\inf_{t\in[0,\bar{\tau}+\epsilon_1]} \Sigma_t \geq b_0' - \epsilon_2 > \gamma_0 \tag{254}$$

by (247), (248), (250) and (252). Then of course, (253) and (254) prove that $\tau_2 \leq \bar{\tau} + \epsilon_1$ and $\{\Sigma_{\tau_2} \geq \gamma_1\}$ holds for all $d \leq \min(\sqrt{\delta}, d_1)$. Equivalently,

$$\omega \in \liminf_{d\to 0}\{\Sigma_{\tau_2} \geq \gamma_1, \tau_2 < \infty\}\,, \tag{255}$$

proving (246).

Next, we show

$$\limsup_{d\to 0}\{\Sigma_{\tau_2} \geq \gamma_1, \tau_2 < \infty\} \cap E \subset \{\bar{B}_{\bar{\tau}} = \gamma_1, \bar{\tau} < \infty\} \cap E\,. \tag{256}$$

Indeed, take $\omega \in \limsup_{d\to 0}\{\Sigma_{\tau_2} \geq \gamma_1, \tau_2 < \infty\} \cap E$, that is a point in the sample space for which there exists a subsequence $d_l \to 0$ such that $\Sigma_{\tau_2} \geq \gamma_1$ for every l. Since $\omega \in E_0$ we know that for all d we have $\tau_2(\omega) \leq T < \infty$. First, we show

$$b_1 \triangleq \max_{0\leq t\leq T} \bar{B}_t \geq \gamma_1\,. \tag{257}$$

Indeed, assuming otherwise and repeating with minor changes the argument leading from (248) to (253), we can show that in this case

$$\sup_{t\in[0,T]} \Sigma_t < \gamma_1 \tag{258}$$

for all sufficiently small d. This contradicts the choice of ω.

We denote

$$t_1 = \inf\{t > 0 : \bar{B}_t = b_1\}\,. \tag{259}$$

Then (257) and continuity of $\bar{B}_t$ imply

$$\bar{\tau} \leq t_1 < \infty\,. \tag{260}$$

We are left only to show that $\bar{B}_{\bar{\tau}} = \gamma_0$ is impossible. If it were so, then $\bar{\tau} < t_1 < T$. Moreover because $\omega \in E_2$ there must exist an $\epsilon_1 > 0$ (similar to (247) and (248)) such that

$$b_0' \triangleq \min_{0\leq t\leq \bar{\tau}+\epsilon_1} \bar{B}_t < \gamma_0\,, \tag{261}$$

and

$$b_1' \triangleq \max_{0 \le t \le \bar{\tau}+\epsilon_1} \bar{B}_t < \gamma_1 \,. \tag{262}$$

Thus, by repeating the argument behind (253) and (254) we can show that for all sufficiently small d we have

$$\sup_{t \in [0, \bar{\tau}+\epsilon_1]} \Sigma_t < \gamma_1 \,, \tag{263}$$

and

$$\inf_{t \in [0, \bar{\tau}+\epsilon_1]} \Sigma_t < \gamma_0 \,, \tag{264}$$

which contradicts the assumption that $\omega \in \limsup_{d \to 0} \{\Sigma_{\tau_2} \ge \gamma_1, \tau_2 < \infty\}$.

Together (246) and (256) prove that

$$\begin{aligned} &\{\bar{B}_{\bar{\tau}} = \gamma_1, \bar{\tau} < \infty\} \cap E \subset \liminf_{d \to 0} \{\Sigma_{\tau_2} \ge \gamma_1, \tau_2 < \infty\} \cap E \subset \\ &\quad \limsup_{d \to 0} \{\Sigma_{\tau_2} \ge \gamma_1, \tau_2 < \infty\} \cap E \subset \{\bar{B}_{\bar{\tau}} = \gamma_1, \bar{\tau} < \infty\} \cap E \,, \end{aligned} \tag{265}$$

which implies that all three sets are equal. By Lemma 14 and the completeness of $\mathcal{H}$ both sets $\liminf_{d \to 0} \{\Sigma_{\tau_2} \ge \gamma_1, \tau_2 < \infty\}$ and $\limsup_{d \to 0} \{\Sigma_{\tau_2} \ge \gamma_1, \tau_2 < \infty\}$ are measurable and computing their probabilities is meaningful. Finally, we have

$$\lim_{d \to 0} \mathbb{P}[\Sigma_{\tau_2} \ge \gamma_1, \tau_2 < \infty] = \lim_{d \to 0} \mathbb{P}[\{\Sigma_{\tau_2} \ge \gamma_1, \tau_2 < \infty\} \cap E] \tag{266}$$

$$= \mathbb{P}[\bar{B}_{\bar{\tau}} = \gamma_1, \bar{\tau} < \infty] \,, \tag{267}$$

where (266) is by Lemma 14 and (267) by (265) and the bounded convergence theorem. ■

Lemma 14: The set E defined in (245) is $\mathcal{H}$-measurable and

$$\mathbb{P}[E] = 1 \,. \tag{268}$$

Proof: By the completeness of $\mathcal{H}$ it is sufficient to prove that all sets E_0, E_1 and E_2 contain a measurable subset of probability 1. To prove

$$\mathbb{P}[E_0] = 1 \,, \tag{269}$$

notice that

$$\sup_{0 < d \le 1} D_t = t \sup_{N \ge t} \frac{1}{N} \sum_{k=1}^{N} L_k \,, \tag{270}$$

and therefore, by the Chernoff bound,

$$\mathbb{P}\left[\sup_{0<d\leq 1} D_t > \frac{t}{4}\right] \leq \sum_{N\geq t} O\left(e^{-a_1 N}\right) \tag{271}$$

$$= O\left(e^{-a_1 t}\right) , \tag{272}$$

for some constant $a_1 > 0$. Hence, for an arbitrary t we have an estimate

$$\mathbb{P}[\bar{B}_t + \sup_{0<d\leq 1} D_t \geq -1] \leq \mathbb{P}\left[\bar{B}_t \geq -1 - \frac{t}{4}\right] + \mathbb{P}\left[\sup_{0<d\leq 1} D_t > \frac{t}{4}\right] \tag{273}$$

$$\leq O\left(e^{-a_1 t}\right) , \tag{274}$$

where (274) is because $\bar{B}_t \sim \mathcal{N}\left(-\frac{t}{2}, \frac{tN_0}{2}\right)$ and (272).

Next, denote

$$\delta_j = \frac{1}{\sqrt{j}} , \tag{275}$$

$$t_n = \sum_{j=1}^{n} \delta_j , \tag{276}$$

$$M_j = \max_{t_j \leq t \leq t_{j-1}} W_t - W_{t_j} , \tag{277}$$

where $W_t = t/2 + \sqrt{\frac{2}{N_0}}\bar{B}_t$ is the standard Wiener process; cf. (62).

Since $t_n \sim 2\sqrt{n}$ and the series $\sum_{n=1}^{\infty} e^{-a_1\sqrt{n}}$ converges, we can apply the Borel-Cantelli lemma via (274) to show that

$$F_1 = \left\{\{B_{t_n} + \sup_{0<d\leq 1} D_{t_n} \geq -1\} \text{ –infinitely often}\right\} \tag{278}$$

has measure zero. Similarly, since $M_j \sim |W_{\delta_j}|$ we have

$$\sum_{j=1}^{\infty} \mathbb{P}[M_j > (2N_0)^{-1}] = \sum_{j=1}^{\infty} 2Q\left(\frac{1}{2N_0\sqrt{\delta_j}}\right) \leq a_3 \sum_{j=1}^{\infty} e^{-a_2\sqrt{n}} < \infty , \tag{279}$$

for some positive constants a_2, a_3. And therefore,

$$F_2 = \left\{M_j > (2N_0)^{-1} \text{ –infinitely often}\right\} \tag{280}$$

also has measure zero. Finally we show that

$$F_1^c \cap F_2^c \subset E_0 . \tag{281}$$

Indeed, for all $t \in [t_j; t_j + \delta_j)$ we have

$$\bar{B}_t + D_t \leq \bar{B}_{t_j} + D_{t_j} + \sqrt{\frac{N_0}{2}} M_j + 2\delta_j , \tag{282}$$

because, from the definition of D_t,

$$|D_{s_1} - D_{s_2}| \leq 2|s_1 - s_2| \,, \tag{283}$$

for all $d > 0$. From (282) for any $\omega \in F_1^c \cap F_2^c$ we have for all sufficiently large t

$$\sup_{0<d\leq 1} \bar{B}_t + D_t \leq -1 + \frac{1}{2} + 2\delta_j \,, \tag{284}$$

where j denotes the index of the unique interval $t \in [t_j; t_{j+1})$. Therefore, for all sufficiently large t we have shown

$$\sup_{0<d\leq 1} \Sigma_t \leq \sup_{0<d\leq 1} \bar{B}_t + D_t < 0 \,, \tag{285}$$

completing the proof of (281) and, hence, of (269).

To show $\mathbb{P}[E_1] = 1$ notice that by the strong Markov property of Brownian motion for any finite stopping time σ according to Blumenthal's zero-one law [33] for

$$F_\sigma = \{\forall \epsilon > 0 \; \exists t_1, t_2 \in (0, \epsilon) \text{ s.t. } \bar{B}_{\sigma+t_1} > \bar{B}_\sigma, \bar{B}_{\sigma+t_2} < \bar{B}_\sigma\} \tag{286}$$

we have

$$\mathbb{P}[F_\sigma] = 1 \,. \tag{287}$$

Since $\sigma_n = \min(\bar{\tau}, n)$ are finite stopping times and $\sigma_n \nearrow \bar{\tau}$, we have

$$E_1 \supset \bigcap_{n=1}^{\infty} F_{\sigma_n} \,. \tag{288}$$

Therefore, $\mathbb{P}[E_1] = 1$ since $\mathbb{P}[F_{\sigma_n}] = 1$ for all $n \geq 1$.

To show

$$\mathbb{P}[E_2] = 1 \tag{289}$$

it is sufficient to show that for every integer $K > 0$

$$\mathbb{P}[\lim_{d\to 0} D_t = 0 \text{ uniformly on} [0; K]] = 1 \tag{290}$$

and to take the intersection of such sets over all $K \in \mathbb{Z}_+$. To prove (290) notice that

$$\begin{aligned}
\mathbb{P}[\limsup_{d\to 0} \sup_{0\leq t\leq K} |D_t| \geq \epsilon] &= \mathbb{P}\left[\limsup_{d\to 0} d^2 \max_{0\leq n\leq \frac{K}{d^2}} \left|\sum_{k=0}^{n} L_k\right| \geq \epsilon\right] && (291)\\
&= \mathbb{P}\left[\limsup_{m\to\infty} \frac{K}{m} \max_{0\leq n\leq m} \left|\sum_{k=1}^{n} L_k\right| \geq \epsilon\right] && (292)\\
&\leq \mathbb{P}\left[\frac{1}{n}\sum_{k=1}^{n} L_k \geq \frac{\epsilon}{K} \text{ –i.o.}\right] + \mathbb{P}\left[\frac{1}{n}\sum_{k=1}^{n} L_k \leq -\frac{\epsilon}{K} \text{ –i.o.}\right] && (293)
\end{aligned}$$

where "i.o." stands for infinitely often. By the strong law of large numbers both probabilities in (293) are zero and we obtain

$$\limsup_{d\to 0} \sup_{0\le t\le K} |D_t| = 0 \quad \text{a.s.}\,, \tag{294}$$

which is equivalent to (290). ■

APPENDIX C

Proof of Theorem 11: We improve upon Schalkwijk-Kailath's scheme by employing Zigangirov's two-phase method [23]. Our construction will depend on the choice of the following quantities (to be optimized later):

- E_0: energy to be used for X_1,
- L: number of channel uses in the first phase,
- E_1: total energy spent in the first phase,
- ρ : auxiliary parameter governing the total energy spent in the second phase.

We assume $\rho > 0$, $E_1 > E_0 > 0$. Using these parameters define two sequences recursively as follows:

$$\sigma_n^2 = \begin{cases} \frac{N_0}{2}\,, & n = 1\,, \\ \sigma_{n-1}^2 \left(1 + \frac{2c_n^2\sigma_{n-1}^2}{N_0}\right)^{-1}, & n \ge 2\,, \end{cases} \tag{295}$$

$$c_n = \begin{cases} \text{undefined}, & n = 1\,, \\ \frac{1}{\sigma_{n-1}}\sqrt{\frac{E_1-E_0}{L}}\,, & n = 2,\ldots,L+1\,, \\ \sqrt{\frac{\rho N_0}{2}}\frac{1}{\sigma_{n-1}}\,, & n \ge L+2\,. \end{cases} \tag{296}$$

From these equations it is easy to see that

$$\sigma_n^2 = \begin{cases} \frac{N_0}{2}\left(1 + \frac{2(E_1-E_0)}{LN_0}\right)^{-n+1}, & n = 1,\ldots,L+1\,, \\ (1+\rho)^{L+1-n}\,\sigma_{L+1}^2\,, & n \ge L+2 \end{cases} \tag{297}$$

and therefore for any $\rho > 0$

$$\lim_{n\to\infty} \sigma_n^2 = 0\,. \tag{298}$$

We now describe the encoding functions $f_n(W, y^{n-1})$ for all n:

1) For $n = 1$, according to the method of Schalkwijk and Kailath [16], we map the message $W \in \{1, \ldots, M\}$ to the interval $[-\sqrt{E_0}, \sqrt{E_0}]$ by means of

$$X_1 = f_1(W) = \sqrt{E_0}\frac{2W - M - 1}{M - 1}\,. \tag{299}$$

2) For $n = 2$ given Y_1 the encoder computes the value of the noise

$$Z_1 = Y_1 - X_1 \tag{300}$$

and sends

$$X_2 = f_2(W, Y_1) = c_2 Z_1\,. \tag{301}$$

3) For $n = 3, \ldots, L + 1$ the encoder proceeds recursively by sending

$$X_n = c_n(Z_1 - \hat{Z}_{n-1})\,, \quad n = 3, \ldots, L + 1\,, \tag{302}$$

where $\hat{Z}_k$ is the minimum mean square error (MMSE) estimate of Z_1 based on the observations $(Y_2, \ldots, Y_k)$:

$$\hat{Z}_k \triangleq \mathbb{E}\left[Z_1 | Y_2^k\right], \qquad k = 1\,, \ldots, L + 1\,. \tag{303}$$

4) For $n \geq L+2$ (the second phase) we modify the Schalkwijk-Kailath scheme by subtracting $\hat{X}_n$:

$$\begin{aligned} X_n &= f_n(W, Y^{n-1}) && (304)\\ &\triangleq c_n(Z_1 - \hat{Z}_{n-1}) - \hat{X}_n && (305)\\ &= c_n\left[q_\delta\left(Y_1 - \hat{Z}_{n-1} + \sqrt{E_0}\right) - \sqrt{E_0} - X_1\right], \quad n = L + 2, \ldots && (306) \end{aligned}$$

where

$$\hat{X}_n \triangleq c_n\left[Y_1 - q_\delta\left(Y_1 - \hat{Z}_{n-1} + \sqrt{E_0}\right) - \sqrt{E_0} - \hat{Z}_{n-1}\right]\,, \tag{307}$$

with $q_d(\cdot)$ being a d-quantization map

$$q_d(x) \triangleq d\left\lceil \frac{1}{d}x - \frac{1}{2}\right\rceil\,, \tag{308}$$

and δ the spacing between adjacent messages in X_1:

$$\delta \triangleq \frac{2\sqrt{E_0}}{M - 1}\,. \tag{309}$$

Additionally, for $k \geq L+2$ the value $\hat{Z}_k$ appearing in (305) is defined as

$$\hat{Z}_k \triangleq \mathbb{E}\left[Z_1 | Y_2^{L+1}, \tilde{Y}_{L+2}^k\right], \quad k = L+2, \ldots, \tag{310}$$

where

$$\tilde{Y}_k = Y_k + \hat{X}_k, \quad k = L+2 \ldots \tag{311}$$

is known at the receiver at time k.

Below we demonstrate that for all $n \geq 1$ we have

$$\mathrm{Var}[Z_1 | Y_2^{\min(L+1,n)}, \tilde{Y}_{L+2}^n] = \sigma_n^2 . \tag{312}$$

Using (298), (312) results in

$$\mathrm{Var}[Z_1 | Y_2^{L+1}, \tilde{Y}_{L+2}^\infty] = 0 . \tag{313}$$

Thus, given Y_1^∞ the decoder computes $(Y_2^{L+1}, \tilde{Y}_{L+2}^\infty)$ and therefore by (313) can estimate Z_1 (and hence $X_1 = Y_1 - Z_1$) exactly:

$$\mathbb{P}[X_1 \neq Y_1 - \mathbb{E}\left[Z_1 | Y_2^{L+1}, \tilde{Y}_{L+2}^\infty\right]] = 0 . \tag{314}$$

The change in the encoding at $n = L+2$ follows the ingenious observation of Zigangirov [23] that as long as one proceeds in Schalkwijk-Kailath mode (i.e., as for $n \leq L+1$) then due to the discreteness of X_1, conditioned on Y_1^{n-1} the input X_n has non-zero bias:

$$\mathbb{E}\left[X_n | Y_1^{n-1}\right] \neq 0 \tag{315}$$

(conditioned on Y_2^{n-1} the bias is zero by construction, of course). Therefore, to save energy it is beneficial to eliminate this bias by subtracting $\mathbb{E}\left[X_n | Y_1^{n-1}\right]$ which then can be added back at the receiver since it knows Y_1^{n-1}. However, calculating $\mathbb{E}\left[X_n | Y_1^{n-1}\right]$ is complicated and instead we considered an approximation to it given by $\hat{X}_n$ in (307). The rationale for such an approximation is to replace X_1, implicit in the definition of X_n in (302), with a naive estimate $q_\delta\left(Y_1 - \hat{Z}_{n-1}\right)$. Note that $\hat{Z}_n$ now is a function of Y_1^n, instead of Y_2^n used in the first phase.

The proof will now proceed in the following steps:

a) show (312) for $n \leq L+1$;

b) show (312) for $n \geq L+2$;

c) show that the total energy spent in the first phase is at most

$$\mathbb{E}\left[\sum_{k=1}^{L+1} |X_k|^2\right] \leq E_1 , \tag{316}$$

d) show that the total energy E_2 spent in the second phase is

$$E_2 \triangleq \sum_{n=L+2}^{\infty} \mathbb{E}\left[|X_n|^2\right] \tag{317}$$

$$= \frac{N_0}{2} \sum_{n=L+2}^{\infty} \rho\, s\left(\frac{\delta}{\sigma_{n-1}}\right), \tag{318}$$

where

$$s(d) \triangleq \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} q_d^2(x) e^{-\frac{x^2}{2}} dx\,. \tag{319}$$

e) conclude the proof by showing that optimization of the choices of E_0, E_1, L and ρ results in

$$\inf_{E_0,E_1,L,\rho} E_1 + E_2 \leq E_z(M)\,, \tag{320}$$

where

$$E_z(M) \triangleq \frac{N_0}{2} + N_0 \log_e \frac{87(M-1)}{32}\,, \tag{321}$$

which is the right-hand side of (178).

a) We prove (312) by induction. For $n = 1$ the statement is obvious. For $2 \leq n \leq L+1$ we have[9]

$$I(Z_1; Y_2^n) = \frac{1}{2} \log \frac{N_0}{2\,\mathrm{Var}[Z_1|Y_2^n]}\,. \tag{322}$$

Suppose (312) is shown for $1, \ldots, n-1$ then

$$I(Z_1; Y_2^n) = I(Z_1; Y_2^{n-1}) + I(Z_1; Y_n|Y_2^{n-1}) \tag{323}$$

$$= I(Z_1; Y_2^{n-1}) + I(X_n; Y_n|Y_2^{n-1}) \tag{324}$$

$$= I(Z_1; Y_2^{n-1}) + \frac{1}{2} \log\left(1 + \frac{2\mathbb{E}\left[|X_n|^2\right]}{N_0}\right) \tag{325}$$

$$= I(Z_1; Y_2^{n-1}) + \frac{1}{2} \log\left(1 + \frac{2c_n^2\,\mathrm{Var}[Z_1|Y_2^{n-1}]}{N_0}\right) \tag{326}$$

$$= \frac{1}{2} \log\left\{\frac{N_0}{2\,\mathrm{Var}[Z_1|Y_2^{n-1}]}\left(1 + \frac{2c_n^2\,\mathrm{Var}[Z_1|Y_2^{n-1}]}{N_0}\right)\right\} \tag{327}$$

$$= \frac{1}{2} \log\left\{\frac{N_0}{2\sigma_{n-1}^2}\left(1 + \frac{2c_n^2\sigma_{n-1}^2}{N_0}\right)\right\} \tag{328}$$

$$= \frac{1}{2} \log \frac{N_0}{2\sigma_n^2}\,, \tag{329}$$

[9]We follow the elegant analysis of the Schalkwijk-Kailath method introduced in [21, p. 18-6].

where (324) expresses the fact that given Y_2^{n-1}, Z_1 is an invertible function of X_n; (325) is because $Y_j = X_j + Z_j$ with Z_j independent of Y_2^{j-1}; (326) is by (302); (327) is by (322); (328) is by the induction hypothesis; and (329) is by (295). The induction step is then proved by comparing (329) and (322).

b) Next, consider $n \geq L+2$. Due to (311) the relationship between $(Z_1 - \hat{Z}_{n-1})$ and $\tilde{Y}_n$ in the second phase is the same as for $(Z_1 - \hat{Z}_{n-1})$ and Y_n in the first phase:

$$\tilde{Y}_n = c_n(Z_1 - \hat{Z}_{n-1}) + Z_n \,, \tag{330}$$

where $Z_1 - \hat{Z}_{n-1}$ is still Gaussian. Thus the proof of the induction step in (322)-(329) holds verbatim by replacing Y_n with $\tilde{Y}_n$ and $\mathbb{E}\left[|X_n|^2\right]$ with $\mathbb{E}\left[|X_n + \hat{X}_n|^2\right]$ for $n \geq L+2$.

c) Note that in the course of the proof we have shown that

$$\mathbb{E}\left[|X_n|^2\right] = c_n^2 \sigma_{n-1}^2 \,, \quad n = 2, \ldots, L+1 \,, \tag{331}$$

and therefore substituting (296) and (297) into (331) and using

$$\mathbb{E}\left[|X_1|^2\right] = \frac{E_0}{3}\frac{M+1}{M-1} \leq E_0 \tag{332}$$

inequality (316) follows.

d) Next we show (318). Since $q_\delta(x) + \delta = q_\delta(x+\delta)$, from (306) we have

$$X_n = c_n q_\delta\left(Z_1 - \hat{Z}_1(Y_2^{n-1})\right) \tag{333}$$

$$\mathbb{E}\left[|X_n|^2\right] = c_n^2 \sigma_{n-1}^2 s\left(\frac{\delta}{\sigma_{n-1}}\right) \,, \tag{334}$$

which trivially implies (318).

e) We are left to show (320). First, we give an upper bound on $s(d)$ for $d \geq 2$:[10]

$$s(d) \leq 2\int_{\frac{d}{2}}^{\infty} \frac{1}{\sqrt{2\pi}}\left(x^2 + xd + \frac{d^2}{4}\right) e^{-\frac{x^2}{2}} dx \tag{335}$$

$$= \frac{d^2}{2} Q\left(\frac{d}{2}\right) + \frac{2d}{\sqrt{2\pi}} e^{-\frac{d^2}{8}} + 2\int_{\frac{d}{2}}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{336}$$

$$\leq \frac{3d}{\sqrt{2\pi}} e^{-\frac{d^2}{8}} + 2\int_{\frac{d}{2}}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{337}$$

$$= \frac{4d}{\sqrt{2\pi}} e^{-\frac{d^2}{8}} + \int_{\frac{d^2}{8}}^{\infty} \sqrt{\frac{1}{\pi y}} e^{-y} dy \tag{338}$$

$$\leq \frac{4d}{\sqrt{2\pi}} e^{-\frac{d^2}{8}} + \frac{4}{d\sqrt{2\pi}} \int_{\frac{d^2}{8}}^{\infty} e^{-y} dy \tag{339}$$

$$\leq \frac{5d}{\sqrt{2\pi}} e^{-\frac{d^2}{8}} , \tag{340}$$

where (335) follows by applying an upper bound

$$|q_d(x)| \leq \left(|x| + \frac{d}{2}\right) 1\{2|x| \geq d\} , \tag{341}$$

(337) is by (41), (338) follows by integrating by parts with $y = \frac{x^2}{2}$, and (340) holds since by assumption $d \geq 2$.

Notice that the dependence of E_2 on E_0, E_1 and L is only through the following parameter

$$\delta_1(E_0, E_1, L) \triangleq \frac{\delta}{\sigma_{L+1}} \tag{342}$$

$$= \frac{2}{M-1}\sqrt{\frac{2E_0}{N_0}}\left(1 + \frac{2(E_1 - E_0)}{LN_0}\right)^{\frac{L}{2}} , \tag{343}$$

where (343) follows from (297) and (309). From now on we write $E_2(\rho, \delta_1)$ to signify the fact that E_2 is implicitly a function of ρ and δ_1.

Next, for any $\delta_1 > 2$ we have

$$\inf_{\rho} E_2(\rho, \delta_1) \leq \frac{20 N_0 \delta_1}{\sqrt{2\pi}} \frac{e^{-\frac{\delta_1^2}{8}}}{\delta_1^2 - 4} . \tag{344}$$

[10]Note that although $\lim_{d\to 0} s(d) = 1$, it is not true that $s(d) \leq 1$. In fact $s(d) > 1$ for a certain interval $d \in (0, d^*)$. This explains why we use subtraction of $\hat{X}_n$ only for $n \geq L+2$. Indeed, without subtraction $\mathbb{E}|X_n|^2 = c_n^2 \sigma_{n-1}^2$ and therefore from (334) we see that it is only sensible to use subtraction when $s(d) \leq 1$, or equivalently when σ_{n-1} is sufficiently small. This is an artifact of the suboptimal approximation of $\mathbb{E}[X_n | Y_1^{n-1}]$ by $\hat{X}_n$. A slightly weaker bound on $s(d)$ follows from [20, Lemma 4.1].

Indeed, consider the following upper bound:

$$E_2(\rho,\delta_1) = \frac{N_0}{2}\sum_{k=0}^{\infty}\rho\, s\left(\delta_1(1+\rho)^{\frac{k}{2}}\right) \tag{345}$$

$$\leq \frac{5N_0\delta_1}{2\sqrt{2\pi}}\sum_{k=0}^{\infty}\rho(1+\rho)^{\frac{k}{2}}e^{-\frac{\delta_1^2}{8}(1+\rho)^k} \tag{346}$$

$$\leq \frac{5N_0\delta_1}{2\sqrt{2\pi}}\sum_{k=0}^{\infty}\rho e^{\frac{k}{2}\log_e(1+\rho)-\frac{\delta_1^2}{8}(1+k\rho)} \tag{347}$$

$$= \frac{5N_0\delta_1}{2\sqrt{2\pi}}e^{-\frac{\delta_1^2}{8}}\sum_{k=0}^{\infty}\rho e^{-\frac{k}{2}\left(\frac{\delta_1^2}{4}\rho-\log_e(1+\rho)\right)} \tag{348}$$

$$= \frac{5N_0\delta_1}{2\sqrt{2\pi}}\rho e^{-\frac{\delta_1^2}{8}}\left(1-e^{-\frac{\delta_1^2\rho}{8}}\sqrt{1+\rho}\right)^{-1}, \tag{349}$$

where (345) is by (297) and (318); (346) is by applying (340); (347) is because $(1+\rho)^k \geq 1+k\rho$, and (349) is because $\frac{\delta_1^2}{4}\rho > \log_e(1+\rho)$ for all $\rho > 0$ and $\delta_1 > 2$. Finally, (344) is obtained by taking $\rho \to 0$ in (349).

Notice now that for E_1 fixed the optimization of δ_1 over E_0 and L is simple:

$$\delta_1^*(E_1) \triangleq \sup_{E_0,L}\delta_1 \tag{350}$$

$$= \sup_{E_0,L}\frac{2}{M-1}\sqrt{\frac{2E_0}{N_0}}\left(1+\frac{2(E_1-E_0)}{LN_0}\right)^{\frac{L}{2}} \tag{351}$$

$$= \frac{2}{M-1}e^{\frac{E_1}{N_0}-\frac{1}{2}} \tag{352}$$

(supremum is attained as $L \to \infty$ and $E_0 \to \frac{N_0}{2}$). In other words, to achieve a certain value of δ_1 we need to expend slightly more than the energy

$$E_1^*(\delta_1) = N_0\left(\frac{1}{2}+\log_e\frac{M-1}{2}\delta_1\right). \tag{353}$$

Thus, we have

$$\inf_{E_0,E_1,L,\rho} E_1 + E_2(\rho,\delta_1) \leq \inf_{E_0,E_1,L:\delta_1>2} E_1 + \frac{20N_0\delta_1}{\sqrt{2\pi}}\frac{e^{-\frac{\delta_1^2}{8}}}{\delta_1^2-4} \tag{354}$$

$$\leq \inf_{\delta_1>2} E_1^*(\delta_1) + \frac{20N_0\delta_1}{\sqrt{2\pi}}\frac{e^{-\frac{\delta_1^2}{8}}}{\delta_1^2-4} \tag{355}$$

$$\leq N_0\left(\frac{1}{2}+\log_e\frac{M-1}{2}+\log_e\frac{87}{16}\right) \tag{356}$$

$$= E_z(M), \tag{357}$$

where (354) is by (344) and restricting to $\{\delta_1 > 2\}$; (355) is by (353); and (356) follows from

$$\inf_{\delta_1>2}\left[\log_e \delta_1 + \frac{20\delta_1}{\sqrt{2\pi}}\frac{e^{-\frac{\delta_1^2}{8}}}{\delta_1^2-4}\right] \le \log_e \frac{87}{16}\,, \tag{358}$$

which is easily verified by taking $\delta_1 = 5$ in the left-hand side. This completes the proof of (320). ■

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