Reconstruction on 2D Regular Grids

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***Abstract*—We investigate the problem of broadcasting a bit on a 2D regular grid. Consider a directed acyclic graph with the structure of a 2D regular grid, which has a single source vertex X at layer 0, and $k+1$ vertices at distance of $k \geq 1$ from X at layer k. Every vertex has outdegree 2, the boundary vertices have indegree 1, and the interior vertices have indegree 2. At time 0, X is given a uniform random bit. At time $k \geq 1$, each vertex in layer k receives bits from its parents in layer $k-1$, where the bits pass through binary symmetric channels with crossover probability $\delta \in \left(0, \frac{1}{2}\right)$. Each vertex with indegree 2 then combines its input bits with a common Boolean processing function to produce its output bit. The goal is to reconstruct X with probability of error less than $\frac{1}{2}$ from all vertices at layer k as $k \to \infty$. Besides their natural interpretation in communication networks, such stochastic processes can be construed as 1D probabilistic cellular automata (PCA) with boundary conditions on the number of sites per layer. Inspired by the "positive rates conjecture" for 1D PCA, we establish that reconstruction of X is impossible for any δ provided that either AND or XOR gates are employed as the common processing function. Furthermore, we show that if certain structured supermartingales exist, reconstruction is impossible for any δ when a common NAND processing function is used. We also provide numerical evidence for the existence of these supermartingales using linear programming.**

I. INTRODUCTION

The problem of reconstruction on two-dimensional (2D) regular grids is a salient specialization of the general problem of reconstruction on directed acyclic graphs (DAGs) [1]–[3]. In the general problem, we are given a bounded indegree DAG with a unique source vertex at layer 0 such that all vertices at layer $k \geq 1$ (at a distance k from the source) only have parents in layer $k-1$. At time 0, the source is given a uniform random bit. At time $k \geq 1$, each vertex in layer k receives noisy bits from its parents, which are corrupted by binary symmetric channels (BSCs) with crossover probability $\delta \in \left(0, \frac{1}{2}\right)$. Moreover, vertices with indegree greater than 1 combine their noisy inputs by applying Boolean processing functions. The broad objective is to determine conditions on the graph topology, the noise level δ, and the choices of processing functions that permit reconstruction of the source bit from all vertices at layer k as $k \to \infty$.

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To address this rather challenging goal, results in the literature analyze fixed classes of DAGs and processing functions. For example, the classical version of the problem concerns reconstruction on a rooted tree T (cf. [4]). Here, it is well-known that the source bit is impossible to reconstruct (i.e., the minimum probability of error converges to $\frac{1}{2}$ as $k \to \infty$) if and only if $(1-2\delta)^2 \operatorname{br}(T) \leq 1$, where $\operatorname{br}(T)$ is the branching number of T [4]–[6]. This key result and its generalizations, cf. [7]–[15], precisely characterize when information about the root bit vanishes completely in tree-structured topologies. On the other hand, [1], [2] study reconstruction on randomly constructed DAGs (which arguably better model real-world communication or social networks since vertices often receive multiple input signals in these scenarios). Specifically, [1], [2] establish *phase transition* results for random DAGs with all indegrees equal to d, and all majority (when $d \geq 3$) or NAND processing functions (when $d = 2$), where reconstruction is possible if δ is less than a critical threshold and layer sizes grow at least logarithmically in the depth, and impossible otherwise. Hence, while layers must grow exponentially for reconstruction to be possible in trees, these results demonstrate the existence of DAGs with bounded indegree and logarithmically growing layer sizes that admit reconstruction. Furthermore, explicit constructions of such DAGs using *expander graphs* are also provided in [1], [2].

In this paper, we consider reconstruction on 2D regular grids where all "interior" vertices use the same processing function. We further motivate this model in section I-A and formalize it in section I-B. Our main contribution is to show that reconstruction is impossible on 2D regular grids for various choices of the common processing function regardless of the noise level δ. In particular, we present our impossibility results for AND and XOR processing functions in sections II and III, respectively. Then, we delineate our partial impossibility result and accompanying numerical simulations for NAND processing functions in section IV. Together, these results convey that reconstruction on 2D regular grids is essentially impossible for all 8 *symmetric* processing functions (due to symmetry in the 2D regular grid model). For brevity, many technical details are deferred to the complete manuscript [16].

A. Motivation

The problem of reconstruction on DAGs analyzes whether the "wavefront of information" broadcasted by a source bit decays irrecoverably as it propagates through the DAG. Besides this canonical communication theoretic interpretation,

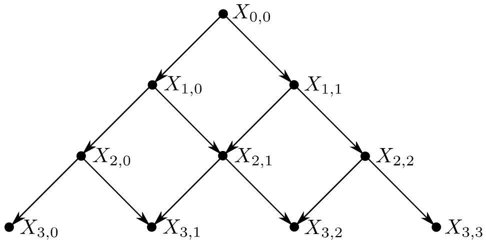

Fig. 1. Diagram of a 2D regular grid.

reconstruction on DAGs is a natural model of *fault-tolerant computation* and storage, because it can be construed as a noisy circuit (with faulty wires and perfect logic gates) that has been constructed to remember a bit, cf. [17]–[22]. Furthermore, reconstruction on certain families of DAGs correspond to well-known models in statistical physics, e.g., trees correspond to studying the extremality of free boundary Gibbs measures of ferromagnetic *Ising models* [4], and regular grids are closely related to spin-flip systems such as *probabilistic cellular automata* (PCA) [23]–[26]. Finally, other special cases of the model represent information flow in biological networks [27]–[30], play a crucial role in *random constraint satisfaction* problems [31]–[34], and are useful in proving converse results for *community detection* in stochastic block models [35].

While this discussion motivates the study of reconstruction on DAGs in general, our work in this paper is particularly inspired by the renowned *positive rates conjecture* for one-dimensional (1D) PCA, cf. [26]. The positive rates conjecture states that "relatively simple" 1D PCA with local interactions and strictly positive noise probabilities are *ergodic* [26]. (Note that known counter-examples either require a lot of states [36] or are non-uniform in time and space [37].) Since reconstruction on a 2D regular grid can be perceived as a 1D PCA with boundary conditions that limit the layer sizes to be $k+1$, we *conjecture* that reconstruction is impossible for 2D regular grids regardless of the noise level δ and the choice of common processing function. We refer readers to [16, Section I-A] for further elaboration of this conjecture. Our main results in sections II, III, and IV make partial progress towards this conjecture and provide strong evidence for it.

B. 2D Regular Grid Model

The *2D regular grid model* consists of an infinite DAG with vertices that are Bernoulli random variables and edges that are independent BSCs. The source vertex of the grid is $X_{0,0} \sim \mathsf{Bernoulli}\big(\frac{1}{2}\big)$, and we let $X_k = (X_{k,0}, \dots, X_{k,k})$ be the vector of $k+1$ vertex random variables at distance $k \in \mathbb{N} \triangleq \{0, 1, 2, \dots\}$ from the source. Furthermore, the 2D regular grid contains the directed edges $X_{k,j} \to X_{k+1,j}$ and $X_{k,j} \to X_{k+1,j+1}$ for every $k \in \mathbb{N}$ and every $j \in \{0, \dots, k\}$. The underlying DAG of such a grid is shown in Figure 1.

To construct a *Bayesian network* on this 2D regular grid, we fix some parameter $\delta \in \big(0, \frac{1}{2}\big)$ and a Boolean processing function $f : \{0,1\}^2 \to \{0,1\}$. Then, for any $k \in \mathbb{N}\backslash\{0\}$, we define $X_{k,0} = X_{k-1,0} \oplus Z_{k,0,2}$ and $X_{k,k} = X_{k-1,k-1} \oplus Z_{k,k,1}$, and for any $k \in \mathbb{N}\backslash\{0,1\}$ and $j \in \{1, \dots, k-1\}$, we define

$$X_{k,j} = f(X_{k-1,j-1} \oplus Z_{k,j,1}, X_{k-1,j} \oplus Z_{k,j,2}) \,, \qquad (1)$$

where $\oplus$ denotes addition modulo 2, and the binary random variables $\{Z_{k,j,i} : k \in \mathbb{N}\backslash\{0\},\, j \in \{0, \dots, k\},\, i \in \{1,2\}\}$ are i.i.d. $\mathsf{Bernoulli}(\delta)$ and independent of $X_{0,0}$. This implies that edges are BSCs with crossover probability δ, and characterizes the conditional distribution of any $X_{k,j}$ given its parents.

The sequence $\{X_k : k \in \mathbb{N}\}$ forms a *Markov chain*, and our goal is to determine whether X_0 can be decoded from X_k as $k \to \infty$. Given X_k for any fixed $k \in \mathbb{N}\backslash\{0\}$, inferring the value of X_0 is a binary hypothesis testing problem with minimum achievable probability of error

$$P_{\mathsf{ML}}^{(k)} \triangleq \mathbb{P}\big(h_{\mathsf{ML}}^k(X_k) \neq X_0\big) = \frac{1}{2}\Big(1 - \big\|P_{X_k}^+ - P_{X_k}^-\big\|_{\mathsf{TV}}\Big), \quad (2)$$

where $h_{\mathsf{ML}}^k : \{0,1\}^{k+1} \to \{0,1\}$ is the *maximum likelihood* decision rule at level k, $P_{X_k}^+$ and $P_{X_k}^-$ are the conditional distributions of X_k given $X_0 = 1$ and $X_0 = 0$, respectively, and $\|\cdot\|_{\mathsf{TV}}$ denotes the *total variation (TV) distance*. Since $P_{\mathsf{ML}}^{(k)}$ is non-decreasing in k (by the data processing inequality for TV distance) and bounded above by $\frac{1}{2}$, its limit exists. Therefore, we say that *reconstruction* (of X_0) *is impossible* if

$$\lim_{k \to \infty} P_{\mathsf{ML}}^{(k)} = \frac{1}{2} \quad \Leftrightarrow \quad \lim_{k \to \infty} \big\|P_{X_k}^+ - P_{X_k}^-\big\|_{\mathsf{TV}} = 0 \qquad (3)$$

(which follows from (2)), and possible otherwise.

II. AND Processing Functions

We first analyze the case where all vertices of the 2D regular grid with two inputs use the AND gate, i.e., $f(x_1, x_2) = x_1 \wedge x_2$ in section I-B, where $\wedge$ denotes the logical AND operation. In this setting, our first main result conveys that reconstruction is impossible for all $\delta \in \big(0, \frac{1}{2}\big)$.

Theorem 1 (AND 2D Regular Grid)**.** *Consider a 2D regular grid model with AND processing functions. Then, for all* $\delta \in \big(0, \frac{1}{2}\big)$*, reconstruction is impossible in the sense of* (3)*.*

Proof Outline. We outline the proof here and refer readers to [16, Section III] for details. We first construct a *monotone Markovian coupling* of the Markov chains $\{X_k^+ : k \in \mathbb{N}\}$ and $\{X_k^- : k \in \mathbb{N}\}$, which are versions of the Markov chain $\{X_k : k \in \mathbb{N}\}$ initialized at $X_0^+ = 1$ and $X_0^- = 0$, respectively. Specifically, we couple these chains to run on the same 2D regular grid with common BSCs; along any edge BSC, e.g., $X_{k,j} \to X_{k+1,j}$, either $X_{k,j}^+$ and $X_{k,j}^-$ are both copied with probability $1 - 2\delta$, or a *shared* independent $\mathsf{Bernoulli}\big(\frac{1}{2}\big)$ bit is produced with probability 2δ (that is used by both $X_{k+1,j}^+$ and $X_{k+1,j}^-$). Since the AND gate is monotone non-decreasing, this coupling is monotone, i.e., $X_{k,j}^+ \geq X_{k,j}^-$ almost surely for all $k \in \mathbb{N}$ and $j \in \{0, \dots, k\}$. For convenience, define the alphabet set $\mathcal{Y} \triangleq \{0_{\mathsf{c}} = (0,0), 1_{\mathsf{u}} = (0,1), 1_{\mathsf{c}} = (1,1)\}$ and the coupled grid variables $\{Y_{k,j} = (X_{k,j}^-, X_{k,j}^+) \in \mathcal{Y} : k \in \mathbb{N},\, j \in \{0, \dots, k\}\}$, where $Y_{0,0} = 1_{\mathsf{u}}$ almost surely. Then, our Markovian coupling of $\{X_k^+ : k \in \mathbb{N}\}$ and $\{X_k^- : k \in \mathbb{N}\}$ is the Markov chain $\{Y_k = (Y_{k,0}, \dots, Y_{k,k}) : k \in \mathbb{N}\}$. In the sequel, we assume that the coupled grid variables index vertices of the 2D regular grid.

We next observe using the maximal coupling characterization of TV distance that

$$\left\|P^+_{X_k} - P^-_{X_k}\right\|_{\mathsf{TV}} \le \mathbb{P}\big(X^+_k \neq X^-_k\big) = 1 - \mathbb{P}\big(X^+_k = X^-_k\big)\,.$$

Since $\{X^+_k = X^-_k\} \subseteq \{X^+_{k+1} = X^-_{k+1}\}$ for all $k \in \mathbb{N}$ (due to our Markovian coupling), we may let $k \to \infty$ and obtain

$$\lim_{k\to\infty} \left\|P^+_{X_k} - P^-_{X_k}\right\|_{\mathsf{TV}} \le 1 - \mathbb{P}(A)\,,$$

where we define the event $A \triangleq \{\exists k \in \mathbb{N}, \forall j \in \{0,\dots,k\}, Y_{k,j} \in \{0_{\mathsf{c}}, 1_{\mathsf{c}}\}\}$. Hence, it suffices to prove that $\mathbb{P}(A) = 1$.

To prove this, we recall a well-known result on *oriented bond percolation* in 2D lattices. Suppose we independently keep each edge of the 2D regular grid "open" with some probability $p \in [0,1]$, and delete it with probability $1-p$. Define the event $\Omega_\infty \triangleq \{$there is an infinite open path starting at $Y_{0,0}\}$, and for every level $k \in \mathbb{N}$, define the random variables

$$R_k \triangleq \sup\{j \in \{0,\dots,k\} : \exists \text{ open path from } Y_{0,0} \text{ to } Y_{k,j}\}\,,$$
$$L_k \triangleq \inf\{j \in \{0,\dots,k\} : \exists \text{ open path from } Y_{0,0} \text{ to } Y_{k,j}\}\,,$$

which are the rightmost and leftmost vertices that are connected to the source. It is proved in [38, Section 3] that the occurrence of Ω_∞ experiences a phase transition phenomenon.

Lemma 1 (Oriented Bond Percolation [38, Section 3])**.** *For the aforementioned bond percolation process on the 2D regular grid, there exists a critical threshold $\delta_{\mathsf{perc}} \in \left(\frac{1}{2}, 1\right)$ such that:*

1) *If $p < \delta_{\mathsf{perc}}$, then $\mathbb{P}_p(\Omega_\infty) = 0$, where $\mathbb{P}_p$ denotes the probability measure defined by the percolation process.*
2) *If $p > \delta_{\mathsf{perc}}$, then $\mathbb{P}_p(\Omega_\infty) > 0$ and for some $\alpha = \alpha(p) > 0$,*
$$\mathbb{P}_p\!\left(\lim_{k\to\infty} \frac{R_k}{k} = \frac{1+\alpha}{2} \text{ and } \lim_{k\to\infty} \frac{L_k}{k} = \frac{1-\alpha}{2}\,\middle|\,\Omega_\infty\right) = 1$$
where $\alpha(p)$ is defined in [38, Section 3, Equation (6)].

We now prove $\mathbb{P}(A) = 1$ by considering two cases.

Case 1: Suppose $1 - 2\delta < \delta_{\mathsf{perc}}$ in our coupled 2D regular grid. Consider a bond percolation process (as described above) with $p = 1 - 2\delta$, where each edge of the grid is open if and only if the corresponding BSC copies its input. Then, by part 1 of Lemma 1, the event Ω^c_∞ occurs almost surely. Moreover, our Markovian coupling ensures that a 1_{u} travels from level k to level $k+1$ only if one of its outgoing edges is open. So, there exists $k \in \mathbb{N}$ such that none of the vertices at level k are 1_{u}'s. Hence, $\Omega^c_\infty \subseteq A$ and we have $\mathbb{P}(A) = 1$, as desired.

Case 2: Suppose $1 - \delta > \delta_{\mathsf{perc}}$ in our coupled 2D regular grid. Consider a bond percolation process (as described above) with $p = 1 - \delta$, where each edge of the grid is open if and only if the corresponding BSC either copies its input or generates 0 as the new shared bit. For $k \in \mathbb{N}\backslash\{0\}$, let B_k be the event that the BSC from $Y_{k-1,0}$ to $Y_{k,0}$ generates a new bit which equals 0. Then, $\mathbb{P}(B_k) = \delta$ and $\{B_k : k \in \mathbb{N}\backslash\{0\}\}$ are mutually independent. So, infinitely many of the events B_k occur almost surely by the second Borel-Cantelli lemma. Furthermore, $B_k \subseteq \{Y_{k,0} = 0_{\mathsf{c}}\}$ for every $k \in \mathbb{N}\backslash\{0\}$.

For every $k \in \mathbb{N}$, let $\mathcal{F}_k$ be the σ-algebra generated by all BSCs before level k. Then, relative to the filtration $\{\mathcal{F}_k : k \in \mathbb{N}\}$, define the sequences of *stopping times*

$$L_i \triangleq \min\{k \ge T_{i-1} + 1 : B_k \text{ occurs}\}\,,$$
$$T_i \triangleq 1 + \max\left\{k \ge L_i : \begin{array}{l}\exists j \in \{0,\dots,k\},\, Y_{k,j} \text{ connected}\\ \text{to } Y_{L_i,0} \text{ by an open path}\end{array}\right\},$$

for all $i \in \mathbb{N}\backslash\{0\}$, where we set $T_0 \triangleq 0$. Here, when $T_{i-1} = \infty$, we let $L_i = \infty$ almost surely, and when $L_i = \infty$, we let $T_i = \infty$ almost surely. (Note that when $L_i < \infty$, $T_i - L_i - 1$ is the length of the longest open path connected to $Y_{L_i,0}$.) Now observe that

$$\begin{aligned}
&\mathbb{P}(\exists k \ge 1,\, T_k = \infty)\\
&= \mathbb{P}(T_1 = \infty) + \sum_{m=2}^{\infty} \mathbb{P}(\exists k \ge 2,\, T_k = \infty | T_1 = m)\mathbb{P}(T_1 = m)\\
&= \mathbb{P}(T_1 = \infty) + \sum_{m=2}^{\infty} \mathbb{P}(\exists k \ge 1,\, T_k + m = \infty)\mathbb{P}(T_1 = m)\\
&= \mathbb{P}(T_1 = \infty) + \mathbb{P}(T_1 < \infty)\mathbb{P}(\exists k \ge 1,\, T_k = \infty)\,,
\end{aligned}$$

where the second equality follows from the fact that for all $m \ge 2$, the random variables $\{(L_i, T_i) : i \ge 2\}$ given $T_1 = m$ have the same probability distribution as the random variables $\{(L_i + m, T_i + m) : i \ge 1\}$. Rearranging this, we get

$$\mathbb{P}(\exists k \ge 1,\, T_k = \infty)\mathbb{P}(T_1 = \infty) = \mathbb{P}(T_1 = \infty)\,.$$

Since $\mathbb{P}(T_1 = \infty) = \mathbb{P}(\Omega_\infty) > 0$ by part 2 of Lemma 1, we have

$$\mathbb{P}(\exists k \ge 1,\, T_k = \infty) = 1\,. \tag{4}$$

For every $k \in \mathbb{N}\backslash\{0\}$, define the events

$$\Omega^{\mathsf{left}}_k \triangleq \{\exists \text{ infinite open path starting at } Y_{k,0}\}\,,$$
$$\Omega^{\mathsf{right}}_k \triangleq \{\exists \text{ infinite open path starting at } Y_{k,k}\}\,.$$

If the event $\{\exists k \ge 1,\, T_k = \infty\}$ occurs, we can choose the smallest $m \in \mathbb{N}\backslash\{0\}$ such that $T_m = \infty$, and for this m, there is an infinite open path starting at $Y_{L_m,0} = 0_{\mathsf{c}}$ (where $Y_{L_m,0} = 0_{\mathsf{c}}$ because B_{L_m} occurs). Hence, using (4), we have

$$\mathbb{P}\big(\exists k \in \mathbb{N},\, \{Y_{k,0} = 0_{\mathsf{c}}\} \cap \Omega^{\mathsf{left}}_k\big) = 1\,,$$

which, by symmetry, implies that

$$\mathbb{P}\big(\exists k, m \in \mathbb{N}, \{Y_{k,0} = Y_{m,m} = 0_{\mathsf{c}}\} \cap \Omega^{\mathsf{left}}_k \cap \Omega^{\mathsf{right}}_m\big) = 1. \tag{5}$$

Finally, consider $k, m \in \mathbb{N}$ such that $Y_{k,0} = Y_{m,m} = 0_{\mathsf{c}}$, and suppose that Ω^{left}_k and $\Omega^{\mathsf{right}}_m$ both happen. For every integer $n > \max\{k, m\}$, define the random variables

$$\hat{R}_n \triangleq \sup\{j \in \{0,\dots,n\} : \exists \text{ open path from } Y_{k,0} \text{ to } Y_{n,j}\}\,,$$
$$\hat{L}_n \triangleq \inf\{j \in \{0,\dots,n\} : \exists \text{ open path from } Y_{m,m} \text{ to } Y_{n,j}\}\,,$$

which are the rightmost and leftmost vertices at level n that are connected to $Y_{k,0}$ and $Y_{m,m}$, respectively. Using part 2 of Lemma 1, we know that almost surely,

$$\lim_{n\to\infty} \frac{\hat{R}_n}{n} = \lim_{n\to\infty} \frac{\hat{R}_n}{n-k} = \frac{1+\alpha(1-\delta)}{2}\,,$$

$$\lim_{n\to\infty} \frac{\hat{L}_n}{n} = \lim_{n\to\infty} \frac{\hat{L}_n - m}{n-m} = \frac{1-\alpha(1-\delta)}{2}\,.$$

This implies that almost surely,

$$\lim_{n\to\infty} \frac{\hat{R}_n - \hat{L}_n}{n} = \alpha(1-\delta) > 0\,,$$

which means that for some sufficiently large level $n^* > \max\{k, m\}$, *the rightmost open path from* $Y_{k,0}$ *meets the leftmost open path from* $Y_{m,m}$, i.e., $\left|\hat{R}_{n^*} - \hat{L}_{n^*}\right| \leq 1$.

To complete the proof, notice that by construction, all the vertices in these two open paths are equal to 0_{c}. Furthermore, due to our Markovian coupling, all vertices at level n^* that are either to left of $\hat{R}_{n^*}$ or to the right of $\hat{L}_{n^*}$ take values in $\{0_{\mathsf{c}}, 1_{\mathsf{c}}\}$. This shows that the event A occurs. Therefore, we get $\mathbb{P}(A) = 1$ using (5), which completes the proof. ■

III. XOR Processing Functions

We next consider the case where all vertices of the 2D regular grid with two inputs use the XOR gate, i.e., $f(x_1, x_2) = x_1 \oplus x_2$ in section I-B. In this setting, our second main result again conveys that reconstruction is impossible for all $\delta \in \left(0, \frac{1}{2}\right)$.

Theorem 2 (XOR 2D Regular Grid)**.** *Consider a 2D regular grid model with XOR processing functions. Then, for all* $\delta \in \left(0, \frac{1}{2}\right)$*, reconstruction is impossible in the sense of* (3)*.*

Theorem 2 is proved in [16, Section IV]. In the 2D regular grid with XOR processing functions, every vertex at level k can be written as a (binary) linear combination of the source bit and all the BSC noise random variables in the grid before level k (i.e., $\{Z_{m,j,i} : m \in \{1, \dots, k\},\, j \in \{0, \dots, m\},\, i \in \{1, 2\}\}$). This linear relationship can be captured by a binary matrix. The main idea of the proof is to perceive this matrix as a parity check matrix of a *linear code*. The problem of inferring $X_{0,0}$ from X_k turns out to be equivalent to the problem of decoding the first bit of a codeword drawn uniformly from this code after observing a noisy version of the codeword. Known properties of bit-wise maximum likelihood decoding of linear codes can then be exploited to complete the proof, as shown in [16, Section IV].

IV. NAND Processing Functions

Finally, we present our partial impossibility result in the setting where all vertices of the 2D regular grid with two inputs use the NAND gate, i.e., $f(x_1, x_2) = \neg(x_1 \wedge x_2)$ in section I-B, where $\neg$ is the logical NOT operation. In particular, inspired by the potential function technique employed in the proof of ergodicity of 1D PCA with noisy NOR gates in [39, Theorem 1], we delineate a sufficient condition for proving impossibility of reconstruction in the 2D regular grid model with NAND processing functions, and provide accompanying numerical evidence that this sufficient condition is actually true.

To this end, we begin with some necessary setup. As before in section II, we couple the Markov chains $\{X_k^+ : k \in \mathbb{N}\}$ and $\{X_k^- : k \in \mathbb{N}\}$ to run on the same 2D regular grid with common BSCs. This produces the Markovian coupling $\{Y_k = (Y_{k,0}, \dots, Y_{k,k}) : k \in \mathbb{N}\}$ with coupled grid variables $\{Y_{k,j} = (X_{k,j}^-, X_{k,j}^+) \in \mathcal{Y} : k \in \mathbb{N},\, j \in \{0, \dots, k\}\}$, where the extended alphabet $\mathcal{Y} \triangleq \{0, 1, u\}$ is slightly different to that in section II. Here, $X_{k,j}^- = X_{k,j}^+ = 0$ if $Y_{k,j} = 0$ and $X_{k,j}^- = X_{k,j}^+ = 1$ if $Y_{k,j} = 1$, but $Y_{k,j} = u$ means that it is *unknown* whether $X_{k,j}^- = X_{k,j}^+$. A more detailed explanation of this Markovian coupling can be found in [16, Section V-A], e.g., the NAND gate can be modified to account for u's. As before, the key takeaway is that to establish impossibility of reconstruction, it suffices to show that the number of u's per layer vanishes at deeper levels of the 2D regular grid. To verify this latter condition, we introduce the class of cyclic potential functions (inspired by [39]), a partial order over these potential functions, and a pertinent linear operator on the space of potential functions in the next definition.

Definition 1 (Cyclic Potential Functions and Related Notions)**.** Given any strings $v_1, \dots, v_m \in \mathcal{Y}^* = \cup_{k \in \mathbb{N}\backslash\{0\}} \mathcal{Y}^k$ and any coefficients $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, we may define a corresponding *cyclic potential function* $w : \mathcal{Y}^* \to \mathbb{R}$ via the formal sum

$$w = \sum_{j=1}^{m} \alpha_j \{v_j\}\,,$$

where curly braces are used to distinguish a string $v \in \mathcal{Y}^*$ from its associated potential function $\{v\} : \mathcal{Y}^* \to \mathbb{R}$. In particular, for every $k \in \mathbb{N}\backslash\{0\}$ and every string $y = (y_0 \cdots y_{k-1}) \in \mathcal{Y}^k$, the cyclic potential function w is evaluated as follows:

$$w[y] \triangleq \sum_{\substack{1 \leq j \leq m:\\ s_j \leq k}} \alpha_j \sum_{i=0}^{k-1} \mathbb{1}\left\{\left(y_{(i)_k} \cdots y_{(i+s_j-1)_k}\right) = v_j\right\}$$

where s_j denotes the length of $v_j \in \mathcal{Y}^{s_j}$, $\mathbb{1}\{\cdot\}$ is the indicator function, and $(i)_k \equiv i \pmod{k}$ for every $i \in \mathbb{N}$. Furthermore, we say that w is *u-only* if the strings $v_1, \dots, v_m$ all contain a u. For any fixed $r \in \mathbb{N}\backslash\{0\}$, we may also define a *partial order* $\succeq_{\mathsf{c}}$ over the set of all cyclic potential functions for which the lengths of the underlying strings (with non-zero coefficients) are bounded by r. Specifically, for any pair of such cyclic potential functions $w_1 : \mathcal{Y}^* \to \mathbb{R}$ and $w_2 : \mathcal{Y}^* \to \mathbb{R}$, we have

$$w_1 \succeq_{\mathsf{c}} w_2 \quad \Leftrightarrow \quad \forall y \in \bigcup_{k \geq r} \mathcal{Y}^k,\;\; w_1[y] \geq w_2[y]\,.$$

Finally, we define the *conditional expectation operator* $\mathcal{E}$ on the space of cyclic potential functions as follows. For any input cyclic potential function w (defined by the formal sum above), $\mathcal{E}$ outputs the cyclic potential function with formal sum

$$\mathcal{E}(w) \triangleq \sum_{j=1}^{m} \alpha_j \sum_{z \in \mathcal{Y}^{s_j+1}} \mathbb{P}\left((Y_{s_j+1,1}, \dots, Y_{s_j+1,s_j}) = v_j \middle| Y_{s_j} = z\right) \{z\}$$

where the probabilities are determined by the aforementioned Markovian coupling $\{Y_k : k \in \mathbb{N}\}$.

Using the concepts shown in Definition 1, our final main result presents a sufficient condition for the impossibility of reconstruction on a 2D regular grid model with NAND processing functions.

TABLE I
LP SOLUTIONS $\alpha^*(\delta) \in \mathbb{R}^{27}$ FOR $r = 4$

	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.1$
(000)	0	0	0	0
(001)	0	0	0	0
$(00u)$	0.0000	0.0171	0.1151	0.1368
(010)	0	0	0	0
(011)	0	0	0	0
$(01u)$	1.9908	1.9223	1.3064	0.7631
$(0u0)$	0.0040	0.0421	0.1477	0.1599
$(0u1)$	1.9904	1.9164	1.2696	0.7283
$(0uu)$	1.9864	1.9528	1.3672	0.8458
(100)	0	0	0	0
(101)	0	0	0	0
$(10u)$	0.0020	0.0358	0.1516	0.1589
(110)	0	0	0	0
(111)	0	0	0	0
$(11u)$	0.9884	1.0010	0.7999	0.5322
$(1u0)$	1.0047	1.3343	1.3745	0.7787
$(1u1)$	1.9958	1.9526	1.4142	0.9025
$(1uu)$	1.9948	1.9610	1.4430	0.9489
$(u00)$	1.0000	1.0000	1.0000	1.0000
$(u01)$	1.0020	1.0057	1.0000	1.0000
$(u0u)$	1.0000	1.0000	1.0000	1.0000
$(u10)$	2.0027	2.2894	2.0237	1.1221
$(u11)$	1.0058	1.3689	1.5929	1.0445
$(u1u)$	1.9881	1.9396	1.4229	1.0000
$(uu0)$	1.0000	1.3358	1.4511	1.0000
$(uu1)$	1.9910	1.9535	1.4682	1.0000
(uuu)	1.9897	1.9499	1.4577	1.0000

Theorem 3 (NAND 2D Regular Grid)**.** *For any $\delta \in \left(0, \frac{1}{2}\right)$, suppose there exists $r \geq 2$, and a cyclic potential function $w_\delta : \mathcal{Y}^* \to \mathbb{R}$ whose formal sum is constructed with strings (with non-zero coefficients) of length at most $r - 1$, such that:*

1) *w_δ is u-only,*
2) *$w_\delta \succeq_{\mathsf{c}} \mathcal{E}(w_\delta)$,*
3) *$w_\delta \succeq_{\mathsf{c}} \{u\}$,*

where $\succeq_{\mathsf{c}}$ is defined using r (as in Definition 1), and $\{u\} : \mathcal{Y}^* \to \mathbb{R}$ is the cyclic potential function consisting of a single string $(u) \in \mathcal{Y}$. Then, reconstruction is impossible on the 2D regular grid model with NAND processing functions in the sense of (3).

Theorem 3 is systematically established in [16, Section V]. Intuitively, the first two conditions in the theorem statement ensure that $\{w_\delta(Y_k) : k \in \mathbb{N}\}$ is a *supermartingale*, and the third condition ensures that this supermartingale upper bounds the total number of uncoupled grid variables at successive levels. Then, using a martingale convergence argument along with some careful analysis of the stochastic dynamics of the coupled 2D regular grid, we can deduce that the number of uncoupled grid variables converges to zero almost surely. Akin to section II, this implies that reconstruction is impossible.

Furthermore, we demonstrate in [16, Section V-E] that for fixed values of δ and r, the problem of finding w_δ satisfying the conditions of Theorem 3 can be posed as a *linear program* (LP). We present some representative MATLAB simulation results in Table I that numerically solve such LPs to construct the cyclic potential functions w_δ^* of Theorem 3 when $r = 4$. Specifically, for different values of δ, Table I displays vectors of coefficients $\alpha^*(\delta) \in \mathbb{R}^{27}$ (rounded to 4 decimal places, and indexed by $\mathcal{Y}^3$) that define w_δ^* via formal sums over all strings in $\mathcal{Y}^3$; the formal sum is constructed by scaling each index in $\mathcal{Y}^3$ with the corresponding value in $\alpha^*(\delta)$. For example, the second column of Table I states that when $\delta = 0.01$, the cyclic potential function, $w_\delta^* = 0.0171\{00u\} + 1.9223\{01u\} + \cdots + 1.9535\{uu1\} + 1.9499\{uuu\}$, satisfies the conditions of Theorem 3 with $r = 4$.

We close this section with three further remarks. Firstly, one can verify the first and third conditions of Theorem 3 from Table I by reading all entries corresponding to indices with either no u's or beginning with a u. Secondly, Table I only presents a small subset of our simulation results for brevity; we have solved LPs for numerous values of $\delta \in \left(0, \frac{1}{2}\right)$. Thirdly, it is worth mentioning that reconstruction is impossible for all choices of processing functions (which may vary between vertices) when $\delta > 0.146446\ldots$ [20, Lemma 2]. (So, there is no need to present LP results for δ larger than this threshold.) Hence, our simulations provide strong computational evidence that reconstruction is impossible on the 2D regular grid with NAND processing functions for all $\delta \in \left(0, \frac{1}{2}\right)$. We again refer readers to [16, Sections II-B and V] for a detailed exposition of the ideas discussed in this section, which can also be extended to prove impossibility of reconstruction on 2D regular grids with other processing functions and ergodicity of 1D PCA.

V. CONCLUSION

To conclude, we emphasize that Theorems 1, 2, and 3 (along with our simulations) make substantial progress towards our conjecture in section I-A that reconstruction is impossible on 2D regular grids for all 16 possible common 2-input Boolean processing functions. To see this, notice that reconstruction is impossible for the two constant functions that always output 0 or 1, because only vertices at the boundary can carry useful information, but these vertices form ergodic Markov chains. The four 2-input processing functions that actually have one input, namely, the identity maps and NOT gates for the first or second input, yield 2D regular grids that are trees. These trees have branching number 1, and hence, reconstruction is impossible [4] (see section I). The six remaining symmetric processing functions are AND, NAND, OR, NOR, XOR, and XNOR. Due to the symmetry of 0's and 1's in our model, we only need to prove the impossibility of reconstruction for three cases: AND, XOR, and NAND. This leaves four asymmetric 2-input processing functions. Once again, due to the symmetry of 0's and 1's, we only need to consider two of these functions. Moreover, due to the symmetry of the edge configuration of our 2D regular grid construction, it suffices to only consider one of these remaining two functions. For example, we may consider the asymmetric 2-input Boolean function defined by the truth table for the *implication relation*, denoted IMP. Therefore, to prove our conjecture in section I-A, we only have to analyze four nontrivial processing functions: AND, XOR, NAND, and IMP. Clearly, Theorems 1 and 2 address the first two cases and Theorem 3 (and our simulations) partially address the third case. So, the two natural future directions are to completely resolve the third and fourth cases.

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