Distance Preserving Maps and Combinatorial Joint Source-channel Coding for Large Alphabets

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***Abstract*—In this paper we present several results regarding distance preserving maps between nonbinary Hamming spaces and combinatorial (adversarial) joint source-channel coding. In an (α, β)-map from one Hamming space to another, any two sequences that are at least α relative distance apart, are mapped to sequences that are relative distance at least β apart. The motivation to study such maps come from (D, δ)-joint source-channel coding (JSCC) schemes, where any encoded sequence must be recovered within a relative distortion D, even in the presence of δ proportion of adversarial errors.**

We provide bounds on the parameters of both (α, β)-maps and (D, δ)-JSCC for nonbinary alphabets. We also provide constructive schemes for both, that are optimal for many cases.

I. Introduction

The problem of combinatorial joint-source channel coding (JSCC) as an analogue to the usual compression and transmission of data over a stochastic channel was first proposed in [8] and then subsequently studied in [7], [9], [11], [12], [14].

Definition 1 ($(D, \delta)_q$-JSCC). *A pair of mappings $f : \mathbb{F}_q^k \to \mathbb{F}_q^n$ and $g : \mathbb{F}_q^n \to \mathbb{F}_q^k$ are called together a $(D, \delta)_q$-JSCC, if $\forall x \in \mathbb{F}_q^k$ and $\forall z \in \mathbb{F}_q^n : \mathrm{wt}(z) \leq \delta n$,*

$$d_H\left(g(f(x) + z), x\right) \leq Dk,$$

where d_H denotes the Hamming distance.

We note that given an encoding function $f(\cdot)$ of a $(D, \delta)_q$-JSCC it has an optimal decoder $g_f^{\mathrm{opt}}(\cdot)$ defined as follows.

$$g_f^{\mathrm{opt}}(z) = \arg\min_{y \in \mathbb{F}_q^k} \max_{x : d_H(z, f(x)) \leq \delta n} d_H(y, x) \ \forall\, z \in \mathbb{F}_q^n.$$

It is straightforward to see that a serial application of a covering code with covering radius Dk followed by an δn-error-correcting code will result in a $(D, \delta)_q$-JSCC (called the *separation scheme*). In particular, it was shown in [8] that such separation schemes are not optimal and can be outperformed.

A natural simplification of the combinatorial JSCC problem is a problem of constructing distance-preserving maps of Hamming spaces [9], [13].

Definition 2 ($(\alpha, \beta)_q$-maps [13]). *A map $f : \mathbb{F}_q^k \to \mathbb{F}_q^n$ is said to be an $(\alpha, \beta)_q$-map if αk and βn are integers and for all $x, x' \in \mathbb{F}_q^k$ we have*

$$d_H\left(x, x'\right) > \alpha k \implies d_H\left(f(x), f(x')\right) > \beta n.$$

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The question arises on what values of β are feasible for a fixed α under various restrictions on k, n, and q. In the case where $\alpha = 0$, this is the classic question of error-correcting codes.

Remark 1 (Connection between $(\alpha, \beta)_q$-maps and $(D, \delta)_q$-JSCC). Note that an encoding map f of a $(D, \delta)_q$-JSCC is also a $(2D, 2\delta)_q$-map. This follows from the fact that for an encoding map f which is not a $(2D, 2\delta)_q$-map, even the optimal decoder g_f^{opt} cannot ensure Hamming distortion at most Dk.

In previous works, as well as in most of the parts of this paper, only asymptotic regime where $n \to \infty$ and k scales linearly with n is considered. We define $R \triangleq \frac{k}{n}$ and $\rho \triangleq \frac{n}{k}$ as the rate of transmission and bandwidth expansion, respectively.

Bounds on the parameters of $(D, \delta)_2$-JSCC (binary alphabet) were proposed in [8], [14]. In a similar spirit, [13] studied the converse bounds for $(\alpha, \beta)_2$-maps (over $\mathbb{F}_2$). In this paper our main contribution is to find impossibility bounds and achievability results for both combinatorial JSCC and (α, β)-maps over nonbinary, possibly large alphabets.

For $(\alpha, \beta)_q$-maps, the main technique we use to get the achievability result is the diametric theorem of Ahlswede and Khachatrian [2]. In the regime of large alphabets, our results can be leveraged to construct good (α, β)-maps out of good error correcting codes (Section II).

For $(D, \delta)_q$-JSCC our results are fourfold. First, we provide a set of converse results that include a list-decoding bound and an information theoretic converse. We also show some converse results for MDS codes as a JSCC (Section III). Secondly, we outline some simple achievability scheme. In particular, we show that for the case of $\rho \leq 1$ the information theoretic converse is optimal for large enough alphabets (Section IV). Next, we consider a generalized definition of $(D, \delta)_q$-JSCC schemes with different input and channel alphabets - we show that a random graph code construction beats the separation scheme in this case (Section V). Lastly, we show that, for combinations of the two common measures of distortions/distances on the Hamming space, namely erasure and Hamming distortions, separation schemes are optimal (Section VI).

II. Distance preserving maps, i.e., (α, β)-maps

In this section we address the question of finding the best parameters that can be achieved for an $(\alpha, \beta)_q$-map. First, we present a simple converse, and subsequently, we present an achievability scheme employing a result by Ahlswede and Khachatrian [2]. For large enough q this scheme is optimal.

Then we utilize Reed-Solomon codes to construct explicit optimal $(\alpha, \beta)_q$-maps for $q > n$.

A. Converse for $(\alpha, \beta)_q$-maps

Theorem 1. *Let* $q \geq 2$. *Then, for an* $(\alpha, \beta)_q$*-map to exist, we must have*

$$h_q(\alpha) \geq 1 - \min\{\rho R^q_{LP1}(\beta), \rho(1-\beta)\}, \quad (1)$$

where, $h_q(D) = D \log_q \frac{q-1}{D} - (1-D)\log_q(1-D)$ *and,* $R^q_{LP1}(\beta) = h_q\left(\frac{q-1}{q} - \beta\frac{q-2}{q} - \frac{2}{q}\sqrt{\beta(1-\beta)(q-1)}\right)$. *Furthermore, for a sufficiently large field size* q, *we must have*

$$\alpha \geq 1 - \rho + \rho\beta + o_q(1). \quad (2)$$

Proof. Let $A_q(n, d)$ be the maximum size of an error-correcting code of length n and minimum distance d over the alphabet $\mathbb{F}_q$. Let f be an $(\alpha, \beta)_q$-map. Assume that $\mathcal{C} \subseteq \mathbb{F}_q^k$ is a code with relative distance α. Encoding each codeword in $\mathcal{C}$ with the map f, we get a set $f(\mathcal{C}) \subseteq \mathbb{F}_q^n$ that is a code with minimum distance βn. This implies that

$$A_q(k, \alpha k) = |f(\mathcal{C})| \leq A_q(n, \beta n). \quad (3)$$

From the Gilbert-Varshamov (GV) bound [10] we know that, $A_q(k, \alpha k) \geq q^{k(1-h_q(\alpha))+o(k)}$. And, the linear programming bound in [1] ensures that $A_q(n, \beta n) \leq q^{nR^q_{LP1}(\beta)+o(n)}$. Now, the bound in (1) follows by using these in (3). We obtain (2) as we have $h_q(a) = a + o_q(1)\ \forall a \in [0, 1]$. □

B. Achievability scheme for $(\alpha, \beta)_q$-map

We now present an achievability scheme to construct an $(\alpha, \beta)_q$-map. One approach to construct such maps is to first cover $\mathbb{F}_q^k$ with configurations of diameter at most αk; then, pack in $\mathbb{F}_q^n$ as many points (codewords) of pairwise distance more than βn as there are configurations in the cover. Subsequently, map configurations to codewords. To obtain good (α, β) properties, it makes sense to look for configurations that contain a large number of points. A natural choice is to cover with Hamming balls - but they are too small when $q > 2$ and, hence, do not give satisfactory (α, β) properties. To find the shape and cardinality of extremal configurations we rely on the diametric theorem of Ahlswede and Khachatrian [2].

We are interested in large subsets with bounded diameter. The cardinality of the largest such subset is

$$N_q(d, k) = \max\{|\mathcal{A}| : \mathcal{A} \subset \mathbb{F}_q^k \text{ s.t. } \mathrm{diam}(\mathcal{A}) \leq d\},$$

where for a set of vectors $\mathcal{A} \subseteq \mathbb{F}_q^n$, $\mathrm{diam}(\mathcal{A}) = \max_{x,y\in\mathcal{A}} d_H(x, y)$ is its diameter. For $x \in \mathbb{F}_q^k$, define $J(x) = \{j : x_j = 0\}$ and

$$\mathcal{U}_i = \{x \in \mathbb{F}_q^k : |J(x) \cap [1, k-d+2i]| \geq k-d+i\}.$$

Note that each set $\mathcal{U}_i$ can be written as a Cartesian product of some $(k-2d+i)$-dimensional ball of radius i with $\mathbb{F}_q^{d-2i}$. In particular, $\mathcal{U}_0$ is a low dimensional cube $\mathbb{F}_q^d$ inside $\mathbb{F}_q^k$. We are now ready to state the diametric theorem:

Proposition 2 (The diametric theorem [2]). *Let* r *be the largest integer such that*

$$k - d + 2r < \min\left\{k+1, k-d+2\frac{k-d-1}{q-2}\right\}.$$

Then $N_q(k, d) = |\mathcal{U}_r|$.

We will make use of the extremal configurations that appear in the theorem in the covering step mentioned above. We state the achievable parameters in the following result. For large q, this result establishes the tightness of Theorem 1.

Theorem 3. *Fix* $q \geq 2$ *and set*

$$\bar{\rho} := \begin{cases} \frac{\rho(q-2)}{q(1-h_q(1/q))} & q > 2 \\ \rho & q = 2 \end{cases}$$

Then for all $\beta \leq 1 - \frac{1}{q}$ *and* $k_i, n_i \to \infty$ *with* $\frac{n_i}{k_i} \to \rho$, *there exists* (α_i, β_i)*-maps* $f : \mathbb{F}_q^{k_i} \to \mathbb{F}_q^{n_i}$ *with* $(\alpha_i, \beta_i) \to (\alpha, \beta)$ *if*

$$\alpha \geq \max\{\frac{2}{q}, 1 - \bar{\rho} + \bar{\rho}h_q(\beta)\}\} \text{ or } h_q(\frac{\alpha}{2}) \geq 1 - \rho + \rho h_q(\beta). \quad (4)$$

Proof. Let $r(k, d)$ be the integer as in the diametric theorem. We cover $\mathbb{F}_q^k$ with $\mathcal{U}_{r(k,d)}$'s. Write

$$\mathcal{U}_{r(k,d)} = B_{r(k,d)} \times \mathbb{F}_q^{d-2r(k,d)}.$$

Define $t(k, d) = k - d + 2r(k, d)$. Now we can mod out the second factor so that covering $\mathbb{F}_q^k$ with translates of $\mathcal{U}_{r(k,d)}$ reduces to covering $\mathbb{F}_q^{t(k,d)}$ with Hamming balls of radius $r(k, d)$. Define the terms,

$$K(d, X) = \min\{m : \cup_{i=1}^m S_i = X, \mathrm{diam}(S_i) = d\},$$

$$W(r, X) = \min\{m : \cup_{i=1}^m B_i = X, \mathrm{rad}(B_i) = r\}.$$

We further define $w(r, X) \triangleq \min\{m : \exists m\text{-dim. linear code } C \subset X \text{ s.t. } r_{cov}(C) \leq r\}$, where, $r_{cov}(\mathcal{A}) = \max_{y\in\mathbb{F}_q^n} \min_{x\in\mathcal{A}} d_H(x, y)$ is the covering radius of the set $\mathcal{A}$. We have $W(r, \mathbb{F}_q^t) \leq q^{w(r,\mathbb{F}_q^t)} \leq q^{t(1-h_q(r/t))+O(\log t)}$ where the second inequality is from [6]. We can thus bound the number of configurations of diameter d needed to cover $\mathbb{F}_q^k$ as follows

$$K(d, \mathbb{F}_q^k) \leq W(r(k, d), \mathbb{F}_q^{t(k,d)}) \leq q^{w(r(k,d),\mathbb{F}_q^{t(k,d)})}$$
$$\leq q^{t(k,d)(1-h_q(\frac{r(k,d)}{t(k,d)}))+O(\log t(k,d))}$$

Using the GV bound, we can see that (α, β) is achievable asymptotically if

$$t(k, \alpha k)(1 - h_q(\frac{r(k, \alpha k)}{t(k, d)})) \leq n(1 - h_q(\beta))$$

holds as $k \to \infty$. By considering the cases of $\alpha \geq 2/q$ and $\alpha < \frac{2}{q}$ separately (omitting details) we arrive at (4). □

Fig. 1 shows the bounds in (1), (2), and (4) for $q = 3^{10}$.

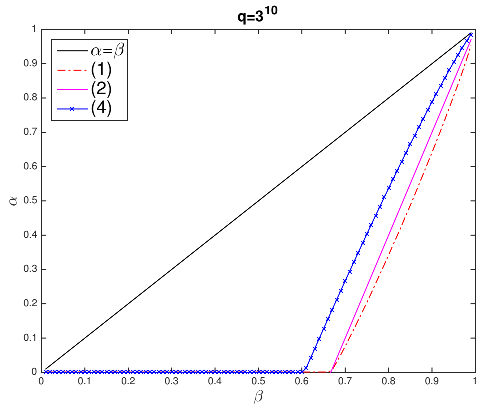

Fig. 1. Converse and achievability schemes for $(\alpha, \beta)_q$-maps.

C. Truncated Reed-Solomon codes

Here we give an explicit family of codes that achieve optimal (α, β)-tradeoffs for $q \geq \rho k$. Set $\bar{\alpha} := 1 - \alpha$ and consider the Reed-Solomon code $f_{RS} : V \to \mathbb{F}_q^{\rho k}$ where V is the subspace of $\mathbb{F}_q^k$ formed by its first $\bar{\alpha}k$ coordinates. Note that f_{RS} is a $(0, 1-\frac{\bar{\alpha}}{\rho})$-map. Now let π_V be the projection map from $\mathbb{F}_q^k$ to V and define the truncated Reed-Solomon (TRS) code $f_{TRS} : \mathbb{F}_q^k \to \mathbb{F}_q^{\rho k}$ as follows: $f_{TRS}(\mathbf{x}) = f_{RS}(\pi_V(\mathbf{x}))$. Any vector $\mathbf{x} \in \mathbb{F}_q^k$ with Hamming weight $\mathrm{wt}(\mathbf{x}) > \alpha k$ gets projected to a non-zero vector in V. Hence $\mathrm{wt}(\mathbf{x}) > \alpha k \implies |f_{TRS}(\mathbf{x})| > (1-\frac{\bar{\alpha}}{\rho})\rho k$, which means that f_{TRS} is a $(\alpha, 1-\frac{\bar{\alpha}}{\rho})$-map. Furthermore, when $q \geq \rho k$, one can check that the bound in (2) is sharp even for finite k. Thus, the TRS parameters are optimal over large enough alphabets.

III. Converses for $(D, \delta)_q$-adversarial JSCC

In this section we obtain the lower bounds on the bandwidth-expansion factor $\rho = \frac{n}{k}$ of a $(D, \delta)_q$-adversarial JSCC. We assume the alphabet size to be large enough for existence of MDS codes (say, $q > n$).

A. List decoding converse

As stated in the Remark 1, the encoding map f of a $(D, \delta)_q$-JSCC is necessarily an $(\alpha = 2D, \beta = 2\delta)_q$ map. Using this fact, the coding converse for the $(\alpha = 2D, \beta = 2\delta)_q$-map provides us the following converse result for $(D, \delta)_q$-JSCC when $\beta = 2\delta < 1$.

Proposition 4. *For $\delta < \frac{1}{2}$, a $(D, \delta)_q$-JSCC must satisfy*

$$\rho = \frac{n}{k} \geq \frac{1-2D}{1-2\delta}, \text{ or } D \geq \frac{2\rho\delta - (\rho - 1)}{2}. \tag{5}$$

We now extend the bound in Proposition 4 for the values of $\delta \in [\frac{1}{2}, 1)$ by using the notion of list decoding. Before presenting the bound, we state the following result which characterizes the maximum size of an (δ, L)-list decodable n-length code $A_{q,L}(n, \delta n)$.

Theorem 5. *Let $R_q(L, \delta) = \frac{1}{n}\log_q A_{q,L}(n, \delta n)$ be the best achievable rate of a code decodable with list size L up to radius δn. Then, we have*

$$R_q(L, \delta) = 1 - \frac{(L+1)\delta}{L} + o_q(1). \tag{6}$$

Proof. The achievability part of this theorem follows from a random choice argument of [5, Sec. 2] - in particular, by taking q to ∞ in there. We omit the details.

For the converse part, we first show that, $A_{q,L}(n, nL/(L+1)) \leq L$. Let us first take any $L+1$ codewords from the code and design a new vector such that, each of the $L+1$ codewords are within distance $nL/(L+1)$ away from this new vector. Indeed, the new vector can be made to agree with any of the $L+1$ codewords in $\frac{n}{L+1}$ coordinates easily. If $A_{q,L}(n, nL/(L+1)) > L$ then there exist a code with $L+1$ codewords that is list decodable with list size at most L up to radius $nL/(L+1)$. But this cannot be true according to the above argument.

Next we will prove $A_{q,L}(n, r) \leq Lq^{n - \frac{L+1}{L}r}$. Let the code $\mathcal{C}$ is (r, L)- list decodable. Let us take a subcode of $\mathcal{C}$ that has fixed first t coordinates. There must exist such a subcode of size $|C|/q^t \leq A_{q,L}(n-t, r)$. Now, we can substitute $t = n - \frac{L+1}{L}r$ in the above to prove the claim. □

We now state the list-decoding bound on a $(D, \delta)_q$-JSCC.

Theorem 6. *Let L be an integer such that $\log L = o(n)$. Then, as $q \to \infty$, a $(D, \delta)_q$-JSCC must have*

$$\rho \geq \frac{1 - \frac{L+1}{L}D}{1 - \frac{L+1}{L}\delta}, \text{ for } \delta \leq \frac{L}{L+1}. \tag{7}$$

Proof. By extending the result from [14, Theorem 6] to nonbinary alphabets, we know that the existence of a $(D, \delta)_q$-JSCC implies that for every L we have,

$$A_{q,L}(k, Dk) \leq LA_{q,L}(n, \delta n). \tag{8}$$

By taking logarithm both side and dividing by n, we obtain the bound in (7). □

B. Information theoretic converse

We now present an information theoretic converse results which works for all values of $\delta \in [0, 1)$. This bound is tighter than both the bounds presented in Section III-A when $n \leq k$. We show in Section IV-A that the information theoretic bound characterizes the exact trade-off for $(D, \delta)_q$-JSCC in the regime of $n \leq k$. The proof follows from a counting argument similar to [8], and we omit it here.

Theorem 7. *For a $(D, k)_q$-JSCC, we must have*

$$\rho = \frac{n}{k} \geq \frac{1-D}{1-\delta} + o(1) \implies D \geq \rho\delta - (\rho - 1). \tag{9}$$

C. MDS codes as $(D, \delta)_q$-JSCC

For large alphabet, MDS codes are optimal when the objective is to recover the source vector with zero distortion against adversarial errors as they achieve the Singleton bound. This begs the question about their applicability as we relax the zero distortion constraint. Here we study a subclass of MDS codes, namely linear systematic codes, and show that they exhibit poor performance as a $(D, \delta)_q$-JSCC. We divide our analysis in two cases: 1) $1 \leq \rho < 2$ and 2) $\rho \geq 2$.

Theorem 8. *Let $\mathcal{C} \subset \mathbb{F}_q^n$ be an linear systematic MDS code with $|\mathcal{C}| = q^k$. Then, the performance of $\mathcal{C}$ as a $(D, \delta)_q$-JSCC satisfies the following.*

1) *For $\rho \in [1, 2)$ and $\delta \geq \frac{\rho-1}{2\rho}$, we have $D \geq \rho\delta$.*
2) *For $\rho \geq 2$ and $\delta \geq \frac{\rho-1}{2\rho}$, we have $D \geq \frac{1}{2}$.*

We omit the proof of this and the subsequent theorem for space constraints.

Note that the requirement of $\delta \geq \frac{\rho-1}{2\rho}$ in the statement of Theorem 8 is necessary as for $\delta < \frac{\rho-1}{2\rho}$, an MDS code can correct all δn errors. For $\delta \geq \frac{\rho-1}{2\rho}$, using a linear systematic code $D = \rho\delta$ can be easily achieved by a simple decoding function where we declare the first k symbols of the received sequence as the source vector.

Finally, we also consider Reed-Solomon (RS) codes with the usual polynomial encoding approach and conclude it to be a poor adversarial JSCC. Note that these codes are non-systematic MDS codes.

Theorem 9. *Let $\mathcal{C}$ be an RS code where codewords are n evaluations of polynomials of degree less than k over $\mathbb{F}_q$ (on n points of $\mathbb{F}_q$) with k message symbols constituting the k coefficients of the polynomials. Then, for sufficiently large field size q and $\delta \geq \frac{\rho-1}{2\rho}$, the $\mathcal{C}$ gives a $(D, \delta)_q$-JSCC with $D \geq \frac{1}{2}$.*

IV. Achievability Schemes for $(D, \delta)_q$-JSCC

In this section we present various $(D, \delta)_q$-JSCC schemes and compare the performance of these schemes with the converse results presented in Section III.

A. Optimality of truncate and transmit scheme for $k \geq n$

When $k \geq n$, consider the scheme of throwing out the last $k - n$ coordinates of a message vector to obtain the corresponding codeword. While decoding, we just append $k - n$ zeros at the end of the received vector. Clearly, the maximum possible distortion that can be introduced in this process is $Dk = \delta n + (k - n)$, which attains the information theoretic bound of (9). Hence, the bound of (9) is tight for the case of $k \geq n$. This prompts us to only focus on devising $(D, \delta)_q$-JSCC schemes for the case $\rho > 1$ going forward.

B. Separation scheme

For the separation scheme, we first compress a k-length message vector within Dk distortion (for large alphabets this can be achieved is by truncating Dk coordinates), and then encode the truncated vector using an n-length MDS code with the minimum distance $2\delta n + 1$. This scheme ensures correct recovery of the truncated vector against at most δn adversarial errors. The minimum distance requirement implies that we have $\frac{n}{k} = \frac{1-D}{1-2\delta}$, which gives us the following.

Proposition 10. *For $\delta < \frac{1}{2}$, the separation scheme achieves,*

$$D = 2\rho\delta - (\rho - 1). \tag{10}$$

C. Repetition scheme with majority decoding

For an odd integer $\rho > 1$, consider a ρ-repetition scheme where we transmit ρ copies of each source symbol. Given a received vector, we decode each of the k source symbols to be that symbol in $\mathbb{F}_q$ which appears at the most number of coordinates among the ρ coordinates corresponding to the repetition of the source symbol.

Proposition 11. *Let ρ be an odd integer. For $\delta < \frac{\rho+1}{2\rho}$, ρ-repetition scheme with majority decoding achieves distortion*

$$D = 2\delta\rho/(\rho + 1). \tag{11}$$

V. JSCC with different source-channel alphabets

So far we have considered adversarial JSCC where both the source symbols and the transmitted (channel) symbols belong to the same alphabet $\mathbb{F}_q$. However, depending on the setting, it is possible and/or desirable to have the channel alphabet different from the source alphabet. In this section we comment on such adversarial JSCC. Let $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$ denote the source and the channel alphabet. Similarly to Definition 1, an encoding map $f : \mathbb{F}_{q_1}^k \to \mathbb{F}_{q_2}^n$ along with a decoding map $g(\cdot)$ defines a $(D, \delta)_{q_1,q_2}$-JSCC if $\forall \mathbf{x} \in \mathbb{F}_{q_1}^k$ and $\forall \mathbf{z} \in \mathbb{F}_{q_2}^n : \mathrm{wt}(\mathbf{z}) \leq \delta n$, we have

$$d_H(g(f(\mathbf{x}) + \mathbf{e}), \mathbf{x}) \leq Dk. \tag{12}$$

In this setting, we define the rate R and the bandwidth expansion factor ρ of the code as follows.

$$R \triangleq \frac{k \log q_1}{n \log q_2}; \quad \rho \triangleq \frac{1}{R} = \frac{n \log q_2}{k \log q_1}. \tag{13}$$

Note that with these modified definitions of R and ρ, for large enough q_1 and q_2, the same converse results presented in Section III continue to hold. Due to lack of space, we only provide a brief sketch here. Note that the coding converse is this setting follows as the existence of an encoding map for a $(D, \delta)_{q_1,q_2}$-JSCC would imply $A_{q_1}(k, 2Dk + 1) \leq A_{q_2}(n, 2\delta n + 1)$, and $A_{q_1,L}(k, Dk) \leq LA_{q_2,L}(n, \delta n)$.

Next, we describe a $(D, \delta)_{q_1,q_2}$-JSCC that has the channel alphabet different from the source alphabet.

A. $(D, \delta)_{q,q^\rho}$-JSCC using bi-regular bipartite graphs

Let ρ be an odd integer. Consider a bi-regular bipartite graph with $k = n$ left vertices, n right vertices and both the left and the right degree equal ρ. We associate each left vertex with a source symbol. Similarly, each right vertex corresponds to a channel symbol. The encoding map consists of the following two steps:

1) Repeat each of the $k = n$ source symbols from $\mathbb{F}_q$ ρ times and assign these ρ copies to ρ edges incident on the left vertex associated with the source symbol.
2) For each right vertex, collect the ρ symbols assigned to ρ edges incident on it and form a symbol belonging to the channel alphabet $\mathbb{F}_{q_2} = \mathbb{F}_{q^\rho}$.

This scheme corresponds to a special case of Alon-Luby transform [3] where we use ρ-repetition code as a local code. For the decoding, we perform the following two steps.

1) For each received symbol, obtain ρ symbol in $\mathbb{F}_q$ and assign these symbols the d edges incident on the corresponding right vertex.
2) For each left vertex, collect the ρ symbols associated to the ρ edges incident on it and perform majority decoding on these ρ symbols.

In order to fully specify the scheme, we need to select a suitable bipartite graph. Towards this, we consider a random ensemble of bipartite graph generated using a random permutations over the set $[\rho n] = \{1, 2, \ldots, \rho n\}$. Here we skip some details of the ensemble. In the following result we show that this ensemble has a bipartite graph which ensures good performance for the $(D, \delta)_{q,q^\rho}$-JSCC described above.

Theorem 12 (Distortion analysis of d-repetition scheme)**.** *Assume that we have*

$$h(D) - \delta d \log x + D \log \sum_{i=\frac{d+1}{2}}^{d} \binom{d}{i} x^i + (1 - D) \log \sum_{i=0}^{\frac{d-1}{2}} \binom{d}{i} x^i < (d - 1)h(\delta), \quad (14)$$

where $x > 0$ is obtained from the following equation.

$$\sum_{i=0}^{\frac{d-1}{2}} \sum_{j=\frac{d+1}{2}}^{d} \binom{d}{i}\binom{d}{j} [\delta d - jD - i(1 - D)] x^{i+j-\frac{d+1}{2}} = 0.$$

Then, for large enough n, there exists a permutation $\pi : [\rho n] \to [\rho n]$ such that the ρ-replication scheme defined by the graph $\mathcal{G}_\pi$ gives us a $(D, \delta)_{q,q^\rho}$-adversarial JSCC.

Proof. The proof follows by adapting the analysis of the proof of Theorem 4.2 in [4]. We omit the details. □

In Fig. 2 we illustrate the performance of this scheme along with the list decoding bounds and other achievability schemes for (D, δ)-JSCC when $\rho = 5$ and $q = 2$.

VI. Combinatorial JSCC for Erasures

Just like Hamming errors and Hamming distance as a measure of distortion, an adversarial erasure channel and/or erasure distortion functions are equally popular and useful models. Recall that the erasure distortion metric $d_E : \mathbb{F}_q \cup \{\star\} \times \mathbb{F}_q \cup \{\star\} \to \{0, 1, \infty\}$ is defined as follows.

$$d_E(x, x) = 0, d_E(x, \star) = 1 \text{ and } d_E(x, y) = \infty, \quad (15)$$

where x and y are two distinct element of the alphabet $\mathbb{F}_q$ and $\star$ corresponds to the erased symbol.

So far in this paper we have only considered the case where the adversary introduces errors in the channel and the measure of distortion is Hamming distance. Table I below summarizes the bounds for all remaining combination of Hamming and erasure measures for large alphabet.

We also note that a separation scheme, that first optimally compress a k-length sequence to endure Dk distortion (in respective measures; for $q > n$ we can just discard Dk coordinates), and then uses an MDS code of required distance, is optimal for all three of the cases. We omit the proof of this claim for space constraints.

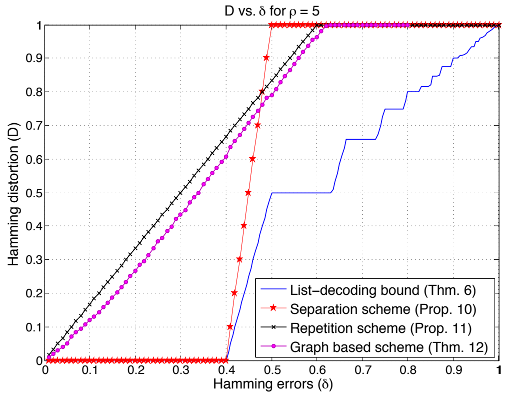

Fig. 2. Converse and achievability schemes for (D, δ)-JSCC.

TABLE I
THE OPTIMUM BANDWIDTH EXPANSION FOR DIFFERENT COMBINATIONS OF ERASURE AND ERRORS

Channel Model	Distortion Measure	Optimum ρ
Hamming Errors (δn)	Erasures (Dk)	$\frac{1-D}{1-2\delta}$
Erasures (δn)	Erasures (Dk)	$\frac{1-D}{1-\delta}$
Erasures (δn)	Hamming Distortion (Dk)	$\frac{1-h_q(D)}{1-\delta}$

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