Upper bound on list-decoding radius of binary codes

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Abstract—Consider the problem of packing Hamming balls of a given relative radius subject to the constraint that they cover any point of the ambient Hamming space with multiplicity at most \( L \). For odd \( L \geq 3 \) an asymptotic upper bound on the rate of any such packing is proven. The resulting bound improves the best known bound (due to Blinovsky’1986) for rates below a certain threshold. The method is a superposition of the linear-programming idea of Ashikhmin, Barg and Litsyn (that was used previously to improve the estimates of Blinovsky for \( L = 2 \)) and a Ramsey-theoretic technique of Blinovsky. As an application it is shown that for all odd \( L \) the slope of the rate-radius tradeoff is zero at zero rate.

Index Terms—Combinatorial coding theory, list-decoding, converse bounds

I. MAIN RESULT AND DISCUSSION

One of the most well-studied problems in information theory asks to find the maximal rate at which codewords can be packed in binary space with a given minimum distance between codewords. Operationally, this (still unknown) rate gives the capacity of the binary input-output channel subject to adversarial noise of a given level. A natural generalization was considered by Elias and Wozencraft [1], [2], who allowed the decoder to output a list of size \( L \). In this paper we provide improved upper bounds on the latter question.

Our interest in bounding the asymptotic tradeoff for the list-decoding problem is motivated by our study of fundamental limits of joint source-channel communication [3]. The best known converse bound for that problem – a straightforward extension of [3, Theorem 7] to lists of size \( L \) – reduces to bounding rate for the list-decoding problem, cf. [4, Theorem 6].

We proceed to formal definitions and brief overview of known results. For a binary code \( C \subseteq \mathbb{F}_2^n \) we define its list-size \( L \) decoding radius as

\[
\tau_L(C) \triangleq \frac{1}{n} \max \{ r : \exists x \in \mathbb{F}_2^n : |C \cap \{ x + B^n_r \}| \leq L \},
\]

where Hamming ball \( B_r^n \) and Hamming sphere \( S^n_r \) are defined as

\[
B_r^n \triangleq \{ x \in \mathbb{F}_2^n : |x| \leq r \},
\]

\[
S_r^n \triangleq \{ x \in \mathbb{F}_2^n : |x| = r \}.
\]

with \( |x| = |\{ i : x_i = 1 \}| \) denoting the Hamming weight of \( x \). Alternatively, we may define \( \tau_L \) as follows:

\[
\tau_L(C) = \frac{1}{n} \left( \min \left\{ \text{rad}(S) : S \in \binom{C}{L+1} \right\} - 1 \right),
\]

where \( \text{rad}(S) \) denotes radius of the smallest ball containing \( S \) (known as Chebyshev radius):

\[
\text{rad}(S) \triangleq \min_{y \in \mathbb{F}_2^n} \max_{x \in S} |y - x|.
\]

The asymptotic tradeoff between rate and list-decoding radius \( \tau_L \) is defined as usual:

\[
\tau^*_L(R) \triangleq \limsup_{n \to \infty} \max_{C : |C| \geq 2^n} \frac{\tau_L(C)}{\log |C|}
\]

\[
R^*_L(\tau) \triangleq \limsup_{n \to \infty} \min_{C : \tau_L(C) \geq \tau n} \frac{1}{n} \log |C|
\]

The best known upper (converse) bounds on this tradeoff are as follows:

- **List size \( L = 1 \):** The best bound to date was found by McEliece, Rodemich, Rumsey and Welch [5]:

\[
R^*_L(\tau) \leq R_{LP2}(2\tau),
\]

\[
R_{LP2}(\delta) \triangleq \min \log 2 - h(\alpha) + h(\beta),
\]

where \( h(x) = -x \log(1-x) - (1-x) \log(1-x) \) and minimum is taken over all \( 0 \leq \beta \leq \alpha \leq 1/2 \) satisfying

\[
\frac{2(\alpha - 1) - \beta(1 - \beta)}{1 + 2\sqrt{\beta(1-\beta)}} \leq \delta.
\]

For rates \( R < 0.305 \) this bound coincides with the simpler bound:

\[
\tau^*_L(R) \leq \frac{1}{2} \delta_{LP1}(R),
\]

\[
\delta_{LP1}(R) \triangleq \frac{1}{2} - \sqrt{\beta(1-\beta)},
\]

\[
R = \log 2 - h(\beta), \quad \beta \in [0, 1/2]
\]

- **List size \( L = 2 \):** The bound found by Ashikhmin, Barg and Litsyn [6] is given as

\[
R^*_L(\tau) \leq \log 2 - h(2\tau) + R_{up}(2\tau, 2\tau),
\]

where \( R_{up}(\delta, \alpha) \) is the best known upper bound on rate of codes with minimal distance \( \delta n \) constrained to live on Hamming spheres \( S^n_{con} \). The expression for \( R_{up}(\delta, \alpha) \) can be obtained by using the linear programming bound

\[1\binom{C}{j}\] denotes the set of all subsets of \( C \) of size \( j \).

\[2\] This result follows from optimizing [6, Corollary 4]. It is slightly stronger than what is given in [6, Corollary 5].
from [5] and applying Levenshtein’s monotonicity, cf. [7, Lemma 4.2(6)]. The resulting expression is

$$R^*(R) \leq \begin{cases} R_{L,R}[2(2\tau)], & \tau \leq \tau_0 \\ \log 2 - h(2\tau) + h(u(\tau)), & \tau > \tau_0, \end{cases}$$  \tag{10}

where \(\tau_0 \approx 0.1093\) and

$$u(\tau) = \frac{1}{2} - \sqrt{\frac{1}{4} - (\sqrt{\tau - 3\tau^2} - \tau)^2}$$

(cf. [7, (9)]).

For list sizes \(L \geq 3\): The original bound of Blinovsky [8] appears to be the best (before this work):

$$\tau^*_L(R) \leq \frac{\sum_{i=1}^{[L/2]} \frac{2i-2}{2i-1}(\lambda(1-\lambda))^i}{R - 1 - h(\lambda)}, \lambda \in (0,1/2]$$ \tag{11}

$$R = 1 - h(\lambda)$$ \quad \tag{12}

Note that [8] also gives a non-constructive lower bound on \(\tau^*_L(R)(R)\). Results on list-decoding over non-binary alphabets are also known, see [9], [10].

In this paper we improve the bound of Blinovsky for lists of odd size and rates below a certain threshold. To that end we will mix the ideas of Ashikhmin, Barg and Litstyn (namely, extraction of a large spectrum component from the code) and those of Blinovsky (namely, a Ramsey-theoretic reduction to study of symmetric subcodes).

To present our main result, we need to define exponent of Krawtchouk polynomial \(K_{\beta}(\xi n) = \exp\{nE_{\beta}(\xi) + o(n)\}\).

For \(\xi \in (1/2 - \sqrt{\beta(1-\beta)}, 1/2 + \sqrt{\beta(1-\beta)})\) the value of \(E_{\beta}(\xi)\) was found in [11]. Here we give it in the following parametric form, cf. [12] or [13, Lemma 4]:

$$E_{\beta}(\xi) = \xi \log(1-\omega) + (1-\xi) \log(1+\omega) - \beta \log \omega$$ \tag{13}

$$\xi = \frac{1}{2}(1-(1-\beta)\omega - \beta\omega^{-1})$$ \tag{14}

where

$$\omega \in \left[ \frac{\beta}{1-\beta}, \sqrt{\frac{\beta}{1-\beta}} \right].$$

Our main result is the following:

**Theorem 1.** Fix list size \(L \geq 2\), rate \(R\) and an arbitrary \(\beta \in (0,1/2)\) with \(h(\beta) \leq R\). Then any sequence of codes \(C_n \subseteq \{0,1\}^n\) of rate \(R\) satisfies

$$\limsup_{n \to \infty} \tau_L(C_n) \leq \max_{\xi_0} \xi_0 g_{j} \left( 1 - \xi_0 \frac{\xi_1}{2\xi_0} \right) + (1-\xi_0) g_{j} \left( \frac{\xi_1}{2(1-\xi_0)} \right),$$ \tag{15}

where maximization is over \(\xi_0\) satisfying

$$0 \leq \xi_0 \leq \frac{1}{2} - \sqrt{(1-\beta)}$$ \tag{16}

and \(j\) ranging over \(\{0,1,3, \ldots, 2k+1, \ldots, L\}\) if \(L\) is odd and over \(\{0,2,3, \ldots, 2k, \ldots, L\}\) if \(L\) is even. Quantity \(\xi_1 = \xi_1(\xi_0, \delta, R)\) is a unique solution of

$$R + h(\beta) - 2E_{\beta}(\xi_0) = h(\xi_0) - \xi_0 h \left( \frac{\xi_1}{2\xi_0} \right) - (1-\xi_0) h \left( \frac{\xi_1}{2(1-\xi_0)} \right),$$ \tag{17}

on the interval \([0,2\xi_0(1-\xi_0)]\) and functions \(g_{j}(\nu)\) are defined as

$$g_{j}(\nu) = \frac{L \nu - \nu \left[ W - j - \nu \right]}{L + j} = \frac{W}{L + j}, \quad W \sim \text{Bino}(L, \nu)$$ \tag{18}

As usual with bounds of this type, cf. [14], it appears that taking \(h(\beta) = R\) can be done without loss. Under such choice, our bound outperforms Blinovsky’s for all odd \(L\) and all rates small enough (see Corollary 3 below). The bound for \(L = 3\) is compared in Fig. 1 with the result of Blinovsky numerically. For larger odd \(L\), the comparison is similar, but the range of rates where our bound outperforms Blinovsky’s becomes smaller, see Table I.

**Table I**

<table>
<thead>
<tr>
<th>List size (L)</th>
<th>Range of rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L = 3)</td>
<td>(0 &lt; R \leq 0.361)</td>
</tr>
<tr>
<td>(L = 5)</td>
<td>(0 &lt; R \leq 0.248)</td>
</tr>
<tr>
<td>(L = 7)</td>
<td>(0 &lt; R \leq 0.184)</td>
</tr>
<tr>
<td>(L = 9)</td>
<td>(0 &lt; R \leq 0.144)</td>
</tr>
<tr>
<td>(L = 11)</td>
<td>(0 &lt; R \leq 0.108)</td>
</tr>
</tbody>
</table>

4 Notice that proofs of each of the two Corollaries below contain a different relaxation of the bound (15), which may appear useful separately.
Corollary 3. Fix arbitrary odd $L \geq 3$. There exists $R_0 = R_0(L) > 0$ such that for all rates $R < R_0$ we have
\[
\tau^*_L(R) \leq g_1(\delta_{LP1}(R)),
\]
where $g_1(\cdot)$ is a degree-$L$ polynomial defined in (18). In particular,
\[
\frac{d}{d\tau} \bigg|_{\tau = \tau^*_L(0)} R^*_L(\tau) = 0,
\]
where the zero-rate radius is $\tau^*_L(0) = \frac{1}{2} - 2^{-L-1} \left( \frac{L}{2^L} \right)$.

We close our discussion with some additional remarks:

1) The bound in Theorem 1 can be slightly improved by replacing $\delta_{LP1}(R)$, that appears in the right-hand side of (16), with a better bound, a so-called second linear-programming bound $\delta_{LP2}(R)$ from [5]. This would enforce the usage of the more advanced estimate of Litsyn [16, Theorem 5] and complicate analysis significantly. Notice that $\delta_{LP2}(R) \neq \delta_{LP1}(R)$ only for rates $R \geq 0.305$. If we focus attention only on rates where new bound is better than Blinovsky’s, such a strengthening only affects the case of $L = 3$ and results in a rather minor improvement (for example, for rate $R = 0.33$ the improvement is $\approx 3 \cdot 10^{-5}$).

2) For even $L$ it appears that $h(\beta) = R$ is no longer optimal. However, the resulting bound does not appear to improve upon Blinovsky’s.

3) When $L$ is large (e.g. 35) the maximum in (15) is not always attained by either $j = 1$ or $\xi_0 = \delta_{LP1}(R)$. It is not clear whether such anomalies only happen in the region of rates where our bound is inferior to Blinovsky’s.

4) The result of Corollary 3 follows by weakening (15) to
\[
\limsup_{n \to \infty} \tau_L(C_n) \leq \max_{j \xi_0} g_j(\xi_0) = \max_j g_j(\delta_{LP1}(R)).
\]
The $R < R_0(L)$ condition is only used to show that the maximum is attained at $j = 1$.

II. Proofs

Several key Lemmas are omitted for space constraints, those can be found in [15].

A. Proof of Theorem 1

Consider an arbitrary sequence of codes $C_n$ of rate $R$. As in [6] we start by using Delsarte’s linear programming to select a large component of the distance distribution of the code. Namely, we apply result of Kalai and Linial [11, Proposition 3.2]: For every $\beta$ with $h(\beta) \leq R$ there exists a sequence $\epsilon_n \to 0$ such that for every code $C$ of rate $R$ there is a $\xi_0$ satisfying (16) such that
\[
A_{\xi_0}(C) \triangleq \frac{1}{|C|} \sum_{x,x' \in C} 1\{ |x - x'| = \xi_0 \} 
\geq \exp\{n(R + h(\beta) - 2E_\beta(\xi_0) + \epsilon_n)\}.
\]

Without loss of generality (by compactness of the interval $[0, 1/2 - \sqrt{\beta(1 - \beta)}]$ and passing to a proper subsequence of codes $C_{n_j}$) we may assume that $\xi_0$ selected in (25) is the same for all blocklengths $n$. Then there is a sequence of subcodes $C'_n$ of asymptotic rate
\[
R' \geq R + h(\beta) - 2E_\beta(\xi_0)
\]
such that each $C'_n$ is situated on a sphere $c_0 + S_{\xi_0}$ surrounding another codeword $c_0 \in C$. Our key geometric result is: If there are too many codewords on a sphere $c_0 + S_{\xi_0}$ then it is possible to find $L$ of them that are inculdable in a small ball that also contains $c_0$. Precisely, we have:

Lemma 4. Fix $\xi_0 \in (0, 1)$ and positive integer $L$. There exist a sequence $\epsilon_n \to 0$ such that for any code $C'_n \subset S_{\xi_0}^n$ of rate $R' > 0$ there exist $L$ codewords $c_1, \ldots, c_L \in C'_n$ such that
\[
\frac{1}{n} \rho(0, c_1, \ldots, c_L) \leq \theta(\xi_0, R', L) + \epsilon_n,
\]
where
\[
\theta(\xi_0, R', L) \triangleq \max_j \theta_j(\xi_0, R', L)
\]
\[
\theta_j(\xi_0, R', L) \triangleq \xi_0 g_j \left( 1 - \frac{\xi_1}{2\xi_0} \right) + (1 - \xi_0) g_j \left( \frac{\xi_1}{2(1 - \xi_0)} \right),
\]
with $\xi_1 = \xi_1(\xi_0)$ found as unique solution on interval $[0, 2\xi_0(1 - \xi_0)]$ of
\[
R' = h(\xi_0) - \xi_0 h \left( \frac{\xi_1}{2\xi_0} \right) - (1 - \xi_0) h \left( \frac{\xi_1}{2(1 - \xi_0)} \right),
\]
functions $g_j$ are defined in (18) and $j$ in maximization (27) ranging over the same set as in Theorem 1. Equipped with Lemma 4 we immediately conclude that
\[
\limsup_{n \to \infty} \tau_L(C_n) \leq \max_{\xi_0 \in [0, \beta]} \theta(\xi_0, R + h(\beta) - 2E_\beta(\xi_0), L).
\]
Clearly, (30) coincides with (15). So it suffices to prove Lemma 4.
B. Proof of Lemma 4

Let $T_L$ be the $(2^L - 1)$-dimensional space of probability distributions on $F^L_2$. If $T \in T_L$ then we have

$$T = (t_v, v \in F^L_2), \quad t_v \geq 0, \quad \sum_v t_v = 1.$$ 

We define distance on $T_L$ to be the $L_{\infty}$ one:

$$\|T - T'\| \triangleq \max_{v \in F^L_2} |t_v - t'_v|.$$ 

Permutation group $S_L$ acts naturally on $F^L_2$ and this action descends to probability distributions $T_L$. We will say that $T$ is symmetric if

$$T = \sigma(T) \iff t_v = t_{\sigma(v)} \quad \forall v \in F^L_2$$

for any permutation $\sigma : [L] \to [L]$. Note that symmetric $T$ is completely specified by $L + 1$ numbers (weights of Hamming spheres in $F^L_2$):

$$\sum_{v:|v|=j} t_v, \quad j = 0, \ldots, L.$$ 

Next, fix some total ordering of $F^L_2$ (for example, lexicographic). Given a subset $S \subset F^L_2$ we will say that $S$ is given in ordered form if $S = \{x_1, \ldots, x_{|S|}\}$ and $x_1 < x_2 \cdots < x_{|S|}$ under the fixed ordering on $F^L_2$. For any subset of codewords $S = \{x_1, \ldots, x_L\}$ given in ordered form we define its joint type $T(S)$ as an element of $T_L$ with

$$t_v \triangleq \frac{1}{n} |\{j : x_1(j) = v_1, \ldots, x_L(j) = v_j\}|,$$

where here and below $y(j)$ denotes the $j$-th coordinate of binary vector $y \in F^n_2$. In this way every subset $S$ is associated to an element of $T_L$. Note that $T(S)$ is symmetric if and only if the $L \times n$ binary matrix representing $S$ (by combining row-vectors $x_j$) has the property that the number of columns equal to $[1, 0, \ldots, 0]^T$ is the same as the number of columns $[0, 1, \ldots, 0]^T$ etc. For any code $C \subset F^L_2$ we define its average joint type:

$$\overline{T}_L(C) = \frac{1}{L! \cdot \binom{|C|}{L}} \sum_{\sigma} \sum_{S \in \binom{[L]}{L}} \sigma(T(S)).$$

Evidently, $\overline{T}_L(C)$ is symmetric.

Our proof crucially depends on a (slight extension of the) brilliant idea of Blinovsky [8]:

**Lemma 5.** For every $L \geq 1$, $K \geq L$ and $\delta > 0$ there exist a constant $K_1 = K_1(L, K, \delta)$ such that for all $n \geq 1$ and all codes $C \subset F^n_2$ of size $|C| \geq K_1$ there exists a subcode $C' \subset C$ of size at least $K$ such that for any $S \in \binom{[L]}{L}$ we have

$$\|T(S) - \overline{T}_L(C')\| \leq \delta. \quad (31)$$

**Remark 1.** Note that if $S' \subset S$ then every element of $T(S')$ is a sum of $2^L$ elements of $T(S)$. Hence, joint types $T(S')$ are approximately symmetric also for smaller subsets $|S'| < L$.

**Proof.** See [15].

Before proceeding further we need to define the concept of an average radius (or a moment of inertia):

$$\overline{\text{rad}}(x_1, \ldots, x_m) \triangleq \min_y \frac{1}{m} \sum_{i=1}^m |x_i - y|.$$ 

Note that the minimizing $y$ can be computed via a per-coordinate majority vote (with arbitrary tie-breaking for even $m$). Consider now an arbitrary subset $S = \{c_1, \ldots, c_L\}$ and define for each $j \geq 0$ the following functions

$$h_j(S) \triangleq \frac{1}{n} \overline{\text{rad}}(0, \ldots, 0, c_1, \ldots, c_L).$$

It is easy to find an expression for $h_j(S)$ in terms of the joint-type of $S$:

$$h_j(S) = \frac{1}{L + j} \left( \mathbb{E}[W] - \mathbb{E}[|2W - L - j|^+] \right) \quad (32)$$

$$\mathbb{P}[W = w] = \sum_{v : |v| = w} t_v, \quad (33)$$

where $t_v$ are components of the joint-type $T(S) = \{t_v, v \in F^L_2\}$. To check (32) simply observe that if one arranges $L$ codewords of $S$ in an $L \times n$ matrix and also adds $j$ rows of zeros, then computation of $h_j(S)$ can be done per-column: each column of weight $w$ contributes

$$\min(w, \frac{L + j - w}{L + j}) = \frac{L + j - w}{L + j}$$

to the sum. In view of expression (32) we will abuse notation and write

$$h_j(T(S)) \triangleq h_j(S).$$

We now observe that for symmetric codes satisfying (31) average-radii $h_j(S)$ in fact determine the regular radius:

**Lemma 6.** Consider an arbitrary code $C$ satisfying conclusion (31) of Lemma 5. Then for any subset $S = \{c_1, \ldots, c_L\} \subset C$ we have

$$\overline{\text{rad}}(0, c_1, \ldots, c_L) - n \cdot \max_j h_j(\overline{T}_L(C)) \leq 2^L (1 + \delta n), \quad (34)$$

where $j$ in maximization (34) ranges over $\{0, 1, 3, \ldots, 2k + 1, \ldots, L\}$ if $L$ is odd and over $\{0, 2, \ldots, 2k, \ldots, L\}$ if $L$ is even.

**Proof.** See [15].

**Lemma 7.** There exist constants $C_1, C_2$ depending only on $L$ such that for any $C \subset F^n_2$ the joint-type $\overline{T}_L(C)$ is approximately a mixture of product Bernoulli distributions$^4$, namely:

$$\left\| \overline{T}_L(C) - \frac{1}{n} \sum_{i=1}^n \text{Bern}^{\otimes L}(\lambda_i) \right\| \leq \frac{C_1}{|C|}, \quad (35)$$

$^4$Distribution $\text{Bern}^{\otimes L}(\lambda)$ assigns probability $\lambda^{v_1}(1 - \lambda)^{L - |v|}$ to element $v \in F^L_2$. 

Proof. See [15].
where \( \lambda_i = \frac{1}{|C|} \sum_{c \in C} 1\{c(i) = 1\} \) be the density of ones in the \( j \)-th column of a \(|C| \times n\) matrix representing the code. In particular,

\[
h_j(T_L(C)) - \frac{1}{n} \sum_j g_j(\lambda_j) \leq \frac{C_2}{|C|},
\]

(36)

where functions \( g_j \) were defined in (18).

**Proof.** See [15]

**Lemma 8.** Functions \( g_j \) defined in (18) are concave on \([0, 1]\).

**Proof.** See [15]

**Proof of Lemma 4.** Our plan is the following:

1) Apply Elias-Bassalygo reduction to pass from \( C_n' \) to a subcode \( C_n'' \) on an intersection of two spheres \( S_{\xi_0 n} \) and \( y + S_{\xi_1 n} \).

2) Use Lemma 5 to pass to a symmetric subcode \( C_n''' \subset C_n'' \).

3) Use Lemmas 7-8 to estimate maxima of average radii \( h_j \) over \( C_n''' \).

4) Use Lemma 6 to transport statement about \( h_j \) to a statement on \( \tau_L(C_n''') \).

We proceed to details. It is sufficient to show that for some constant \( C = C(L) \) and arbitrary \( \delta > 0 \) estimate (26) holds with \( \epsilon_n = C\delta \) whenever \( n \geq n_0(\delta) \). So we fix \( \delta > 0 \) and consider a code \( C' \subset S_{\xi_0 n} \subset \mathbb{F}_q^n \) with \( |C'| \geq \exp(nR' + o(n)) \).

Note that for any \( r \), even \( m \) with \( m/2 \leq \min(r, n - r) \) and arbitrary \( y \in S_{\xi_0 m} \) intersection \( \{y + S_{\xi_0 m}\} \cap S_r \) is isometric to the product of two lower-dimensional spheres:

\[
\{y + S_{\xi_0 m}\} \cap S_r \cong S_r^{m/2} \times S_{m/2}^{n-r}.
\]

(37)

Therefore, we have for \( r = \xi_0 n \) and valid \( m \):

\[
\sum_{y \in S_{\xi_0 m}} |\{y + S_{\xi_0 m}\} \cap C'| = |C'| \left( \frac{\xi_0 n}{\xi_0 n - m/2} \right) \left( n(1 - \xi_0)/m/2 \right).
\]

Consequently, we can select \( m = \xi_1 n - o(n) \), where \( \xi_1 \) defined in (29), so that for some \( y \in S_{\xi_0 m} \):

\[
|\{y + S_{\xi_0 m}\} \cap C'| > n.
\]

Note that we focus on solution of (29) satisfying \( \xi_1 < 2\xi_0(1 - \xi_0) \). For some choices of \( R, \delta \) and \( \xi_0 \) choosing \( \xi_1 > 2\xi_0(1 - \xi_0) \) is also possible, but such a choice appears to result in a weaker bound.

Next, we let \( C'' = \{y + S_{\xi_0 m}\} \cap C' \). For sufficiently large \( n \) the code \( C'' \) will satisfy assumptions of Lemma 5 with \( K \geq \frac{n}{2} \).

Denote the resulting large symmetric subcode \( C'' \).

Note that because of (37) column-densities \( \lambda_i \)'s of \( C'' \), defined in Lemma 7, satisfy (after possibly reordering coordinates):

\[
\sum_{i=1}^{\xi_0 n} \lambda_i = \xi_1 n/2 + o(n), \quad \sum_{i > \xi_0 n} \lambda_i = \xi_1 n/2 + o(n).
\]

Therefore, from Lemmas 7-8 we have

\[
h_j(T_L(C'')) \leq \xi_0 g_j \left( 1 - \frac{\xi_1}{2\xi_0} + \frac{1 - \xi_0}{2(1 - \xi_0)} + \epsilon'_n + \frac{C_1}{|C'|} \right),
\]

(38)

where \( \epsilon'_n \to 0 \). Note that by construction the last term in (38) is \( O(\delta) \). Also note that the first two terms in (38) equal \( \theta_j \) defined in (27).

Finally, by Lemma 6 we get that for any codewords \( c_1, \ldots, c_L \in C'' \), some constant \( C \) and some sequence \( \epsilon''_n \to 0 \) the following holds:

\[
\frac{1}{n} \text{rad}(0, c_1, \ldots, c_L) \leq \theta(\xi_0, R', L) + \epsilon''_n + C\delta.
\]

By the initial remark, this concludes the proof of Lemma 4. \( \square \)

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**References**


