Converse and duality results for combinatorial source-channel coding in binary Hamming spaces

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***Abstract*—This article continues the recent investigation of combinatorial joint source-channel coding. For the special case of a binary source and channel subject to distortion measured by Hamming distance, the lower (converse) bounds on achievable source distortion are improved for all values of channel noise. Operational duality between coding with bandwidth expansion factors ρ and $\frac{1}{\rho}$ is established. Although the exact value of the asymptotic noise-distortion tradeoff curve is unknown (except at $\rho = 1$), some initial results on inter-relations between these curves for different values of ρ are shown and lead to statements about monotonicity and continuity in ρ.**

I. Introduction

The joint source-channel problem seeks to encode the user data in such a way that the noise applied to the encoded string did not lead to excessive distortion of the original data. The combinatorial joint source-channel (CJSCC) problem seeks to provide answers under the assumption that the noise is taken in the worst-case sense. Namely, for this paper we will focus on binary Hamming source/channel and correspondingly, the (D, δ) CJSCC is a pair of encoder and decoder such that addition of any string of (normalized) Hamming weight up to δ does not lead to post-decoding distortion larger than D, as measured by (normalized) Hamming distance.

The CJSCC problem and a framework for analysis were originally introduced in [1] and expanded in [2]. For thorough motivation and background on the problem we refer the reader to [2]. Here we only recall perhaps the more surprising observation from [1] about asymptotic sub-optimality of separated schemes in CJSCC.

In binary Hamming space, the adversarial source problem is a covering problem and the adversarial channel problem is a packing problem. For the covering problem, the asymptotically optimal covering has been found exactly, see e.g. [3]. The packing problem is addressed extensively in [4] and an exact asymptotic solution is still open. The best known lower bound is the Gilbert-Varshamov bound and the best known upper bound is the MRRW bound [5]. As such, these bounds, and the exact solution for the covering problem, characterize separation based schemes for the BSC. The observation that spurred our interest in this problem was made in [1], where it was shown that some simple CJSCC (such as repetition) achieve performance *strictly* better than any separated scheme for certain values of parameters.

In this paper, we extend the previous work in several directions. First, new converse bounds are proved that together improve the state-of-the-art for all values of bandwidth expansion factor $\rho > 0$ and channel parameters $0 < \delta < 1$, cf. Fig. 2. One interesting implication is the following (perhaps counter-intuitive) conclusion: For certain values of (D, δ) increasing the redundancy factor ρ can lead to decrease in performance. For example, this holds for all $\delta > \frac{1}{2}$, cf. Section III-A.

Second, an exact operational duality is established between the CJSCC problems at ρ and at $\frac{1}{\rho}$. In particular, this allows us to extend our directory of basic CJSCC codes and close, for example, the question of the largest channel noise δ for which distortion D is still less than 1. (This critical value, that is always greater than $\frac{1}{2}$, establishes the threshold at which the adversarial binary channel becomes fully useless for conveying binary-coded information.) For more details, see (20).

The structure of the paper is as follows. Converse bounds occupy Section III. For $\delta \geq 1/2$, a converse relating coverings in the source and channel spaces, Section III-A, establishes that D is bounded below by a function monotonically *increasing* in ρ and $\rho = 1$ is strictly optimal in the region $\rho \geq 1$. For $1/4 \leq \delta < 1/2$, a converse relating packings in the source and channel spaces, Section III-B, demonstrates that, for all $\rho > 0$, $D \gtrsim \delta$ and, for linear codes, Section III-C, $D \geq \delta$. In Section III-D a further stronger converse bound is proved for all schemes based on the idea of repeating a small code, cf. [6]. Sections IV and V introduce the concepts of $\rho \leftrightarrow \frac{1}{\rho}$ duality and the composition of CJSCCs, respectively. Finally, Section VI concludes with numerical comparisons and discussions.

II. Preliminaries

A. Hamming Space

For the BSC, the alphabets of interest are all binary Hamming space and the adversary is restricted to outputs whose hamming distance to the input is bounded according to the channel parameter. The notation for the n fold product of the field of two elements $\mathbb{F}_2^n$ is used for n dimensional Hamming space and Euclidean notation $|\cdot|$ is used for both the Hamming distance and weight.

Given a set $S \subset \mathbb{F}_2^n$ its Chebyshev radius is the radius of the smallest Hamming ball containing all of its points,

$$\mathrm{rad}(S) = \min_{y \in \mathbb{F}_2^n} \max_{x \in S} d(x, y),$$

a point y_0 achieving this minimum is called a Chebyshev center, and its covering radius is the radius of the smallest covering by points in S

$$r_{\mathrm{cov}}(S) = \max_{y \in \mathbb{F}_2^n} \min_{x \in S} d(x, y).$$

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These two quantities satisfy an important relation

$$\mathrm{rad}(S) = n - r_{\mathrm{cov}}(S), \tag{1}$$

which follows from the following property of binary Hamming space: for all $x \in \mathbb{F}_2^n$ and $r \in \mathbb{R}$

$$B(x,r)^c = B(\bar{x}, \lceil n - r - 1 \rceil), \tag{2}$$

where $B(x,r) := \{y \in \mathbb{F}_2^n : |x-y| \le r\}$ and $\bar{x}$ is the entrywise binary negation.

There are also some combinatorial quantities of interest:

- $K(n,r)$ – minimal number of points covering $\mathbb{F}_2^n$ with radius r balls;
- $A(n,d)$ – maximal number of points with distance between any two points greater than d;
- $A_L(n,r)$ – the maximal number of points such that any ball of radius r contains at most L points[1].

B. Basic Definitions

Definition 1. *A pair of maps $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ and $g : \mathbb{F}_2^n \to \mathbb{F}_2^k$ is a $(k, n\,; D, \delta)$ CJSCC if*

$$|f(x) - y| \le \delta n \quad \Longrightarrow \quad |x - g(y)| \le Dk,$$

or equivalently $D(\delta\,; k, n, f, g) \le D$, where

$$D(\delta\,; k, n, f, g)k := \max_{(x,y):|f(x)-y|\le\delta n} |x - g(y)|\,.$$

In the sequel the k and n may be dropped when understood from the context. Moreover, the notation $D(\delta\,; h)$ is used when h is either an *encoder* or *decoder*, an encoder being a map from the source space to the channel space and a decoder being a map from the channel space to the source space. In the interest of notational consistency, typically, an encoder is denoted with an f, a decoder with a g, the source dimensional is k and the channel dimension is n.

Definition 2. *The optimal distortion for a $(k, n\,; D, \delta)$ CJSCC is*

$$D^*(\delta\,; k, n) := \min_{f,g} D(\delta\,; f, g),$$

with minimization over $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ and $g : \mathbb{F}_2^n \to \mathbb{F}_2^k$.

Asymptotically we allow the user to choose the optimal sequence of source and channel dimensions.

Definition 3. *For bandwidth expansion factor $\rho > 0$, the asymptotically optimal CJSCC is*

$$D^*(\delta\,; \rho) := \inf_{\{k_m\},\{n_m\}} \liminf_{m\to\infty} D(\delta\,; k_m, n_m),$$

where the infimum is over subsequences of the natural numbers such that

$$\lim_{m\to\infty} \frac{n_m}{k_m} = \rho.$$

Remark 4. Note that $(D^*(\delta\,; \rho), \delta)$ is just a lower boundary of all asymptotically achievable (D, δ) with bandwidth expansion factor ρ. Formally, we say (D, δ) is asymptotically achievable if there exists a sequence of $(k_m, n_m\,; D_m, \delta_m)$ CJSCC such that $D_m \to \delta$ and $\delta_m \to \delta$ and (3). We call a pair of integer sequences $(\{k_m\}, \{n_m\})$ satisfying (3) to be an admissible ρ source-channel sequence.

[1]Note: $A_1(n,r) = A(n, 2r+1)$.

The following is a simplified characterization of the CJSCC performance of encoders and decoders, due to [1].

i) For all $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$

$$D(\delta\,; f)k = \max_{y\in\mathbb{F}_2^n} \mathrm{rad}\left(f^{-1}B_{\delta n}(y)\right). \tag{3}$$

ii) For all $g : \mathbb{F}_2^n \to \mathbb{F}_2^k$

$$D(\delta\,; g)k = \max_{x\in\mathbb{F}_2^k} \min_{z\in\mathbb{F}_2^n} \max_{y\in B_{\delta n}(z)} d(g(y), x).$$

In the sequel an encoder $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ (resp. decoder $g : \mathbb{F}_2^n \to \mathbb{F}_2^k$) may be called a $(k, n\,; D, \delta)$ CJSCC if $D(\delta\,; f) \le Dk$ (resp. $D(\delta\,; g) \le Dk$).

III. Converse Bounds

This section serves primarily to extend the known converse bounds of [1] and any converses explicitly named reference converses given therein. A common theme for these bounds is to study the behavior of intrinsic combinatorial objects under the action of a CJSCC, e.g. coverings and packings, and asymptotic results are obtained by analyzing the limit of the normalized rate for such objects. In particular, the information theoretic converse (IT) and asymptotic coding converse (CC) are

$$D_{\mathrm{IT}}(\delta\,; \rho) := \begin{cases} h^{-1}(|1 - \rho(1 - h(\delta))|^+) & 0 \le \delta \le \frac{1}{2} \\ 1/2 & 1/2 < \delta \le 1 \end{cases}$$

$$D_{\mathrm{CC}}(\delta\,; \rho) := \begin{cases} \frac{1}{2}h^{-1}(1 - \rho R_{\mathrm{MRRW}}(2\delta)) & 0 \le \delta \le \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \le \delta \le \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} < \delta \le 1 \end{cases}, \tag{4}$$

where, $h : [0, 1/2] \to [0, 1]$, $h(x) := -x\log x - (1-x)\log(1-x)$ with base 2 logarithms and

$$R_{\mathrm{MRRW}}(\delta) := \min_{0\le u\le 1-2\delta} 1 + \hat{h}(u^2) - \hat{h}(u^2 + 2(1+u)\delta),$$

where $\hat{h}(u) := h((1 - \sqrt{1-u})/2)$. The maximum of these two lower bounds represents the current state of the art, and our contribution is an improvement for all δ and ρ, excluding the combination of $\delta \le 1/2$ and $\rho \le 1$.

A. Covering Converse

The following serves as both an extension of the information theoretic converse to $\delta > 1/2$ and non-asymptotic strengthening.

Theorem 5. *(Covering Converse) If a $(k, n\,; D, \delta)$ CJSCC exists, then*

$$K(k, (1-D)k - 1) \ge K(n, (1-\delta)n - 1) \tag{5}$$

$$K(k, Dk) \le K(n, \delta n) \tag{6}$$

Proof: Let $C \subset \mathbb{F}_2^n$ be a minimal $K(n, \delta n)$ covering. Partition $\mathbb{F}_2^n$ into $\{U_c : c \in C\}$ with $\mathrm{rad}(U_c) \le \delta n$ for all c. By the CJSCC condition, $\{f^{-1}U_c\}$ is a partition of $\mathbb{F}_2^k$ with $\mathrm{rad}(f^{-1}U_c) \le Dk$. For each c choose c' to be the minimizer achieving $\mathrm{rad}(f^{-1}U_c)$. Let $C' = \{c'\}$, then $r_{\mathrm{cov}}(C') \le Dk$

and thusly $K(k, Dk) \le |C'| = |C| = K(n, \delta n)$. The second statement follows by Theorem 9. ∎

Asymptotically this yields a lower-bound on $D^*(\delta\,;\rho)$ given by the following function:

$$D_{\text{cov}}(\delta;\rho) = \begin{cases} h^{-1}(|1-\rho(1-h(\delta)|^+) & \delta < \frac{1}{2} \\ 1-h^{-1}(|1-\rho(1-h(\delta))|^+) & \delta \ge \frac{1}{2} \end{cases}.$$

It should be noted that, for $1/2 < \delta \le 1$, $D_{\text{cov}}(\delta\,;\rho)$ is monotonically *increasing* in ρ with $D_{\text{cov}}(\delta\,;\rho) > \delta$ for $\rho > 1$ and

$$\lim_{\rho\to 0} D_{\text{cov}}(\delta\,;\rho) = \tfrac{1}{2} \quad \lim_{\rho\to\infty} D_{\text{cov}}(\delta\,;\rho) = \begin{cases} 0 & 0 \le \delta < \frac{1}{2} \\ \frac{1}{2} & \delta = \frac{1}{2} \\ 1 & \frac{1}{2} < \delta \le 1 \end{cases}.$$

Combined with and (18) this shows

$$\lim_{\rho\to 0} D^*(\delta\,;\rho) = \tfrac{1}{2} \quad \forall\, \delta < \tfrac{1}{2} \quad \lim_{\rho\to\infty} D^*(\delta\,;\rho) = 1 \quad \forall\, \delta > \tfrac{1}{2}.$$

B. Packing Converse

The coding converse has a natural extension to multiple packings.

Theorem 6. *Let f be a $(k, n\,; D, \delta)$ CJSCC. If an L-multiple packing of radius Dk exists in $\mathbb{F}_2^k$, then its image under f is an L-multiple packing of radius δn and*

$$A_L(k, Dk) \le L A_L(n, \delta n).$$

Proof: Let C be an L-multiple packing of radius Dk. Suppose $f(C)$ is not an L-multiple packing of radius δn. Then there exists $y_0 \in \mathbb{F}_2^n$ such that $|f(C) \cap B_{\delta n}(y_0)| > L$. By construction $\text{rad}(f(C) \cap B_{\delta n}(y_0)) \le \delta n$. Thus there exists x_0 such that $f^{-1}(f(C) \cap B_{\delta n}(y_0)) \subset B_{Dk}(x_0)$. For all $c_0 \in C$,

$$\begin{aligned} f(c_0) \in f(C) \cap B \quad &\Longrightarrow \quad c_0 \in f^{-1}(f(C) \cap B) \\ &\Longrightarrow \quad c_0 \in C \cap f^{-1}(f(C) \cap B). \end{aligned}$$

Hence $|C \cap B_{Dk}(x_0)| \ge |C \cap f^{-1}(f(C) \cap B_{\delta n}(y_0))| \ge |f(C) \cap B_{\delta n}(y_0)| > L$, a contradiction. The bound follows from $|f^{-1} f((c_0))| \le L$. ∎

With $L = 1$, Theorem 6 is asymptotically equivalent to (4), and the novelty here is using it for $\delta > 1/4$ or $L > 1$. Blinovsky showed explicit upper and lower bounds

$$R_{\text{ach}}(L, \delta) + o(1) \le \frac{1}{n} \log A_L(n, n\delta) \le R_{\text{conv}}(L, \delta) + o(1).$$

The upper bound was improved in [7] for $L = 2$ and [8] for odd L. As per the numerical evaluations given in Section VI, for $0 \le \delta < 1/4$ the best bound is given by $L = 2$.

The following "staircase" converse shows that coding with greater than unit bandwidth expansion factor probably yields no improvement in the region $1/4 < \delta < 1/2$.

Proposition 7. *Let $\rho > 0$.*

i) (Plotkin-Levenshtein) Provided an infinite sequence of Hadamard matrices exists in k-space, for all $m \in \mathbb{N}$,

$$D^*\left(\left(\frac{1}{2}\frac{m}{2m-1}\right)^+ ;\rho\right) \ge \frac{1}{2}\frac{m}{2m-1}. \tag{7}$$

ii) (Blinovsky) For all $\ell \in \mathbb{N}$,

$$D^*\left(\frac{1}{2}\left(1-\binom{2\ell}{\ell}2^{-2\ell}\right)^- ;\rho\right) \ge \frac{1}{2}\left(1-\binom{2\ell}{\ell}2^{-2\ell}\right).$$

Proof: (Sketch)

i) Evaluate the coding converse using the Plotkin-Levenshtein solution to $A(n, d)$, [4] ch. 7.3.
ii) Evaluate the endpoint for Blinovsky's upper and lower bounds for ranging values of L.

∎

C. Linear Encoder Converse

A linear $(k, n\,; D, \delta)$ CJSCC is a $n \times k$ matrix $A \in \mathbb{F}_2^{n\times k}$ and satisfies, for all $x \in \mathbb{F}_2^k$,

$$\text{wt}(x) \ge 2Dk + 1 \quad \Longrightarrow \quad \text{wt}(Ax) \ge 2\lfloor \delta n \rfloor + 1. \tag{8}$$

For linear encoders we can sharpen the double staircase result of the previous section:

Theorem 8. *For all $\rho > 0$ and $1/4 \le \delta \le 1/2$, the asymptotic distortion for linear encoders satisfies*

$$D^*_{lin}(\delta\,;\rho) \ge \delta.$$

Proof. Boundary cases are handled by (7), so assume $\delta \in (\frac{1}{4}, \frac{1}{2})$. Let A_k be a sequence of linear $(k, \lfloor \rho k \rfloor\,; D_k, \delta)$ CJSCC with $\underline{D} = \liminf_{k\to\infty} D_k$. Let X^k be i.i.d. Bernoulli(q) vector with $q > \max(2\underline{D}, \frac{1}{2})$. By the law of large numbers

$$\limsup_{k\to\infty} \mathbb{P}[|X^k| \ge 2D_k k + 1] = 1\,. \tag{9}$$

On the other hand, we have

$$\begin{aligned} \mathbb{P}[|X^k| \ge 2D_k k + 1] &\le \mathbb{P}[|A_k X^k| \ge 2\lfloor \delta \lfloor \rho k \rfloor \rfloor + 1] && (10) \\ &\le \frac{\mathbb{E}[|A_k X^k|]}{2\lfloor \delta \lfloor \rho k \rfloor \rfloor + 1} && (11) \\ &\le \frac{q\rho k}{2\delta\rho k - 3}, && (12) \end{aligned}$$

where (10) is from (8), (11) is by Chebyshev, and (12) follows because for arbitrary coefficients a_j we have

$$\mathbb{P}[\sum a_j X_j = 1] \le q$$

when $q \ge \frac{1}{2}$ and thus the average Hamming weight of the vector $A_k X^k$ does not exceed $q\rho k$.Putting together (9) and (12) we conclude that $q \ge 2\delta$, and thusly $\underline{D} \ge \delta$. □

D. Repetition Scheme Converse

An L-repetition scheme $f^{\oplus L} : \mathbb{F}_2^{Lk} \to \mathbb{F}_2^{Ln}$ is the L times concatenation of a based code $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$. Previous results [1] demonstrated that repeating a small base code may yield good CJSCC. The asymptotic performance of L-repetition schemes is characterized in [6, Thm. 2] where it is shown that, for all $\rho > 0$, $k \in \mathbb{N}$ and $f : \mathbb{F}_2^k \to \mathbb{F}_2^{\lfloor \rho k \rfloor}$, the limit function

$$D(\delta\,; f^\infty) := \lim_{L\to\infty} D(\delta\,; f^{\oplus L})$$

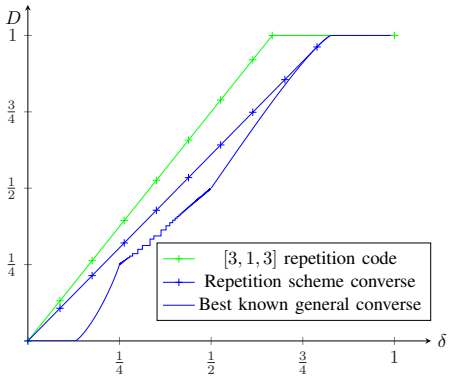

Fig. 1. Repetition code (17) and repetition converse (13) for $\rho = 3$.

exists and is concave in δ. The concavity in δ and the covering converse, $D_{\text{cov}}(\delta\ \rho)$, yield the following lower bound on the asymptotic performance of any repetition scheme:

$$D(\delta\,;f^{\oplus\infty}) \geq \begin{cases} \frac{D_{\text{cov}}(\delta_0\,;\rho)}{\delta_0}\delta & 0 \leq \delta \leq \delta_0 \\ D_{\text{cov}}(\delta\,;\rho) & \delta_0 < \delta < \theta_\rho \\ 1 & \theta_\rho \leq \delta \leq 1 \end{cases}, \qquad (13)$$

where δ_0 is the unique solution in $1/2 < \delta < \theta_\rho$ to

$$\delta D'_{\text{cov}}(\delta\,;\rho) - D_{\text{cov}}(\delta\,;\rho) = 0.$$

See Fig. 2 for an illustration for $\rho = 3$.

IV. Duality

A. Dual Problem

As introduced, the combinatorial joint source-channel problem seeks the minimal distortion for a given channel parameter. Conversely, the dual problem asks for the largest admissible channel parameter for a given distortion. This optimization is defined as follows:

$$\begin{aligned} \delta^*(D\,;k,n)n + 1 &:= \max_{f,g} \min_{(x,y)\,:\,d(x,g(y))\geq Dk+1} d(f(x),y) \\ &= \max_{g} \min_{x\in\mathbb{F}_2^k} r_{\text{cov}}(g^{-1}B_{Dk}^C(x)) \\ &= \max_{f} \min_{y\in\mathbb{F}_2^n} \max_{x\in\mathbb{F}_2^k} \min_{z\in fB_{Dk}^C(x)} d(z,y), \end{aligned}$$

where $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ and $g : \mathbb{F}_2^n \to \mathbb{F}_2^k$.

The following theorem relates the primal and dual CJSCC problems operationally:

Theorem 9. *(Duality) Let $k, n, kD, n\delta$ be integers. There exists a $(k, n; D, \delta)$ CJSCC if and only if there exists a $(n, k; D_1, \delta_1)$ CJSCC with*

$$D_1 = 1 - \delta - \frac{1}{n}, \quad \delta_1 = 1 - D - \frac{1}{k}.$$

Remark 10. Optimizing over the CJSCC we get:

$$\delta^*(D\,;k,n) = 1 - D^*(1 - D - 1/k\,;n,k) - 1/n.$$

Speaking asymptotically, the point (D, δ) is achievable at bandwidth ρ if and only if $(1-\delta, 1-D)$ achievable at bandwidth $\frac{1}{\rho}$. That is, the (D, δ) regions at ρ and $\frac{1}{\rho}$ are related by reflection in the diagonal $(0, 1) - (1, 0)$.

Proof: Consider $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ and $g : \mathbb{F}_2^n \to \mathbb{F}_2^k$ comprising a $(k, n; D, \delta)$ code. Equivalently, for all $s \in \mathbb{F}_2^n$ we have:

$$\forall e' \in \mathbb{F}_2^k, |e'| > kD : |f(g(s) + e') - s| > n\delta\,. \qquad (14)$$

Define the new encoder/decoder pair as follows:

$$f_1(s) \triangleq g(s)\,, \quad g_1(x) \triangleq 1^n + f(x + 1^k)\,, \quad s \in \mathbb{F}_2^n, x \in \mathbb{F}_2^k\,,$$

where 1^k and 1^n are the all-one vectors with respective dimensions. Let $e \in \mathbb{F}_2^k$ s.t. $|e| < k - kD$, then we have

$$\begin{aligned} |g_1(f_1(s) + e) - s| &= |1^n + f(g(s) + e + 1^k) - s| \\ &= n - |f(g(s) + e') - s| \\ &< n - n\delta\,, \end{aligned} \qquad (15)$$

where we defined $e' \triangleq e + 1^k$ and applied (14). Clearly, (15) shows that (f_1, g_1) defines a $(n, k; D_1, \delta_1)$ CJSCC. ∎

V. $D - \delta$ trade-off as a function of ρ

In the information theoretic setting there is both monotonicity and continuity in ρ. This section partially extends these properties to the combinatorial setup. A basis for this analysis is the performance of CJSCCs combined by composition.

Lemma 11. *(Composition of Encoders)* Let $k, m, n \in \mathbb{N}$ and $0 \leq \delta \leq 1$. For all $f_1 : \mathbb{F}_2^k \to \mathbb{F}_2^m$ and $f_2 : \mathbb{F}_2^m \to \mathbb{F}_2^n$

$$D(\delta\,;f_2 \circ f_1) \leq D(D(\delta\,;f_2)\,;f_1)$$

and

$$D^*(\delta\,;k,n) \leq D^*(D^*(\delta\,;m,n)\,;k,m).$$

Proof: Let g_1 and g_2 be the optimal Chebyshev decoders. Then $d((f_2 \circ f_1)(x), y) \leq \delta n$ implies $d(f_1(x), g_2(y)) \leq D(\delta\,;f_2)m$ implies $d(x, (g_1 \circ g_2)(y)) \leq D(D(\delta\,;f_2)\,;f_1)k$. The second statement follows immediatley from the first by using the optimal encoders. ∎

Of particular interest is the canonical admissible ρ source-channel sequence $(k, \lfloor \rho k \rfloor)$. To facilitate in the analysis of such sequences we define upper and lower limits

$$\begin{aligned} \overline{E}(\delta\,;\rho) &:= \limsup_{k\to\infty} D(\delta\,;k,\lfloor\rho k\rfloor) \\ \underline{E}(\delta\,;\rho) &:= \liminf_{k\to\infty} D(\delta\,;k,\lfloor\rho k\rfloor). \end{aligned}$$

The notation $E(\delta\,;\rho)$ is used in statements that apply to both.

An immediate application of the composition Lemma shows that $E(\delta\,;\rho)$ is more or less impervious to small deviations in ρ and provides a limited monotonicity result.

- For all $\rho > 0$ and $a, b \in \mathbb{N}$

$$\overline{E}(\delta^-\,;\rho) \leq \limsup_{k\to\infty} D(\delta\,;k+a,\lfloor\rho k\rfloor + b) \leq \overline{E}(\delta^+\,;\rho)$$

$$\underline{E}(\delta^-\,;\rho) \leq \liminf_{k\to\infty} D(\delta\,;k+a,\lfloor\rho k\rfloor + b) \leq \underline{E}(\delta^+\,;\rho).$$

- If $\rho, \tau > 0$ and $\limsup_{k\to\infty} D(\delta\,;\lfloor\tau k\rfloor, \lfloor\rho k\rfloor) \leq \delta$, then $\overline{E}(\delta\,;\rho) \leq \overline{E}(\delta^+\,;\tau)$.
- If $\rho, \tau \in \mathbb{Q}$, then

$$\limsup_{k\to\infty} D(\delta\,;\lfloor\tau k\rfloor, \lfloor\rho k\rfloor) \leq \overline{E}(\delta^+\,;\rho/\tau).$$

VI. Discussion

In this section we discuss how our converse results compare against simple achievability results.

A. Basic CJSCCs

The following is a collection of basic CJSCCs that we will compare our converse bounds against:

- (Pseudo)-identity code $I_{k,n}$: maps $k \to \min\{k,n\}$ bits followed by $n - \min\{k,n\}$ zeros. The distortion of the (pseudo)-identity map is
$$D(\delta\,; I_{k,n}) = (\delta n + \max\{0, k-n\})/k. \tag{16}$$
- Repetition code $R_{\rho,k}$: for $\rho \in \mathbb{N}$ each bit is repeated ρ times. For odd ρ the distortion of the ρ-repetition code is better than (16):
$$D(\delta\,; R_{\rho,k})k = \frac{\lfloor \delta\rho k \rfloor}{\lceil \rho/2 \rceil}. \tag{17}$$
- Dual repetition codes: For ρ equal to reciprocal of the odd integer, one may define a small code $f_1 : \mathbb{F}_2^{\frac{1}{\rho}} \to \mathbb{F}_2$ to be a majority vote. Repeating this code asymptotically achieves
$$D = 1 - (1-\delta)\frac{1+\rho}{2}\,.$$
This is an improvement over the pseudo-identity for large δ.
- Separated code $S_{M,k,n}$: Given a covering $C_1 \subset \mathbb{F}_2^k$ (of radius kD) and a packing $C_2 \subset \mathbb{F}_2^n$ (of radius δn) of equal cardinality M, the separation code takes $x \in \mathbb{F}_2^k$, finds the closest point in C_1 and outputs a corresponding point from C_2. Asymptotically, these codes achieve [1, Sec. III-C],
$$D = \begin{cases} h^{-1}(|1-\rho(1-h(2\delta))|^+) & 0 \le \delta < \frac{1}{4}\,, \\ \frac{1}{2} & \frac{1}{4} \le \delta < \frac{1}{2} \end{cases}. \tag{18}$$
- Dual separated codes $S'_{M,k,n}$[2]: Given the packing $C_1 \subset \mathbb{F}_2^k$ of radius $k(1-D)$ and covering $C_2 \subset \mathbb{F}_2^n$ of radius $n(1-\delta)$ of the same cardinality M, the encoder takes $x \in \mathbb{F}_2^k$, finds the closest point in C_1 and outputs a corresponding point in C_2.
To verify that this construction indeed yields a $(k,n;D,\delta)$ CJSCC we will use (3). Indeed, by (2) every ball of radius $n\delta$ must miss at least one point of C_2. Thus, $f^{-1}B(y,n\delta)$ must exclude a ball of radius $k(1-D)$, and thus again by (2) is contained in a ball of radius kD, QED.
Asymptotically, these codes achieve:
$$D = 1 - \frac{1}{2}h^{-1}(|1-\rho(1-h(\delta))|^+), \qquad \frac{1}{2} \le \delta \le 1\,. \tag{19}$$

B. Comparison for $\rho = 3$

Figure 2 gives the best known converse and achievability bounds for bandwidth expansion factor $\rho = 3$. The dotted black line represents the uncoded or $\rho = 1$ case where the identity scheme is optimal. Deviation from this line is of interest.

The achievability bound is given as follows:

- for $0 \le \delta < .185$ the best code is the separated code (18).

[2]The name reflects the fact that these codes can be constructed by applying Theorem 9 to $S_{M,n,k}$.

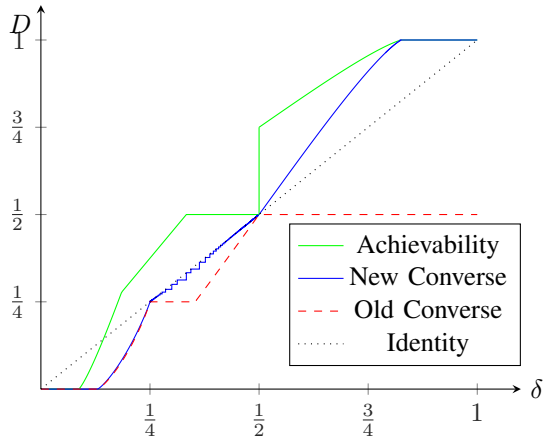

Fig. 2. State of the art for achievability and converse bounds when $\rho = 3$.

- for $.184 \le \delta < 1/3$ – the $[3,1,3]$-repetition code (17).
- for $1/3 \le \delta < 1/2$, – the separated code with $M = 2$, see (18))
- for $1/2 \le \delta < 1$, – the dual separated code (19).

The converse bound is given as follows:[3]

- for $0 \le \delta \le 1/4$, the best bound is Theorem 6 using $L = 2$ and the upper bound from [7],
- for $1/4 < \delta \le 1/2$, the interlacing of the bounds in Proposition 7 (double staircase) and,
- for $1/2 < \delta \le 1$, Theorem 5 the covering converse.

We also note that together the dual-separated codes and the covering converse establish that the supremum of δ for which distortion $D < 1$ is asymptotically achievable at bandwidth expansion factor ρ is given by
$$\delta^*(1\,;\rho) = 1 - h^{-1}\left(\left|1 - \frac{1}{\rho}\right|^+\right) \tag{20}$$

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[3]*Added in print:* The bound in the first interval is superceded by [9, Theorem 1], while the double staircase can be replaced by the straight line by [9, Theorem 3].