HYPERCONTRACTIVITY OF SPHERICAL AVERAGES IN HAMMING SPACE

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Abstract. Consider the linear space of functions on the binary hypercube and the linear operator S_δ acting by averaging a function over a Hamming sphere of radius δn around every point. It is shown that this operator has a dimension-independent bound on the norm $L_p \to L_2$ with $p = 1 + (1-2\delta)^2$. This result evidently parallels a classical estimate of Bonami and Gross for $L_p \to L_q$ norms for the operator of convolution with a Bernoulli noise. The estimate for S_δ is harder to obtain since the latter is neither a part of a semigroup, nor a tensor power. The result is shown by a detailed study of the eigenvalues of S_δ and $L_p \to L_2$ norms of the Fourier multiplier operators Π_a with symbol equal to a characteristic function of the Hamming sphere of radius a (in the notation common in boolean analysis $\Pi_a f = f^{=a}$, where $f^{=a}$ is a degree-a component of function f). A sample application of the result is given: Any set $A \subset \mathbb{F}_2^n$ with the property that $A + A$ contains a large portion of some Hamming sphere (counted with multiplicity) must have cardinality a constant multiple of 2^n.

Key words. Hamming space, hypercontractivity, Krawtchouk polynomials, Fourier analysis on hypercube, additive combinatorics

1. Main result and discussion. Consider a linear space $\mathcal{L}$ of functions on n-dimensional Hamming cube $f : \mathbb{F}_2^n \to \mathbb{C}$. We endow $\mathcal{L}$ with the following norms and an inner product:

$$\|f\|_p \triangleq \mathbb{E}^{\frac{1}{p}}[|f(X)|^p], \qquad 1 \le p \le \infty, \tag{1.1}$$

$$(f, g) \triangleq \mathbb{E}[f(X)\bar{g}(X)], \tag{1.2}$$

where X is uniform on $\mathbb{F}_2^n$. For any linear operator $T : \mathcal{L} \to \mathcal{L}$ we define

$$\|T\|_{p\to q} \triangleq \sup_{f\in\mathcal{L}} \frac{\|Tf\|_q}{\|f\|_p}.$$

Let $Z = (Z_1, \dots, Z_n)$ be a random element of $\mathbb{F}_2^n$ with components independent and identically distributed (i.i.d.) according to Bern(δ) distribution: $\mathbb{P}[Z_i = 1] = 1 - \mathbb{P}[Z_i = 0] = \delta$. For the following operator

$$N_\delta f(x) \triangleq \mathbb{E}[f(x + Z)], \qquad x \in \mathbb{F}_2^n, 0 \le \delta \le 1 \tag{1.3}$$

the so-called "hypercontractive" inequality was established by Bonami [3], Gross [12] and others (see [22, Chapter 9, notes] for the history):

$$\|N_\delta f\|_q \le \|f\|_p, \qquad \forall q \ge p \ge 1, p - 1 \ge (q-1)(1-2\delta)^2, p, q \ge 1. \tag{1.4}$$

There are a number of applications of hypercontractive inequalities. For example, we mention an early result in information theory [1], which has recently become known as the "small-set expansion". A number of applications in theoretical computer science are presented in [22, Chapter 9-10]. One of the pillars of the analysis

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of boolean functions, the KKL lemma [14], is an ingenious application of (1.4). Hypercontractivity is also an indispensable tool in probability for analyzing mixing of Markov chains [6] and isoperimetry [21, Theorem 3.4].

In this paper we analyze the $L_p \to L_2$ norm for an operator S_δ of averaging over a Hamming sphere $\mathbb{S}_{\delta n}$. Specifically, for $x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ denote the Hamming weight of x and the Hamming sphere centered at zero as

$$|x| \triangleq |\{j : x_j = 1\}| \tag{1.5}$$

$$\mathbb{S}_j \triangleq \{x : |x| = j\} \,. \tag{1.6}$$

The operator S_δ is defined as follows:

$$S_\delta f(x) \triangleq \binom{n}{j}^{-1} \sum_{y \in \mathbb{F}_2^n, |y|=j} f(x+y) \,,$$

where $j = \lceil \delta n \rceil$ if $\delta < 1/2$ and $j = \lfloor \delta n \rfloor$ if $\delta \ge 1/2$. In other words, we may write

$$S_\delta f \triangleq \frac{f * 1_{\mathbb{S}_j}}{|\mathbb{S}_j|} \,,$$

where $*$ denotes the convolution

$$f * g(x) \triangleq \sum_{y \in \mathbb{F}_2^n} f(x-y) g(y) \,.$$

This definition ensures $S_\delta f(x) = S_{1-\delta} f(\bar{x})$ for $\delta \neq \frac{1}{2}$, where $\bar{x} = (1-x_1, \ldots, 1-x_n)$.

Our main result is that S_δ satisfies an inequality entirely similar to N_δ, namely:

$$\|S_\delta f\|_2 \le C_\delta \|f\|_p \,, \qquad \forall p \ge 1 + (1-2\delta)^2, \delta \neq \frac{1}{2} \,, \tag{1.7}$$

where the crucial part is that $C_\delta > 1$ does not depend on dimension n. Note also that the constant cannot be tightened to 1. Indeed, taking $f = 1_{even}$ to be the characteristic function of the set of all even-weight vectors we get

$$\|S_\delta\|_{p \to 2} \ge 2^{\frac{1}{2} - \frac{1}{p}} \,, \qquad 1 \le p \le 2 \,, 0 < \delta < 1 \,,$$

regardless of dimension n. More precisely, we show the following.

Theorem 1.1. *Consider the set $F \subset [0,1] \times [1,2]$*

$$F = \{(\delta, p) : p \ge 1 + (1-2\delta)^2, 0 \le \delta \le 1, 1 < p \le 2\} \,.$$

For every compact subset K of F there exists a constant $C = C(K)$ such that for all $(\delta, p) \in K$, $n \ge 1$ and $f : \mathbb{F}_2^n \to \mathbb{C}$ we have

$$\|S_\delta f\|_2 \le C \|f\|_p \,. \tag{1.8}$$

Conversely, for any $(\delta, p) \notin F$ there is $E > 0$ such that

$$\sup_f \frac{\|S_\delta f\|_2}{\|f\|_p} \ge e^{nE + o(n)} \,, \qquad n \to \infty \tag{1.9}$$

with the exception of $\delta = 1/2, p = 1$ for which we have

$$\sup_f \frac{\|S_{1/2}f\|_2}{\|f\|_1} = 2^{n/2}\binom{n}{\lfloor n/2 \rfloor}^{-\frac{1}{2}} \sim \left(\frac{\pi n}{2}\right)^{\frac{1}{4}} . \tag{1.10}$$

Remark: The constants that can be extracted from our proof method (after numerical evaluations) are as follows: for $\delta \le 0.16$ we have $C = \sqrt{2}$, while for larger δ we can take C to be arbitrarily close to $\sqrt{2}$ for sufficiently large n.

The full proof is given in Section 3, while here we give a high-level sketch. We note first that the standard methods for showing hypercontractivity do not apply since they require the operator to be a tensor power or be part of a semigroup. The semigroup could be continuous-time, as in [6], or discrete-time as in [20], but S_δ is a member of neither. Instead, our proof proceeds by noticing that S_δ and N_δ commute and are self-adjoint, hence have common orthogonal eigenspaces (given by the Fourier transform, also known as degree-d components). Consequently, decomposing a function $f = \sum_j f_j$ into sum of its projections on eigenspaces we have from (1.4):

$$\|N_\delta f\|_2^2 = \sum_j \lambda_j(N_\delta)^2 \|f_j\|_2^2 \le \|f\|_p^2 . \tag{1.11}$$

Writing a similar expansion for S_δ we have

$$\|S_\delta f\|_2^2 = \sum_j \lambda_j(S_\delta)^2 \|f_j\|_2^2 . \tag{1.12}$$

If we had that $\lambda_j(S_\delta) \le \lambda_j(N_\delta)$, then we could just upper bound (1.12) with (1.11) and conclude the proof. It turns out that such estimate does hold but only for a range of j, and thus the bulk of the proof consists of showing that contribution to (1.12) of the eigenspaces outside of this range is small. This part crucially depends on a curious relation between norms of certain Fourier-multiplier operators on $\mathbb{F}_2^n$ and eigenvalues of S_δ. The corresponding estimates that bound energies in the degree-a components of functions on the hypercube are, perhaps, of independent interest.

1.1. Discussion. Why would one conjecture that S_δ is hypercontractive? Note that [6, Theorem 3.7] shows that a discrete time Markov chain on state space $\mathcal{X}$ and whose kernel satisfies hypercontractive inequality, mixes in time of order $O(\log\log|\mathcal{X}|)$. For S_δ, this Markov chain is a non-standard random walk on a hypercube $\mathbb{F}_2^n$ which jumps by a distance exactly δn at each step. A simple coupling argument shows that indeed such a random walk must mix in time $O(\log n)$, therefore giving some probabilistic intuition as to why Theorem 1.1 might hold.

We note that our main goal was to show an $O(1)$ estimate for $\|S_\delta\|_{p\to q}$. Indeed, a $O(\sqrt{n})$ estimate is much easier:

Theorem 1.2. *For any δ and $p \ge 1 + (q-1)(1-2\delta)^2$ we have*

$$\|S_\delta\|_{p\to q} = O(\sqrt{n}) .$$

Proof. Assuming without loss of generality that $f \ge 0$ it is easy to see from Stirling's formula that

$$\frac{1}{\binom{n}{\delta n}} \sum_{|y|=\delta n} f(x+y) \le O(\sqrt{n}) \sum_{|y|=\delta n} f(x+y)\delta^{|y|}(1-\delta)^{n-|y|} .$$

Then extending summation to all of y we get

$$S_\delta f(x) \le O(\sqrt{n}) N_\delta f(x) \qquad \forall x \in \mathbb{F}_2^n\,.$$

The result then follows from (1.4). $\square$

The importance of having an $O(1)$ estimate for the $p \to q$ norm is due to the following general result of Semenov and Shneiberg [26], which generalized earlier results of Fefferman and Segal [8, 25]. Semenov and Shneiberg showed that if T is any operator with $\|T\|_{p\to q} < \infty$ then for all $\epsilon < \epsilon_0 = \epsilon_0(p, q, \|T\|_{p\to q})$ we have

$$\|(1-\epsilon)\mathbb{E} + \epsilon T\|_{p\to q} = 1\,,$$

provided that $\mathbb{E} \circ T = T \circ \mathbb{E}$, $T1 = 1$ and $(\mathbb{E}\, f)(x) \stackrel{\triangle}{=} \mathbb{E}\,[f(X)]$. The key point is that ϵ_0 only depends on T through the norm $\|T\|_{p\to q}$. Paired with our Theorem 1.1 this allows to establish that certain permutation-invariant (or S_n-equivariant) operators in Hamming space have $L_p \to L_q$ norm equal to 1.

1.2. Application: sumsets in Hamming space. Our original interest in hypercontractivity was motivated by a remarkably simple solution it yields to a problem that the author attempted to solve using more conventional semi-definite programming (SDP), compare Sections IV in [23] and [24]. Here is an application of the new result (Theorem 1.1) similar in spirit:

Corollary 1.3. *For every $\epsilon \in (0,1)$ there are constants $C_1, C_2 > 0$ such that for any dimension n and any set $A \subset \mathbb{F}_2^n$ we have*

$$\sup_{j\in[\epsilon n,(1-\epsilon)n]} \frac{2^n(1_A * 1_A, 1_{\mathbb{S}_j})}{|\mathbb{S}_j||A|} \ge \lambda \quad \Longrightarrow \quad |A| \ge C_1 \lambda^{C_2} 2^n\,.$$

In other words, $\mathbb{P}[X+Y \in A] \ge \lambda$ implies $|A| \ge C_1\lambda^{C_2}2^n$, where (X,Y) is uniform on $A \times \mathbb{S}_j$.

Remark: It is known that any linear subspace $V \subset \mathbb{F}_2^n$ which contains a $\Omega(1)$-fraction of any $\mathbb{S}_{\delta n}$ must have co-dimension $O(1)$ (in $n \to \infty$). This corollary is a generalization: if a sumset $A + A$ contains a λ-fraction of any Hamming sphere $\mathbb{S}_j$ (counted with multiplicity normalized by $|A|$) then the set must be of cardinality $\Omega(2^n)$.

Proof. We prove a stronger statement:

$$\left(\phi * \phi, \frac{1_{\mathbb{S}_j}}{|\mathbb{S}_j|}\right) \ge \lambda\|\phi\|_2^2 \quad \Longrightarrow \quad \frac{\|\phi\|_2^2}{\|\phi\|_1^2} \le \frac{1}{C_1}\lambda^{-C_2}\,, \tag{1.13}$$

from which the result follows by taking $\phi = 1_A$. To show (1.13) denote $\delta = \frac{j}{n}$ and consider the chain

$$\begin{aligned}
\lambda\|\phi\|_2^2 &\le \left(\phi * \phi, \frac{1_{\mathbb{S}_j}}{|\mathbb{S}_j|}\right) && (1.14)\\
&= (\phi, S_\delta \phi) && (1.15)\\
&\le \|\phi\|_2 \|S_\delta\phi\|_2 && (1.16)\\
&\le C\|\phi\|_2\|\phi\|_p\,, \qquad p = 1 + (1-2\epsilon)^2 < 2 && (1.17)\\
&\le C\|\phi\|_2 \|\phi\|_1^{\frac{2}{p}-1} \|\phi\|_2^{2-\frac{2}{p}} && (1.18)
\end{aligned}$$

where (1.16) is Cauchy-Schwarz, (1.17) is from Theorem 1.1, and (1.18) is from log-convexity of $\frac{1}{p} \mapsto \|\phi\|_p$. Rearranging terms yields (1.13). □

In fact, this corollary can be interpreted in terms of the Frankl-Rödl graphs $\mathtt{FR}^n_\gamma$, which are defined on the vertex set $\mathbb{F}_2^n$ with $v \sim v'$ if $|v - v'| = (1-\gamma)n$. Denoting by $E(A, A)$ the number of internal edges of a set A, our corollary says

$$|A| \le \mu 2^n \implies E(A, A) \le C_1' \mu^{C_2'} |\mathbb{S}_{\gamma n}|\, |A|\,.$$

In the regime of constant μ this is essentially tight. Indeed, an estimate in the opposite direction has been obtained by Benabbas, Hatami and Magen [2] (see [16, Section 5] for a public account of these results):

$$|A| \ge \mu 2^n \implies E(A, A) \ge \left((\mu/2)^{\frac{1}{\gamma}} - o_n(1)\right) 2^n |\mathbb{S}_{\gamma n}|\,, \tag{1.19}$$

provided $\gamma < 1/2$. In particular, this implies that if A is an independent set of $\mathtt{FR}^n_\gamma$ (so that $E(A, A) = 0$) we must have $|A| \le o(1)2^n$. This is a weak form of the famous Frankl-Rödl theorem [9] showing that $\alpha(\mathtt{FR}^n_\gamma) \le (2 - \epsilon(\gamma))^n$, where $\alpha(\cdot)$ denotes the maximal independent set of the graph. Similar to our result, (1.19) was obtained by employing a reverse hypercontractivity result of Borell [4], which states

$$\|N_\delta f\|_q \ge \|f\|_p, \qquad \forall -\infty < q < p < 1, p - 1 \le (q-1)(1-2\delta)^2\,, \tag{1.20}$$

for any $f > 0$. Note that (1.20) cannot be extended to S_δ, but in [2] the authors show that the eigenvalues of N_δ and $\frac{1}{2}(S_\delta + S_{\delta+1/n})$ are similar enough that the latter operator is almost reverse-hypercontractive. We will further discuss results of [2] below.

1.3. Hypercontractivity and SDP. Part of our motivation to study hypercontractivity is that it may be employed as an improvement to the method of semi-definite programming (SDP) relaxation in various constraint satisfaction problems. For example, the best known bound [19] on the size of error correcting codes in Hamming space are obtained by the SDP relaxation of Delsarte [5], and there has long been interest in using hypercontractivity to improve the SDP relaxation, see [15].

The relation between hypercontractivity and SDP has also been known in the computer science literature.[1] For example, [11] shows that any (fixed) number of rounds of Lovász-Schrijver SDPs is unable to prove a bound better than $\alpha(\mathtt{FR}(m, \gamma)) < (\frac{1}{2} - \epsilon)2^m$, whereas we know from [9] that $\alpha(FR(m, \gamma)) < (2 - \epsilon)^m$. At the same time, [2] shows that reverse hypercontractivity proves $\alpha(FR(m, \gamma)) < o(2^m)$. Following up on the latter, [16] shows that reverse hypercontractivity itself is provable in a sum-of-squares (SOS) proof system, thereby showing that $\alpha(\mathtt{FR}(m, \gamma)) < o(2^m)$ is provable via Lasserre's SOS algorithm of a fixed (but dependent on γ) degree.

This section gives another example where (direct, as opposed to reverse) hypercontractivity supersedes SDP methods. We mention that while the previously mentioned examples deal with integer-programming problems, our example below is inherently "continuous".

Define $B_\delta(x) = \delta^{|x|}(1-\delta)^{n-|x|}$ to be a distribution function of an iid Bernoulli noise. For $\lambda \in (0, 1)$ we define

$$V_n(\lambda) = \max\left\{\frac{(\phi, \phi)}{(\phi, 1)^2} : \phi \ge 0, (\phi * \phi, B_\delta) \ge \lambda \|\phi\|_2^2\right\} \tag{1.21}$$

[1] This paper was originally written before some of the discussed results were published. We thank the reviewers for pointing out these references.

An argument entirely similar to (1.16)-(1.18) invoking Bonami-Gross (1.4) instead of Theorem 1.1 demonstrates[2]

$$V_n(\lambda) \le \lambda^{-s} \tag{1.22}$$

for some $s > 0$ and all dimensions n.

Note that the problem in (1.21) is completely "L_2" and thus escaping to L_p space in order to solve it looks somewhat unusual. Indeed, a more natural approach (at least to us) would be to apply Fourier analysis or an SDP relaxation. Here is the "spectral gap" type of argument: Since the second-largest eigenvalue of N_δ equals $(1-2\delta)$ we get

$$(\phi_0, N_\delta \phi_0) \le (1-2\delta)\|\phi_0\|^2 ,$$

where $\phi_0 = \phi - (\phi, 1)$. Simple manipulations then imply

$$V_n(\lambda) \le \frac{2\delta}{\lambda - (1-2\delta)} , \qquad \text{if } \lambda > (1-2\delta) .$$

This proves a correct estimate of $O(1)$ but only for large values of λ.

An improvement of this method comes with the use of an SDP relaxation. The latter is obtained by considering $\psi = \phi * \phi$ and retaining only the non-negative definiteness property of ψ. I.e. we have the following upper bound:

$$V_n(\lambda) \le SDP(n,\lambda) \triangleq \max\left\{ 2^n \frac{(\psi, B_0)}{(\psi, 1)} : \psi \ge 0 \,, \psi \succeq 0 \,, (\psi, B_\delta) \ge \lambda(\psi, B_0) \right\}$$

where $B_0(x) = 1\{x = 0\}$ and $\psi \succeq 0$ denotes that $f \mapsto f * \psi$ is a non-negative definite operator. It can be shown that[3]

$$SDP(n,\lambda) = O(1) , \qquad \lambda > (1-2\delta)^2 ,$$

while for smaller values of λ $SDP(n,\lambda)$ grows polynomially in n. Thus, while SDP improves on the "spectral-gap" argument, it is still unable to yield the correct estimate of $V_n(\lambda)$ for the entire range of λ.

2. Auxiliary results.

2.1. Notation. For $x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ define $\bar{x} \triangleq (1 - x_1, \ldots, 1 - x_n)$. For each $j = 1, \ldots, n$ let

$$\chi_j(x_1, \ldots, x_n) \triangleq 1_{\{x_j = 0\}} - 1_{\{x_j = 1\}} .$$

Define the characters, indexed by $v \in \mathbb{F}_2^n$,

$$\chi_v(x) \triangleq \prod_{j : v_j = 1} \chi_j(x) = (-1)^{\langle v, x \rangle} ,$$

[2] The original question was to check whether there exists a small set $A \subset \mathbb{F}_2^n$ such that $\mathbb{P}[X + X' = Z] \ge \lambda \mathbb{P}[X + X' = 0]$, where $X \perp\!\!\!\perp X' \sim$ uniform on A and $Z \sim \text{Bern}(\delta)$. Bound (1.22) shows any such set occupies a non-vanishing fraction of $\mathbb{F}_2^n$.

[3] These observations were made in collaboration with Prof. A. Megretski.

where $\langle v, x\rangle = \sum_{j=1}^n v_j x_j$ is a non-degenerate bi-linear form on $\mathbb{F}_2^n$. The Fourier transform of $f : \mathbb{F}_2^n \to \mathbb{C}$ is

$$\hat{f}(\omega) \triangleq \sum_{x\in\mathbb{F}_2^n} \chi_\omega(x) f(x) = 2^n (f, \chi_\omega)\,, \qquad \omega \in \mathbb{F}_2^n\,.$$

L_p-norms are monotonic

$$\|f\|_p \le \|f\|_{p_1}, \qquad p \le p_1\,. \tag{2.1}$$

and satisfy the Young inequality:

$$\|f * g\|_p \le 2^n \|f\|_q \|g\|_r \qquad \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}\,, 1 \le p, q, r \le \infty \tag{2.2}$$

For the size of Hamming spheres we have

$$|\mathbb{S}_{\delta n}| = \binom{n}{\lfloor \delta n\rfloor} = e^{nh(\delta) - \frac{1}{2}\ln n + O(1)}\,, \qquad n \to \infty \tag{2.3}$$

where the estimate is a consequence of Stirling's formula, $O(1)$ is uniform in δ on compact subsets of $(0, 1)$ and

$$h(\delta) = -\delta \ln \delta - (1-\delta)\ln(1-\delta)\,. \tag{2.4}$$

Furthermore, for all $0 \le j \le n$

$$e^{nh(\frac{j}{n})} \sqrt{\frac{1}{2n}} \le |\mathbb{S}_j| < e^{nh(\frac{j}{n})} \tag{2.5}$$

and for $1 \le j \le n-1$, cf. [10, Exc. 5.8],

$$e^{nh(\frac{j}{n})} \sqrt{\frac{n}{8j(n-j)}} \le |\mathbb{S}_j| \le e^{nh(\frac{j}{n})} \sqrt{\frac{n}{2\pi j(n-j)}} \tag{2.6}$$

2.2. Asymptotics of Krawtchouk polynomials. Krawtchouk polynomials are defined as Fourier transforms of Hamming spheres:

$$K_j(x) \triangleq \widehat{1_{\mathbb{S}_j}}(x) = \sum_{k=0}^n (-1)^k \binom{|x|}{k}\binom{n-|x|}{j-k} \tag{2.7}$$

Since $K_j(x)$ only depends on x through its Hamming weight $|x|$, we will abuse notation and write $K_j(2)$ to mean value of K_j at a point with weight 2, etc.

Some useful properties of K_j, cf. [18]:

$$K_j(x) = (-1)^j K_j(n-x) \tag{2.8}$$

$$K_j(x) = (-1)^x K_{n-j}(x) \tag{2.9}$$

$$\frac{K_j(x)}{K_j(0)} = \frac{K_x(j)}{K_x(0)} \tag{2.10}$$

$$K_j(0) = \|K_j\|_2^2 = |\mathbb{S}_j| = \binom{n}{j}\,, \tag{2.11}$$

$$K_j(x) = \sum_{|v|=j} \chi_v(x) \tag{2.12}$$

It is also well-known that $K_j(x)$ has j simple real roots. For $j \le n/2$ all of them are in the following interval, see [18, eq. (71)]:

$$\frac{n}{2} - \sqrt{j(n-j)} \le x \le \frac{n}{2} + \sqrt{j(n-j)}\,.$$

For large n the above bounds become tight, so that for $j = \delta n$ the location of the first root is at roughly

$$\xi_{crit}(\delta) \triangleq \frac{1}{2} - \sqrt{\delta(1-\delta)}\,.$$

The following gives a convenient non-asymptotic estimate of the magnitude of $K_j(x)$:

Lemma 2.1. *For all $x, j = 0, \ldots, n$ we have*

$$|K_j(x)| \le e^{nE_{j/n}(x/n)}\,, \tag{2.13}$$

where the function $E_\delta(\xi) = E_{1-\delta}(\xi)$ and for $\delta \in [0, 1/2]$:

$$E_\delta(\xi) = \begin{cases} \frac{1}{2}\left(h(\delta) + \ln 2 - h(\xi)\right)\,, & \xi_{crit}(\delta) \le \xi \le 1 - \xi_{crit}(\delta) \\ \phi(\xi, \omega)\,, & \xi = \frac{1}{2}(1 - (1-\delta)\omega - \delta\omega^{-1})\,, \end{cases} \tag{2.14}$$

where in the second case ω ranges in

$$\omega \in \left[-\sqrt{\frac{\delta}{1-\delta}}, -\frac{\delta}{1-\delta}\right] \cup \left[\frac{\delta}{1-\delta}, \sqrt{\frac{\delta}{1-\delta}}\right]$$

and

$$\phi(\xi, \omega) \triangleq \xi \ln|1-\omega| + (1-\xi)\ln|1+\omega| - \delta \ln|\omega|\,. \tag{2.15}$$

Remark: Exponent $E_\xi(\delta)$ was derived in [15] for $\xi \le \xi_{crit}(\delta)$. Subsequently, a refined asymptotic expansion for all $\xi \in [0, 1]$ was found in [13]:

$$K_{\delta n}(\xi n) = \frac{O(1)}{\sqrt{n}} e^{nE_\delta(\xi)}\,, \tag{2.16}$$

where the $O(1)$ term is $\theta(1)$ for $\xi \le \xi_{crit}$, while for $\xi \in [\xi_{crit}, 1/2]$ the factor $O(1)$ is oscillating and may reduce the exponent for a few integer points $x \in [\xi_{crit} n, (1 - \xi_{crit})n]$, which are close to one of the roots of $K_j(\cdot)$.

Proof. Following [13][4] we have

$$K_j(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}} (1-z)^x (1+z)^{n-x} z^{-j} \frac{dz}{z}\,, \tag{2.17}$$

where integration is over an arbitrary circle $\mathcal{C}$ with center at $z = 0$. The derivative of the function $(1-z)^x(1+z)^{n-x}z^{-j}$ is zero when

$$n - 2x = (n-j)z + jz^{-1}\,. \tag{2.18}$$

[4]Note that $K_j(\cdot)$ in [13] corresponds to $(-1)^j K_j(\cdot)$ in this paper.

Due to (2.9) it is sufficient to consider $j \le n/2$. Among the two solutions of (2.18) denote by ω the unique one with smallest $|z|$ and $\Im(z) \ge 0$. Set, for convenience

$$\xi = x/n, \qquad \delta = j/n \in [0, 1/2]$$

and note that we have the following relation between ω and ξ

$$\omega = \frac{1}{2(1-\delta)}\left(1 - 2\xi - \operatorname{sgn}(1-2\xi)\cdot\sqrt{(1-2\xi)^2 - 1 + (1-2\delta)^2}\right) \tag{2.19}$$

$$1 - 2\xi = (1-\delta)\omega + \frac{\delta}{\omega}\,. \tag{2.20}$$

As ξ ranges from 0 to 1 the saddle point ω traverses the path

$$\omega : \frac{\delta}{1-\delta} \to \sqrt{\frac{\delta}{1-\delta}} \to -\sqrt{\frac{\delta}{1-\delta}} \to -\frac{\delta}{1-\delta}\,,$$

where the middle segment is along the arc $e^{i\phi}\sqrt{\frac{\delta}{1-\delta}}, \phi \in [0, \pi]$; Corresponding to these corner points ξ ranges as follows

$$\xi : 0 \to \xi_{crit} \to 1 - \xi_{crit} \to 1\,.$$

It is more convenient to reparameterize the answer in terms of location of the saddle-point ω. If we take $\mathcal{C}$ to be the circle passing through ω, then as shown in [13, (3.4) and paragraph after (3.19)] the maximum

$$\max_{z\in\mathcal{C}} \left|(1-z)^x (1+z)^{n-x} z^{-j}\right|$$

is attained at $z = \omega$ and is equal to $e^{nE_\delta(\xi)}$, where

$$E_\delta(\xi) = \phi(\xi, \omega)\,, \tag{2.21}$$

and ξ is a function of ω defined via (2.20). Thus, upper-bounding the integrand $\{\cdot\}$ in (2.17) by the maximal value and noting that for any circle

$$\oint_{\mathcal{C}} \left|\frac{dz}{z}\right| \le 2\pi$$

we conclude that (2.13) holds.

It remains to show the simplified expression in (2.14) for $\xi \in [\xi_{crit}, 1 - \xi_{crit}]$. To that end, notice that such ξ corresponds to

$$\omega = e^{i\phi}\sqrt{\frac{\delta}{1-\delta}}, \qquad \phi \in [0, \pi]\,.$$

Substituting this ω into (2.21) we see that (2.14) is equivalent to

$$\xi \ln \frac{|1-\omega|}{\sqrt{\xi}} + (1-\xi)\ln\frac{|1+\omega|}{\sqrt{1-\xi}} = \frac{1}{2}\ln\frac{2}{1-\delta}\,. \tag{2.22}$$

But for ω on the arc we have

$$\frac{|1-\omega|}{\sqrt{\xi}} = \frac{|1+\omega|}{\sqrt{1-\xi}} = \sqrt{\frac{2}{1-\delta}}\,,$$

thus verifying (2.22) and (2.14). □

Some of the properties of $E_\delta(\xi)$ are summarized below (see Fig. 2.1 for an illustration):

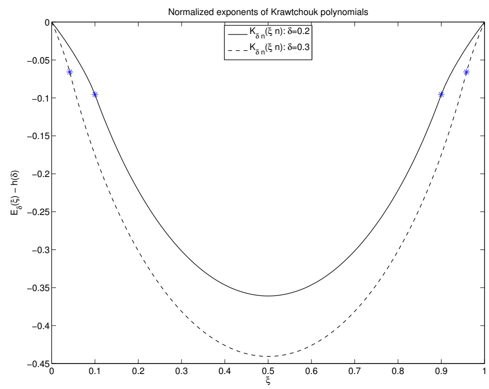

Fig. 2.1. *The exponent of* $\frac{K_{\delta n}(\xi n)}{K_{\delta n}(0)}$ *is equal to* $E_\delta(\xi) - h(\delta)$. *The figure compares these exponents for two values of* δ. *Asterisks mark the interval* $[\xi_{crit}, 1-\xi_{crit}]$ *containing all the roots of* $K_{\delta n}(\cdot)$. *In this interval* $K_{\delta n}(\cdot)$ *is oscillatory.*

.

1. $(\delta, \xi) \mapsto E_\delta(\xi)$ is continuous on $[0,1]\times[0,1]$ and has two symmetries: $E_\delta(\xi) = E_{1-\delta}(\xi)$, $E_\delta(\xi) = E_\delta(1-\xi)$.
2. $E_\delta(0) = E_\delta(1) = h(\delta)$, $E_\delta(1/2) = h(\delta)/2$
3. $E_{1/2}(\xi) = \ln 2 - h(\xi)/2$
4. $E_\delta(\xi) = h(\delta) - h(\xi) + E_\xi(\delta)$
5. $\xi \mapsto E_\delta(\xi)$ is monotonically decreasing on $[0, 1/2]$ and has continuous derivative on $[0,1]$.
6. $\delta \mapsto E_\delta(\xi)$ is monotonically increasing on $[0, 1/2]$.
7. $\delta \mapsto E_\delta(\xi) - h(\delta)$ is monotonically decreasing on $[0, 1/2]$.
8. For fixed δ and all $\xi \le \xi_{crit}(\delta)$ we have

$$E_\delta(\xi) \le \xi \ln(1-2\delta) + h(\delta)\,. \tag{2.23}$$

We will also need a more refined estimate for $K_j(x)$ when x is small:

Lemma 2.2. *For* $j \le n/2$ *and* $0 \le x \le n\xi_{crit}(j/n) = n/2 - \sqrt{j(n-j)}$ *we have*

$$\frac{K_j(x)}{K_j(0)} \le \left(1 - \frac{2j}{n}\right)^x\,. \tag{2.24}$$

Remark: With the additional factor $O(\sqrt{n})$ the estimate (2.24) follows from (2.13). Lemma 2.2 establishes the crucial relation between spectra of operators N_δ and S_δ powering Theorem 1.1.

Proof. In the mentioned range of x the polynomial $K_j(x)$ is monotonically decreasing since $K_j(0) > 0$ and all roots are to the right of x. Hence, for any $x + 1 \le n\xi_{crit}(j/n)$ we have

$$0 \le \frac{K_j(x+1)}{K_j(x)} < 1\,. \tag{2.25}$$

On the other hand, e.g. [18, (15)], $K_j(\cdot)$ satisfies a three-term recurrence

$$(n-x)K_j(x+1) - (n-2j)K_j(x) + xK_j(x-1) = 0\,. \tag{2.26}$$

Dividing by $nK_j(x)$ we get

$$\frac{K_j(x+1)}{K_j(x)} = \left(1 - \frac{2j}{n}\right) - \frac{x}{n}\left(\frac{K_j(x-1)}{K_j(x)} - \frac{K_j(x+1)}{K_j(x)}\right) \tag{2.27}$$
$$\leq \left(1 - \frac{2j}{n}\right), \tag{2.28}$$

where (2.28) is from (2.25). The (2.24) then follows by iterating (2.28). □

Note that for $j \approx \frac{n}{2}$ conditions of Lemma 2.2 are not satisfied for any x. For such j we prove another (somewhat loose) estimate below.

Lemma 2.3. *Fix arbitrary $\theta_1 \in (0, 1/2)$. Then for all x, j such that*

$$n - 2j \leq n\theta_1, \tag{2.29}$$
$$0 \leq x \leq 1 + \frac{\theta_1}{1+\theta_1^2}(n\theta_1 - (n-2j)) \tag{2.30}$$

we have

$$\left|\frac{K_j(x)}{K_j(0)}\right| \leq \theta_1^x\,. \tag{2.31}$$

Proof. Denote $\theta = 1 - 2\frac{j}{n} \leq \theta_1$. Clearly (2.31) holds for $x = 0$. From (2.27) and (2.29) it also holds for $x = 1$. Let the induction hypothesis be that (2.31) holds for $x \leq x_0$. Then

$$\left|\frac{K_j(x_0+1)}{K_j(0)}\right| = \left|\frac{n\theta}{n-x_0}\frac{K_j(x_0)}{K_j(0)} - \frac{x_0}{n-x_0}\frac{K_j(x_0-1)}{K_j(0)}\right| \tag{2.32}$$
$$\leq \frac{n\theta}{n-x_0}\theta_1^{x_0} + \frac{x_0}{n-x_0}\theta_1^{x_0-1}, \tag{2.33}$$

where (2.32) is from (2.26) and (2.33) is by induction hypothesis. Finally, it is easy to see that whenever $n - x_0 > 0$ it holds that

$$x_0 \leq n\frac{\theta_1}{1+\theta_1^2}(\theta_1 - \theta) \qquad \iff \qquad \frac{n\theta}{n-x_0}\theta_1^{x_0} + \frac{x_0}{n-x_0}\theta_1^{x_0-1} \leq \theta_1^{x_0+1},$$

which concludes the proof of (2.31) for $x = x_0 + 1$. □

On the other extreme, for small values of j we can extend Lemma 2.2 to the whole range $0 \leq x \leq \frac{n}{2}$:

Lemma 2.4. *There exist $C_1 \geq 1$ and $\delta_0 \in (0, 1)$ such that for all $0 \leq j \leq \delta_0 n$ we have*

$$\left|\frac{K_j(x)}{K_j(0)}\right| \leq C_1 \cdot \left(1 - \frac{2j}{n}\right)^x, \qquad 0 \leq x \leq \frac{n}{2}.$$

Remark: In fact, one can show the statement with $C_1 = 1$ and $\delta_0 = 0.16$. This is achieved by carefully following constants in the analysis and showing that $\max_{\delta\in[0,\delta_0]}$ over the right-hand side of (2.34) is ≤ 1 for $n \geq 300$. For smaller n the statement is checkable numerically, e.g. by running the recurrence (2.26) for normalized functions $\frac{K_j(x)}{K_j(0)(1-2\delta)^x}$ (to avoid large numbers).

Proof. For $j = 0$ the inequality is trivial. For $x \le \xi_{crit}(j/n)$ it follows from Lemma 2.2. Thus, it is sufficient to consider $x \ge \xi_{crit}(j/n)$, $j \ge 1$. Denote $\delta = j/n$. Then from Lemma 2.1 and (2.6) we have for all $n \ge 1$:

$$\left|\frac{K_j(x)}{K_j(0)\,(1-2\delta)^x}\right| \le \sqrt{8(1-\delta)} \cdot e^{n(f(\delta)-\frac{1}{2}h(\delta))}\sqrt{n\delta}\,, \tag{2.34}$$

where

$$f(\delta) = \max_{\xi\in[\xi_{crit}(\delta),1/2]} \frac{1}{2}(\ln 2 - h(\xi)) - \xi\ln(1-2\delta)\,.$$

From convexity of the function under maximization, we conclude

$$f(\delta) = \frac{\ln 2}{2} - \frac{1}{2}\min\left(h(\xi_{crit}(\delta)) + 2\xi_{crit}(\delta)\ln(1-2\delta), \ln 2(1-2\delta)\right)\,.$$

Taking derivative at $\delta = 0$ we conclude that for some $\delta_0' > 0$ we have

$$h(\xi_{crit}(\delta)) + 2\xi_{crit}(\delta)\ln(1-2\delta) \le \ln 2(1-2\delta)\,, \qquad \forall\delta\in[0,\delta_0']\,.$$

Consequently, for such δ

$$f(\delta) = \frac{1}{2}\left(\ln 2 - h(\xi_{crit}(\delta))\right) - \xi_{crit}(\delta)\ln(1-2\delta)\,.$$

Evidently, f is continuously differentiable and

$$f(\delta) = 2\delta + o(\delta), \qquad \delta\to 0\,.$$

Therefore for some $\delta_0 \in (0,\delta_0']$ we must have

$$f(\delta) - \frac{1}{2}h(\delta) < 0\,, \qquad \forall\delta\in(0,\delta_0]\,.$$

The statement of the Lemma then follows with $C_1 = \max(1, \sqrt{8}C_1')$, where C_1' is the finite supremum found in the following Lemma. □

Lemma 2.5. *Let $\alpha, \delta_0, C > 0$ and f – a continuous function on $[0,\delta_0]$ with $f(0) = 0$, derivative (one-sided at 0) bounded by C and satisfying*

$$f(\delta) - \alpha h(\delta) < 0\,, \qquad \forall\delta\in(0,\delta_0]\,. \tag{2.35}$$

Then

$$\sup_{n\ge 1}\max_{\delta\in[0,\delta_0]} e^{n(f(\delta)-\alpha h(\delta))}\sqrt{n\delta} < \infty\,. \tag{2.36}$$

Proof. Under conditions of the theorem there exists $0 < \delta_1 < \delta_0$ such that

$$f(\delta) \le \frac{\alpha}{2}h(\delta)\,, \qquad \forall\delta\in[0,\delta_1]\,.$$

Thus we have

$$\max_{\delta\in[0,\delta_1]} n(f(\delta) - \alpha h(\delta)) + \frac{1}{2}\ln(\delta n) \le \frac{1}{2}\max_{\delta\in[0,\delta_1]} -\alpha n h(\delta) + \ln\delta n \tag{2.37}$$

$$\le \frac{1}{2}\max_{\delta\in[0,\delta_1]} \alpha n\delta\ln\delta + \ln(\delta n)\,. \tag{2.38}$$

Without loss of generality we may assume $\delta_1 < \frac{1}{e}$ and $n > \frac{e^2}{\alpha}$. In this case, maximization in (2.38) is attained at $\delta^* \in (0, \frac{1}{n\alpha})$. Consequently, upper-bounding the first term by zero and second by $\ln(\frac{1}{n\alpha} \cdot n)$ we get

$$\frac{1}{2} \max_{\delta \in [0,\delta_1]} \alpha n \delta \ln \delta + \ln(\delta n) \le \frac{-\ln \alpha}{2}.$$

On the other hand, from (2.35) and continuity we get

$$\max_{\delta \in [\delta_1, \delta_0]} f(\delta) - \alpha h(\delta) = -C_2 < 0.$$

Therefore, putting both bounds together

$$\max_{n \ge 1, \delta \in [0,\delta_0]} e^{n(f(\delta) - \alpha h(\delta))} \le \max\left(\frac{1}{\sqrt{\alpha}}, \sup_n \sqrt{\delta_0 n} e^{-C_2 n}\right) < \infty.$$

□

Remark: Reference [2] establishes the following estimate:

$$\left|\frac{1}{2}\left(\frac{K_c(n)}{K_c(0)} + \frac{K_{c-1}(n)}{K_{c-1}(0)}\right) - \left(1 - \frac{2c}{n}\right)^n\right| \le O(\max(n^{-\frac{1}{5}}, \frac{n}{c^2} \log^2 \frac{c^2}{n})),$$

for all $e^2\sqrt{n}c \le \frac{n}{2}$. This result is incomparable to ours: it bounds deviation from $(1 - \frac{2c}{n})^n$ on both sides, albeit much less precisely.

Finally, for illustrating tightness of the bounds in the next section we will need the following Lemma, proved in the Appendix. It is not used in the proof of Theorem 1.1.

Lemma 2.6. *L_p norms of Krawtchouk polynomials are given asymptotically by the following parametric formula: Let $\omega \in [0,1]$ then for $p \ge 2$*

$$\|K_{\lfloor \delta n \rfloor}\|_p = \exp\left\{n\left(\frac{h(\xi) - \ln 2}{p} + \phi(\xi,\omega)\right) + O(\log n)\right\}, \qquad n \to \infty \tag{2.39}$$

$$c = \frac{(1+\omega)^p - (1-\omega)^p}{(1+\omega)^p + (1-\omega)^p} \tag{2.40}$$

$$\xi = \frac{1-c}{2} = \frac{1}{2}(1 - (1-\delta)\omega - \delta\omega^{-1}) \tag{2.41}$$

$$\delta = \frac{c\omega - \omega^2}{1 - \omega^2} \tag{2.42}$$

and $\phi(\xi,\omega)$ is given by (2.15). For $p \le 2$ we have

$$\|K_{\lfloor \delta n \rfloor}\|_p = \exp\left\{\frac{n}{2} h(\delta) + O(\log n)\right\}, \tag{2.43}$$

as $n \to \infty$ along a subsequence such that both $\lfloor \delta n \rfloor$ and n are even.

2.3. Norms of Fourier projection operators. The Fourier projection operators Π_a are defined as

$$\widehat{\Pi_a f} \triangleq \hat{f} \cdot 1_{\mathbb{S}_a} \qquad a = 0, 1, \ldots, n, \tag{2.44}$$

or, equivalently,

$$\Pi_a f \triangleq 2^{-n} f * K_a.$$

On the other hand from Young's inequality (2.2) we have for any convolution operator:

$$\|\phi * (\cdot)\|_{1\to 2} = 2^n \|\phi\|_2 .$$

Thus we have

$$\|\Pi_a\|_{1\to 2} = \sqrt{\binom{n}{a}} . \tag{2.45}$$

Also, we note that

$$\|\Pi_a\|_{p\to q} = \|\Pi_{n-a}\|_{p\to q} ,$$

and thus we only consider $a \le \frac{n}{2}$ below.

Estimates for other $L_p \to L_2$ follow from Bonami-Gross inequality (1.4) and complex interpolation:

LEMMA 2.7. *For any $1 \le p \le 2$ and $0 \le a = n\delta \le \frac{n}{2}$ we have*

$$\|\Pi_a\|_{p\to 2} \le \begin{cases} (p-1)^{-\frac{a}{2}}, & p > p^* , \\ (p^*-1)^{-\frac{(1-s)a}{2}} \binom{n}{a}^{\frac{s}{p}-\frac{s}{2}}, & \frac{1}{p} = \frac{1-s}{p^*} + s, 0 \le s \le 1 \end{cases} \tag{2.46}$$

where $p^* = p^*(a) = 2$ if $\frac{h(\delta)}{\delta} \le 2$, and otherwise $p^* \in (1,2)$ is a solution of

$$p^* - \ln(p^* - 1) = \frac{h(\delta)}{\delta} .$$

We also have two weaker bounds

$$\|\Pi_a\|_{p\to 2} \le (p-1)^{-\frac{a}{2}} , \tag{2.47}$$

$$\|\Pi_a\|_{p\to 2} \le \binom{n}{a}^{\frac{1}{p}-\frac{1}{2}} . \tag{2.48}$$

Remark: The estimate (2.47) has been the basis of Kahn-Kalai-Linial results [14], so we refer to (2.47) as KKL bound. Note that $p^*(a) = 2$ corresponds to $a > 0.3093n$, and then bound (2.46) coincides with (2.48).

Proof. From Riesz-Thorin interpolation [7, Section VI.10.8], we know that the map $\frac{1}{p} \mapsto \|\Pi_a\|_{p\to 2}$ is log-convex. Thus (2.46) follows from (2.47) and (2.48) by convexification (the value of p^* is chosen to minimize the resulting exponent when $a = \delta n$). Thus, it is sufficient to prove (2.47) and (2.48). The second one again follows from interpolating between (2.45) and $\|\Pi_a\|_{2\to 2} = 1$. For the first one notice that for any τ we have

$$N_\tau \Pi_a = \Pi_a N_\tau = (1-2\tau)^a \Pi_a .$$

And thus from (1.4) with $(1-2\tau)^2 = p-1$ we get

$$\|\Pi_a f\|_2 = |1-2\tau|^{-a} \|\Pi_a N_\tau f\|_2 \le |1-2\tau|^{-a} \|N_\tau f\|_2 \le |1-2\tau|^{-a} \|f\|_p .$$

□

To verify the tightness of our bounds we derive a simple lower bound by considering permutation invariant functions:

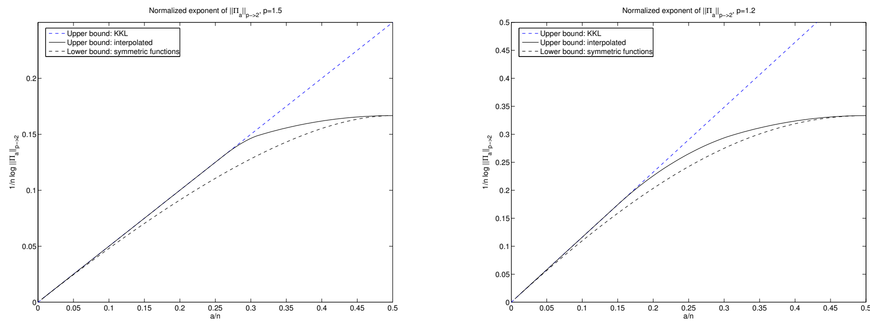

FIG. 2.2. *Exponent of* $\|\Pi_a\|_{p\to 2}$ *as a function of* a *for two values of* p. *Two upper bounds correspond to Kahn-Kalai-Linial* (2.47) *and the interpolated one* (2.46). *The lower bound is given by considering only permutation invariant functions (cf. Lemmas* 2.6 *and* 2.8*).*

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LEMMA 2.8. *For any* $a \in \{0, \dots, n\}$ *and any* $q, p \geq 1$ *we have*

$$\|\Pi_a\|_{p\to q} \geq \frac{\|K_a\|_q \|K_a\|_{p'}}{\|K_a\|_2^2},$$

where $p' = \frac{p}{p-1}$ *is the Hölder conjugate.*

Proof. The lower bound is shown by optimizing over a class of permutation invariant functions

$$f(x) = K_a(x) + \sum_{j\neq a}^{n} c_j K_j(x) \triangleq K_a(x) + \Phi(x),$$

where $\Phi \perp K_a$. Note that

$$\inf_{\Phi\perp K_a} \|f\|_p = \inf_{\Phi\perp K_a} \sup_{g:\|g\|_{p'}\leq 1} (K_a + \Phi, g) \tag{2.49}$$

$$= \inf_{\Phi\perp K_a} \sup_{g-\text{sym.}:\|g\|_{p'}\leq 1} (K_a + \Phi, g) \tag{2.50}$$

$$= \sup_{g-\text{sym.}:\|g\|_{p'}\leq 1} \inf_{\Phi\perp K_a} (K_a + \Phi, g) \tag{2.51}$$

$$= \left(K_a, \frac{K_a}{\|K_a\|_{p'}}\right) = \frac{\|K_a\|_2^2}{\|K_a\|_{p'}}, \tag{2.52}$$

where (2.49) is by duality $(L_p)^* = L_{p'}$, (2.50) states the obvious fact that supremization can be restricted to permutation-symmetric g, (2.51) is by Kneser's minimax theorem [17] (for bi-affine function over $X \times Y$ with X convex-compact, Y convex and f upper semi-continuous on X) and (2.52) is because the inner inf can only be finite if g belongs to the one-dimensional subspace spanned by K_a, i.e. $g = cK_a$ for a suitable c.

Since $\Pi_a(K_a + \Phi) = K_a$ we conclude that

$$\|\Pi_a\|_{p\to q} \geq \frac{\|K_a\|_q}{\inf_{\Phi\perp K_a} \|K_a + \Phi\|_p} = \frac{\|K_a\|_q \|K_a\|_{p'}}{\|K_a\|_2^2}$$

as claimed. □

On Fig. 2.2 we compare the upper and lower bounds on $\|\Pi_a\|_{p\to 2}$ as a ranges from 0 to $n/2$ for two values of p. We note that KKL bound (2.47) is significantly suboptimal for small p and large a. For example, for $a > 0.3093n$ the bound (2.48) is strictly better than KKL.

Before proceeding to the proof of the main result, we need one last estimate relating magnitude of Krawtchouk polynomials (in the oscillating strip) to the norms of projectors Π_a.

LEMMA 2.9. *Fix arbitrary $0 < \delta_0 < \Delta < 1/2$. Then there exist constants $C_1', C_2 > 0$ such that for all $n \ge 1$, all $j \in [\delta_0 n, \Delta n]$ and all*

$$\frac{n}{2} - \sqrt{j(n-j)} \le x \le \frac{n}{2} + \sqrt{j(n-j)}$$

we have

$$\left|\frac{K_j(x)}{K_j(0)}\right| \cdot \|\Pi_j\|_{p(\frac{j}{n})\to 2} \le C_1'\sqrt{n}e^{-C_2 n} \tag{2.53}$$

where $p(\delta) = 1 + (1-2\delta)^2$.

Proof. Let $\xi = \frac{a}{n}$ and $\delta = \frac{j}{n}$. From symmetry, we can and will assume $\xi \le \frac{1}{2}$. Since ξ is restricted to critical strip of Krawtchouk polynomial $K_{\delta n}$ from Lemma 2.1, bound (2.6) and Lemma 2.7 it is sufficient to show

$$\max_{\delta_0\le\delta\le\Delta}\ \max_{\xi:(1-2\xi)^2+(1-2\delta)^2\le 1} \frac{1}{2}(\ln 2 - h(\xi) - h(\delta)) + \pi(p(\delta),\xi) \le -C_2 < 0\,, \tag{2.54}$$

where $p(\delta) = 1 + (1-2\delta)^2$ and

$$\frac{1}{p} \mapsto \pi(p,\xi)$$

is the convexification of the function (cf. Lemma 2.7)

$$\frac{1}{p} \mapsto \min\left\{-\frac{\xi}{2}\ln(p-1), (\frac{1}{p}-\frac{1}{2})h(\xi)\right\}\,. \tag{2.55}$$

To show (2.54) we first change variable δ to $p = p(\delta) = 1 + (1-2\delta)^2$. Set

$$p_0 = 1 + (1-2\Delta)^2\,, \tag{2.56}$$

$$p_1 = 1 + (1-2\delta_0)^2\,. \tag{2.57}$$

Then (2.54) is equivalent to (we also interchange the maxima in ξ and δ):

$$\max_{\xi:(1-2\xi)^2\le 2-p_0}\ \max_{p:p_0\le p\le\min(p_1,2-(1-2\xi)^2)} \eta(\xi,p) + \frac{\ln 2 - h(\xi)}{2} \le -C_2 < 0 \tag{2.58}$$

where

$$\eta(\xi,p) \triangleq \pi(p,\xi) - \frac{1}{2}h\left(\frac{1-\sqrt{p-1}}{2}\right)\,.$$

By construction, $\frac{1}{p} \mapsto \pi(p,\xi)$ is convex. Taking derivatives one can show that $h\left(\frac{1-\sqrt{p-1}}{2}\right)$ is concave in $\frac{1}{p}$. Thus, the maximization over p in (2.58) is applied to a convex function and therefore must be achieved at one of the boundaries. Consequently, to verify (2.58) it is sufficient to show the following three strict inequalities :

$$\max_{\xi:(1-2\xi)^2\le 2-p_0} \eta(\xi,p_0)+\frac{\ln 2-h(\xi)}{2}<0 \tag{2.59}$$

$$\max_{\xi:(1-2\xi)^2\le 2-p_1} \eta(\xi,p_1)+\frac{\ln 2-h(\xi)}{2}<0 \tag{2.60}$$

$$\max_{\xi:2-p_1\le(1-2\xi)^2\le 2-p_0} \eta(\xi,2-(1-2\xi)^2)+\frac{\ln 2-h(\xi)}{2}<0 \tag{2.61}$$

(the maximum value of the three left-hand sides is then taken to be $-C_2$). The first two are verified as follows: From (2.55) we have

$$\pi(p,\xi)\le -\frac{\xi}{2}\ln(p-1)\,.$$

Plugging this upper bound in (2.59) we arrive at the optimization

$$\max_{\xi:(1-2\xi)^2\le 2-p} -\frac{\xi}{2}\ln(p-1)-\frac{1}{2}h(\xi)\,.$$

Equating derivative in ξ to zero, we find solution $\xi^*(p)=1-\frac{1}{p}$. Since for $p>1$ we have $(1-2\xi^*(p))^2<2-p$ this is also the maximizer. Consequently, substituting $\xi=\xi^*(p)$ we get

$$\max_{(1-2\xi)^2\le 2-p}\eta(\xi,p)+\frac{\ln 2-h(\xi)}{2}\le -\frac{\xi^*(p)}{2}\ln(p-1)+\frac{1}{2}\left[\ln 2-h(\xi^*(p))-h\left(\frac{1-\sqrt{p-1}}{2}\right)\right]$$

Function of a single variable p on the right is continuous, non-positive and attains zero only at the endpoints of $p\in[1,2]$. Since both p_0 and p_1 belong to the interior of $[1,2]$, this completes the proof of (2.59) and (2.60).

To show (2.61) we apply the bound in (2.55) (without convexification):

$$\max_{\xi}\eta(\xi,2-(1-2\xi)^2)+\frac{\ln 2-h(\xi)}{2}\le \max_{\xi}\frac{1}{2}f(\xi) \tag{2.62}$$

where maximization is over

$$2-p_1\le(1-2\xi)^2\le 2-p_0 \tag{2.63}$$

and $f(\xi)$ is defined as

$$\begin{aligned} f(\xi) &\triangleq \min\left\{\left(\frac{(1-2\xi)^2}{2-(1-2\xi)^2}\right)h(\xi),-\xi\ln(4\xi(1-\xi))\right\}\\ &\quad+\ln 2-h(\xi)-h\left(\frac{1}{2}-\sqrt{\xi(1-\xi)}\right) \end{aligned} \tag{2.64}$$

The minimum in this expression selects the first term for $\xi\in[\xi^*,1/2]$ and second term otherwise, where $\xi^*\approx 0.3082$ is the solution of

$$8\xi^2(1-\xi)\ln\xi+(2\xi-(1-2\xi)^2)\ln(1-\xi)+2\xi(2-(1-2\xi)^2)\ln 2=0$$

in the interior of $(0,1/2)$. Furthermore, function in (2.64) is non-positive, continuous and attains zero only at $\xi=0,\frac{1}{2}$ both of which are excluded by the constraints (2.63). Thus (2.61) holds. □

3. Proof of Theorem 1.1. Denote the boundary of F as

$$p(\delta) \triangleq 1 + (1-2\delta)^2 .$$

Note that every compact subset K' of F is contained in $F \cap \{p \geq p_0\}$ for sufficiently small p_0 and in turn in some

$$K = (F \cap \{\delta : |1-2\delta| \geq \theta\}) \cup \{(\delta, p) : |1-2\delta| \leq \theta, p \geq p_0\} \tag{3.1}$$

for sufficiently small θ. In particular, we may choose θ so small that $p_0 > 1+\theta^2$. Next note that

$$(f * 1_{\mathbb{S}_{n-a}})(x) = (f * 1_{\mathbb{S}_a})(\bar{x})$$

and thus estimates for S_δ and $S_{1-\delta}$ coincide asymptotically. Due to this symmetry and thanks to the monotonicity (2.1) of norms, to prove the theorem it is sufficient to prove the following pair of statements, corresponding to the boundary of K:

S1. (critical estimate for $\delta < 1/2$) For each δ there is C_δ such that for all $n \geq 1$ and all functions f we have

$$\|S_\delta f\|_2 \leq C_\delta \|f\|_{p(\delta)} , \tag{3.2}$$

and function $\delta \mapsto C_\delta$ is bounded on each $[0, \Delta], \Delta < 1/2$.

S2. (subcritical estimate around $\delta = 1/2$) For any $p > 1$ and sufficiently small θ (in particular, $p > 1 + \theta^2$) there is C such that for all $\delta \in [(1-\theta)/2, 1/2]$, $n \geq 1$ and functions f we have

$$\|S_\delta f\|_2 \leq C \|f\|_p \tag{3.3}$$

First we show S1. In accordance with (2.7)

$$\|S_\delta f\|_2^2 = \sum_{a=0}^{n} \left| \frac{K_{\delta n}(a)}{K_{\delta n}(0)} \right|^2 \|f_a\|_2^2 , \tag{3.4}$$

where we denoted

$$f_a \triangleq \Pi_a f .$$

The scheme of our proof is illustrated by Fig. 3.1:

1. First, we show that summation in (3.4) can be truncated to $a \leq \frac{n}{2}$.
2. Second, we show that for small values of δ eigenvalues of S_δ are upper-bounded by a constant multiple of eigenvalues of N_δ defined in (1.3). This is the content of Lemma 2.4.
3. Third, for larger values of δ we show that although eigenvalues of S_δ can be exponentially larger than those of N_δ, such eigenvalues correspond to large a for which $\frac{\|f_a\|_2}{\|f\|_p}$ is exponentially smaller.

For the first step note that any f can be written as

$$f = f_{even} + f_{odd} ,$$

where each of the summands is supported on vectors $x \in \mathbb{F}_2^n$ of even/odd weight. Note that $S_\delta f_{even}$ and $S_\delta f_{odd}$ are also of opposite parity. Thus,

$$\|S_\delta f\|_2^2 = \|S_\delta f_{even}\|_2^2 + \|S_\delta f_{odd}\|_2^2 .$$

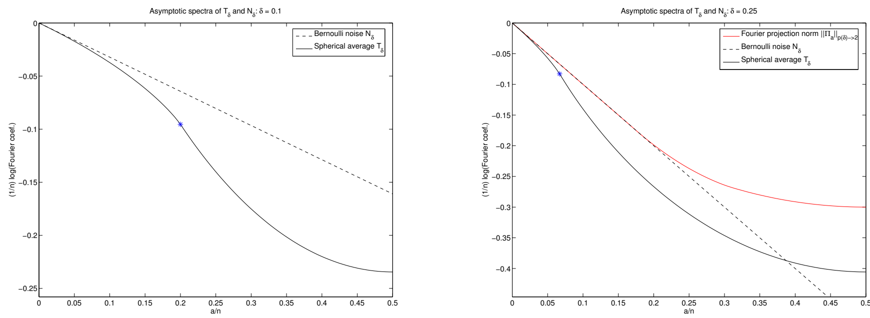

FIG. 3.1. *Comparison of exponents of a-th eigenvalue of S_δ and N_δ. For larger δ we also show the negative of the exponent of $\|\Pi_a\|_{p(\delta)\to 2}$, $p(\delta) = 1 + (1-2\delta)^2$. As before asterisks denote the critical value $\xi_{crit}(\delta)$, i.e. the smallest root of Krawtchouk polynomial $K_{\delta n}(\cdot)$.*

On the other hand, we have

$$\left(\|f_{even}\|_p^2 + \|f_{odd}\|_p^2\right)^{\frac{1}{2}} \le \left\|\sqrt{f_{even}^2 + f_{odd}^2}\right\|_p \tag{3.5}$$

$$= \|f\|_p\,, \tag{3.6}$$

where (3.5) is from Minkowski's inequality and (3.6) is because the supports of f_{even} and f_{odd} are disjoint. Thus, if (3.2) is established for both odd and even functions then (3.2) follows for all functions with the same constant C.

Note that for both odd and even functions we have

$$|\hat{f}(\omega)| = |\pm \hat{f}(\bar{\omega})| = |\hat{f}(\bar{\omega})|\,.$$

and for any such f from (3.4) and (2.8) we get

$$\|S_\delta f\|_2^2 \le 2 \sum_{0\le a\le n/2} \left|\frac{K_{\delta n}(a)}{K_{\delta n}(0)}\right|^2 \|f_a\|_2^2\,. \tag{3.7}$$

In the remaining we show that (3.7) is upper-bounded by $C\|f\|_{p(\delta)}$ uniformly in f and $\delta \le \Delta < 1/2$. For all $\delta \in [0, \delta_0]$ from Lemma 2.4 we have

$$\|S_\delta f\|_2^2 \le 2C_1^2 \sum_{0\le a\le n/2} (1-2\delta)^{2a} \|f_a\|_2^2 \tag{3.8}$$

$$= 2C_1^2 \|N_\delta f\|_2^2 \tag{3.9}$$

$$\le 2C_1^2 \|f\|_{p(\delta)}^2\,, \tag{3.10}$$

where the last step follows from Bonami-Gross (1.4). For $\delta \in [\delta_0, \Delta]$ we have from Lemma 2.2

$$\left|\frac{K_{\delta n}(a)}{K_{\delta n}(0)}\right| \le (1-2\delta)^a\,, \qquad 0 \le a \le n\xi_{crit}(\delta)\,. \tag{3.11}$$

On the other hand, for $a \in [n\xi_{crit}(\delta), n/2]$ we have the estimate given by Lemma 2.9. Putting together (3.11) and (2.53) we get similar to (3.10):

$$\|S_\delta f\|_2^2 \le 2C_1^2 \|N_\delta f\|_2^2 + 2\|f\|_{p(\delta)}^2 \sum_{a\in[n\xi_{crit}(\delta),n/2]} (C_1')^2 n e^{-2C_2 n} \tag{3.12}$$

$$\le 2C_1^2 \|N_\delta f\|_2^2 + 2(C_1')^2 \|f\|_{p(\delta)}^2 \cdot n^2 e^{-2C_2 n} \tag{3.13}$$

$$\le 2(C_1^2 + (C_1')^2 n^2 e^{-2C_2 n}) \|f\|_{p(\delta)}^2 \,, \tag{3.14}$$

where in the last step we applied (1.4). Since constants C_1' and C_2 only depend on δ_0 and Δ we finish the proof of (3.2) and of statement S1.

We proceed to statement S2. Showing (3.3) is significantly simpler since $p > p(\delta)$ this time. Take $\theta_1 = \sqrt{p-1} > \theta$ and $\delta_1 = \frac{1-\theta_1}{2}$. Let

$$\xi_1 \triangleq \frac{\theta_1}{1+\theta_1^2}(\theta_1 - \theta)$$

and assume that θ is so small that $\xi_{crit}(\delta) < \xi_1$ for all $\delta \in [\frac{1-\theta}{2}, \frac{1}{2}]$. Then, on one hand, for all $0 \le a \le n\xi_1$ and all $\delta \in [\frac{1-\theta}{2}, \frac{1}{2}]$ we have from Lemma 2.3:

$$\left|\frac{K_j(a)}{K_j(0)}\right| \le (1-2\delta_1)^a \,.$$

Thus, from (1.4) we get

$$\sum_{a\in[0,n\xi_1]} \left|\frac{K_j(a)}{K_j(0)}\right|^2 \|f_a\|_2^2 \le \|N_{\delta_1} f\|_2^2 \le \|f\|_p^2 \,. \tag{3.15}$$

On the other hand, for $a > n\xi_1$ we have for some $C_1, E > 0$:

$$\left|\frac{K_j(a)}{K_j(0)}\right| \cdot \frac{\|f_a\|_2}{\|f\|_p} \le C_1 \sqrt{n} e^{-nE} \,, \qquad \forall a \in [n\xi_1, \frac{n}{2}] \tag{3.16}$$

Indeed, from Lemma 2.1 and (2.48) the exponent of the left-hand side of (3.16) is upper-bounded by

$$\frac{1}{2}(\ln 2 - h(\delta)) + \left(\frac{1}{p} - 1\right) h(\xi) \,, \qquad \xi \triangleq \frac{a}{n},\ \delta \triangleq \frac{j}{n}$$

since $\xi \in (\xi_{crit}(\delta), 1/2]$. The largest value is attained when $\delta = \frac{1-\theta}{2}$ and $\xi = \xi_1$, yielding

$$\frac{1}{2}(\ln 2 - h(\delta)) + \left(\frac{1}{p} - 1\right) h(\xi) \le \frac{1}{2}\left(\ln 2 - h(\frac{1-\theta}{2})\right) + \left(\frac{1}{p} - 1\right) h\left(\frac{\theta_1(\theta_1-\theta)}{1+\theta_1^2}\right) .$$

Since $p > 1$ as $\theta \to 0$ the function on the right-hand side becomes negative. Thus the exponent of left-hand side in (3.16) is negative for sufficiently small θ.

Estimating the sum in (3.7) via (3.15) and (3.16) we get similar to (3.14) that

$$\|S_\delta f\|_2^2 \le 2(1 + (C_1)^2 n^2 e^{-2En}) \|f\|_p^2 \qquad \forall \delta \in [\frac{1-\theta}{2}, \frac{1}{2}] \,.$$

This completes the proof of (3.3) and statement S2.

We proceed to lower bounds on $\|S_\delta\|_{p\to 2}$. To show (1.9) consider function

$$f(x) = \prod_{j=1}^{n}(1+\epsilon\chi_j) = \sum_{t=0}^{n}(1+\epsilon)^{n-t}(1-\epsilon)^t 1_{\mathbb{S}_t} = \sum_{k=0}^{n}\epsilon^k K_k(x)\,.$$

On one hand,

$$\|f\|_p = \left(\frac{(1+\epsilon)^p}{2}+\frac{(1-\epsilon)^p}{2}\right)^{\frac{n}{p}} \tag{3.17}$$

$$= e^{n\frac{p-1}{2}\epsilon^2+o(\epsilon^2)}\,, \quad \epsilon\to 0 \tag{3.18}$$

On the other hand, from Lemma 2.1 and (2.16) we have

$$\|S_\delta f\|_2^2 = \sum_{a=0}^{n} e^{2n\left(E_\delta(\frac{a}{n})-h(\delta)+\frac{a}{n}\ln\epsilon+\frac{1}{2}h(\frac{a}{n})\right)+o(n)}\,, \tag{3.19}$$

where we also used

$$\|f_a\|_2 = \epsilon^a\binom{n}{a}^{\frac{1}{2}} = e^{a\ln\epsilon+nh(\frac{a}{n})+o(n)}\,.$$

For convenience, set $\xi=\frac{a}{n}$. Then it is not hard to show from (2.14) that

$$E_\delta(\xi)-h(\delta) = \xi\ln(1-2\delta)+o(\xi)\,.$$

Then setting $\xi=\epsilon^2(1-2\delta)^2$ we find that

$$E_\delta(\xi)-h(\delta)+\xi\ln\epsilon+\frac{1}{2}h(\xi) = \frac{(1-2\delta)^2}{2}\epsilon^2+o(\epsilon^2)\,, \qquad \epsilon\to 0$$

Thus from (3.19) and (3.18) we get

$$\liminf_{n\to\infty}\frac{1}{n}\ln\frac{\|S_\delta f\|_2}{\|f\|_p} \ge \frac{(1-2\delta)^2-(p-1)}{2}\epsilon^2+o(\epsilon^2)\,.$$

Evidently, for $p<1+(1-2\delta)^2$ the norm $\|S_\delta\|_{p\to 2}$ grows exponentially in dimension.

Finally, estimate (1.10) follows from Young's inequality (2.2):

$$\|S_{1/2}f\|_2 \le 2^n\|f\|_1\frac{\|1_{\mathbb{S}_{n/2}}\|_2}{|\mathbb{S}_{n/2}|} \tag{3.20}$$

$$= 2^n\cdot\left(2^{-n/2}\binom{n}{\lfloor n/2\rfloor}^{-1/2}\right)\|f\|_1 \tag{3.21}$$

$$= (1+o(1))\left(\frac{\pi n}{2}\right)^{\frac{1}{4}}\|f\|_1 \tag{3.22}$$

This upper-bound is tight as $f(x)=1\{x=0\}$ shows.

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Appendix A. Proof of Lemma 2.6.

Proof. Let $j = \lfloor \delta n \rfloor$ and note that from Plancherel we have

$$\|K_j\|_2 = \sqrt{|\mathbb{S}_j|} = \exp\left\{\frac{n}{2}h(\delta) + O(\log n)\right\} . \tag{A.1}$$

Consequently, we only consider $p \neq 2$ from now on.

The lemma is shown by analyzing with exponential precision the expression

$$\|K_j\|_p^p = \sum_{a=0}^{n} 2^{-n}\binom{n}{a}|K_j(a)|^p , \tag{A.2}$$

so that

$$nE(p,\delta) \le \ln \|K_j\|_p^p \le \ln(n+1) + nE(p,\delta) ,$$

where

$$E(p,\delta) \triangleq \frac{1}{n}\max_{a\le n/2} \ln\binom{n}{a} - n\ln 2 + p\ln|K_j(a)| , \tag{A.3}$$

and we used the symmetry to restrict analysis to $a \le n/2$. We will show below that for $p > 2$ the term exponentially dominating this sum occurs at $a \le n\xi_{crit}(j/n)$, while for $p < 2$ the dominating term is at $a = n/2$.

First, consider $p > 2$. From Lemma 2.1, we have

$$E(p,\delta) \le \max_{0\le\xi\le 1/2} h(\xi) - \ln 2 + pE_\delta(\xi) + O\left(\frac{\log n}{n}\right) . \tag{A.4}$$

In the regime $\xi_{crit}(\delta) \le \xi \le 1/2$ we have

$$h(\xi) - \ln 2 + pE_\delta(\xi) = \frac{1}{2}(h(\delta) - \ln 2) + (1 - p/2)h(\xi) ,$$

which is decreasing in ξ, and hence we may restrict maximization in (A.4) to $\xi \le \xi_{crit}(\delta)$. We introduce parametrization $\xi = \xi(\omega)$ as in (2.14), with

$$\frac{\delta}{1-\delta} \le \omega \le \sqrt{\frac{\delta}{1-\delta}} .$$

Then using identity

$$\frac{d}{d\omega}\phi(\xi(\omega),\omega) = \xi'(\omega)\ln\frac{1-\omega}{1+\omega} \tag{A.5}$$

we get that derivative of the expression under the max in (A.4) is

$$\frac{d}{d\omega}(\cdots) = \xi'(\omega)\left(\ln\frac{1-\xi}{\xi} + p\ln\frac{1-\omega}{1+\omega}\right) . \tag{A.6}$$

It is clear that this function is strictly increasing as ω ranges in (A.5). For the right endpoint in (A.5) we have $\xi = 0$ and thus the derivative tends to $-\infty$, for the left endpoint, notice that when $p = 2$ and $\omega = \sqrt{\frac{\delta}{1-\delta}}$ the expression (A.6) is exactly zero and thus > 0 for $p > 2$. So there does exist a unique $\omega^*(p, \delta)$ such that (A.6) equals zero. Instead of finding the function $\omega^*(p, \delta)$ and $\xi^* = \xi(\omega^*)$ we fix an arbitrary value $\omega \in [0, 1]$ and find the δ for which $\omega^*(p, \delta) = \omega$. This gives expression for $\delta = \delta(\omega)$ given in (2.42). Plugging the values $\delta = \delta(\omega)$ and $\xi^* = \xi^*(\delta(\omega), \omega)$ into (A.4) we conclude that

$$E(p, \delta) \le h(\xi^*) - \ln 2 + E_\delta(\xi^*) + O\left(\frac{\log n}{n}\right),$$

where furthermore $E_\delta(\xi^*) = \phi(\xi, \omega)$. This completes proof of the upper bound in (2.39).

To prove a matching lower bound, notice that for any fixed δ we have argued that $\omega = \sqrt{\frac{\delta}{1-\delta}}$ yields a positive value of (A.6). Consequently, the optimal value of ξ^* in (A.4) is always $< \xi_{crit}(\delta) - \epsilon$ for some $\epsilon = \epsilon(p, \delta) > 0$. Thus, taking $a = \lfloor \xi^* n \rfloor$, we can apply the result of [15, Section IV] establishing

$$K_j(a) = \exp\{nE_\delta(\xi^*) + O(\log n)\},$$

which shows that $E(p, \delta) \ge h(\xi^*) - \ln 2 + pE_\delta(\xi^*) + O\left(\frac{\log n}{n}\right)$ matching the previous upper bound.

We now prove (2.43). The upper bound follows from $\|K_j\|_p \le \|K_j\|_2$ and (A.1). For the lower bound, assume j and n are even. From (2.8) we have $K_k(n/2) = 0$ for any odd k, and thus from (2.10), we have that roots of $K_{n/2}(\cdot)$ are precisely all odd integers in $[n]$, so that

$$K_{n/2}(x) = c \prod_{m=1}^{n/2} (x - 2m - 1),$$

where constant c is found from $K_{n/2}(0) = \binom{n}{n/2}$. Applying (2.10) again, we find

$$K_j(n/2) = \frac{\binom{n}{j}}{\binom{n}{n/2}} K_{n/2}(j).$$

When j is even, $K_j(n/2)$ is non-zero, so analyzing this similar to proof of Stirling formula we get

$$K_j(n/2) = \exp\{nh(\delta)/2 + O(\log n)\}.$$

The lower bound in (2.43) then follows from, cf. (A.2),

$$\|K_j\|_p^p \ge 2^{-n} \binom{n}{n/2} |K_j(n/2)|^p.$$

□

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