Channel coding: finite blocklength results

Plan:

1. Overview on the example of BSC

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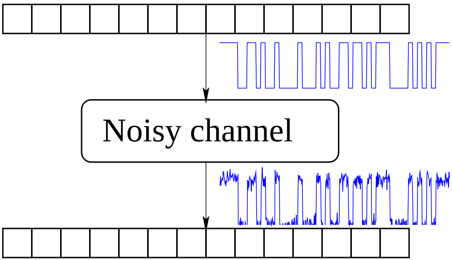
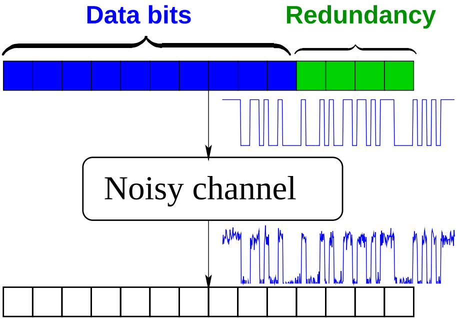
2. Converse bounds
3. Achievability bounds
4. Channel dispersion
5. Applications: performance of real-world codes
   Extensions: codes with feedback

Abstract communication problem



Goal: Decrease corruption of data caused by noise

Channel coding: principles

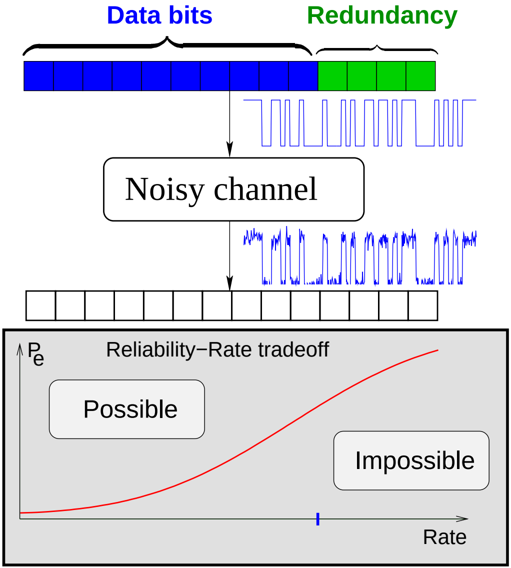


Goal: Decrease corruption of data caused by noise

Solution: *Code* to diminish probability of error P_e.

Key metrics: Rate and P_e

Channel coding: principles



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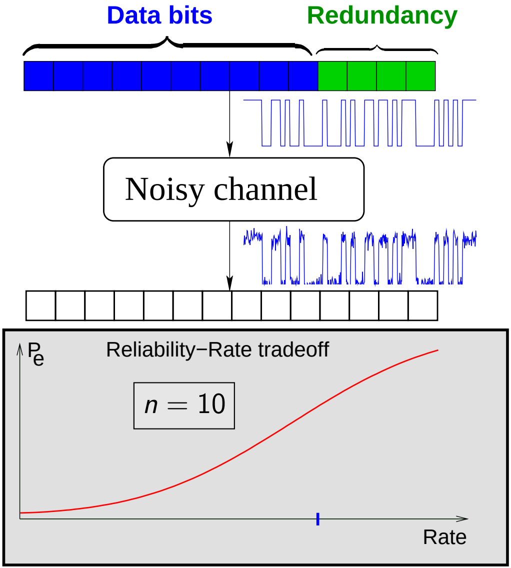
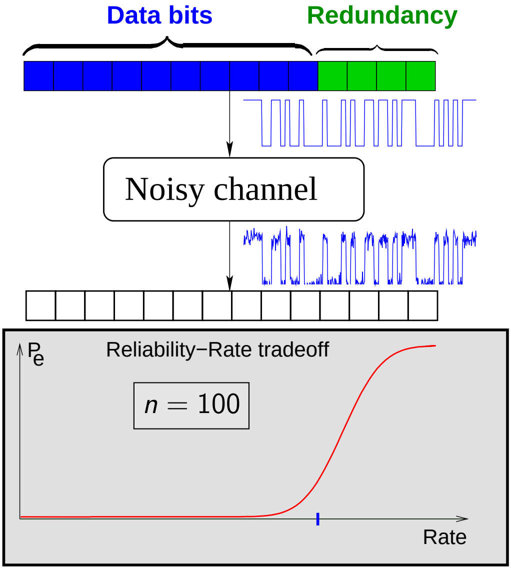


Decreasing P_e further:

1. More redundancy
   Bad: loses rate
2. Increase blocklength!

Channel coding: principles



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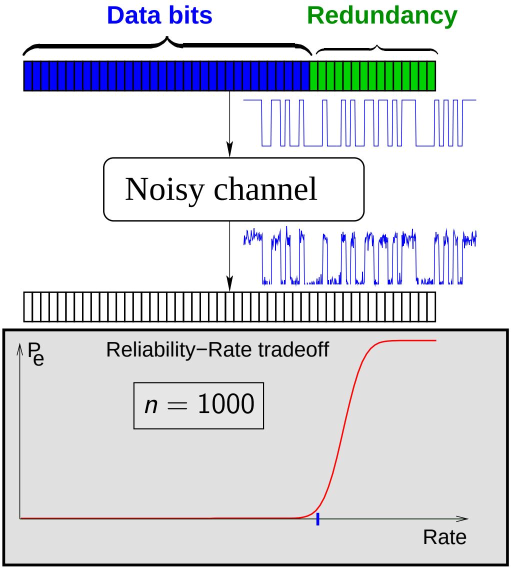

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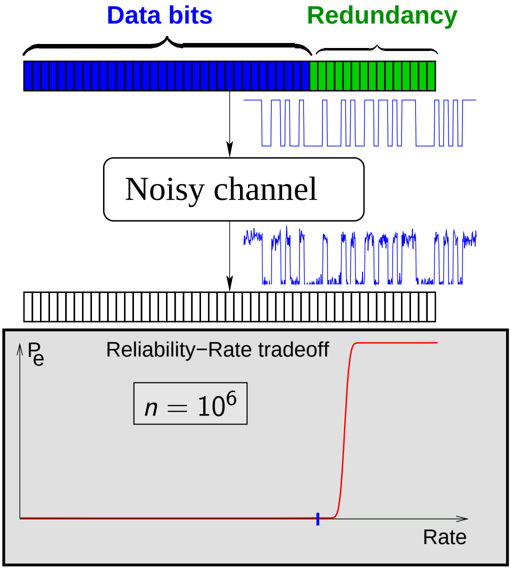

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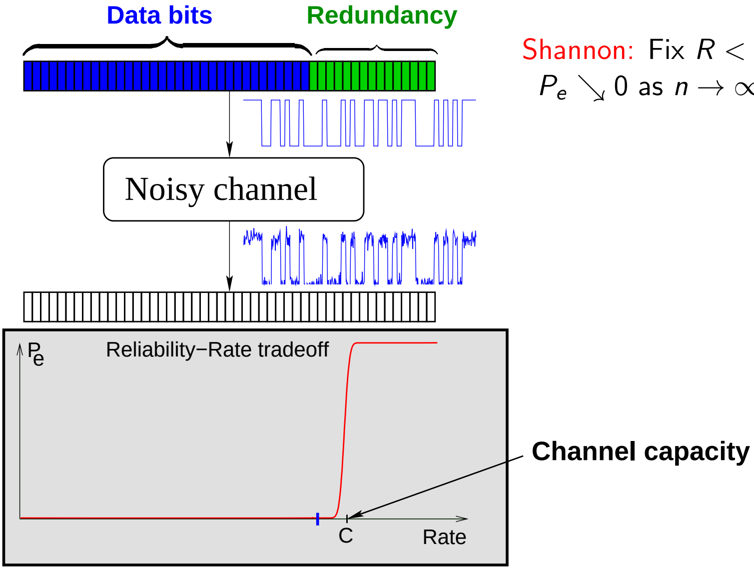
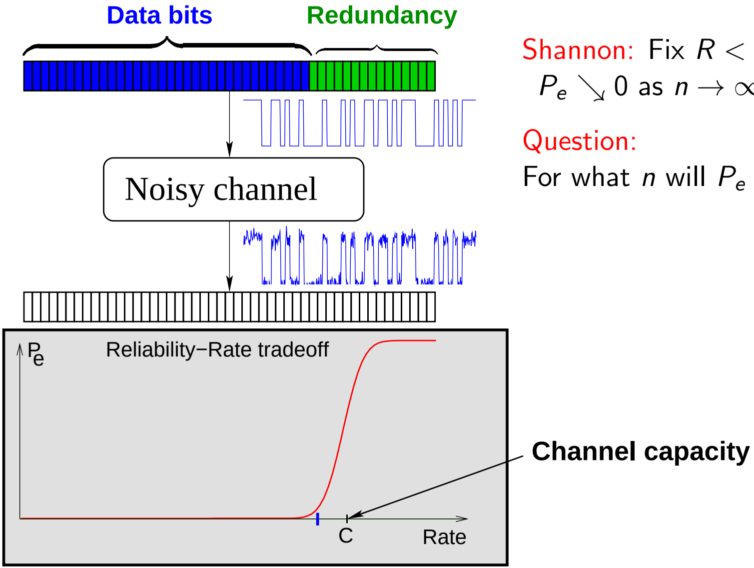
Channel coding: Shannon capacity



Shannon: Fix $R < C$

$P_e \searrow 0$ as $n \to \infty$

Channel coding: Shannon capacity



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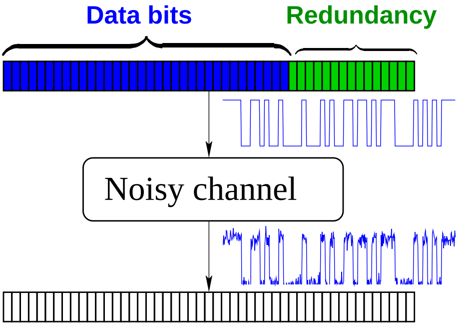
Question:

For what n will $P_e < 10^{-3}$?

Channel coding: Gaussian approximation



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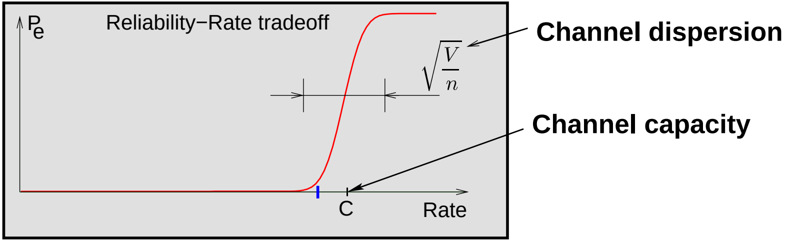

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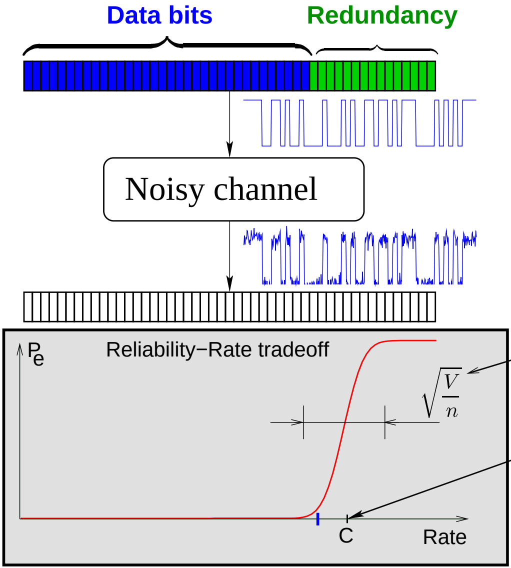


Shannon: Fix $R < C$

$P_e \searrow 0$ as $n \to \infty$

Question:

For what n will $P_e < 10^{-3}$?

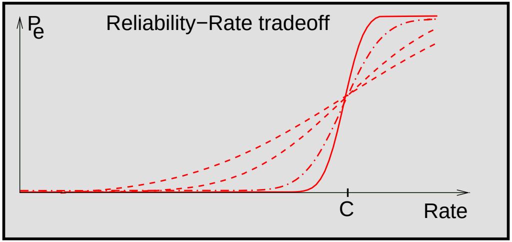
Answer:

$$n \gtrsim \mathrm{const} \cdot \frac{V}{C^2}$$

Channel dispersion

Channel capacity

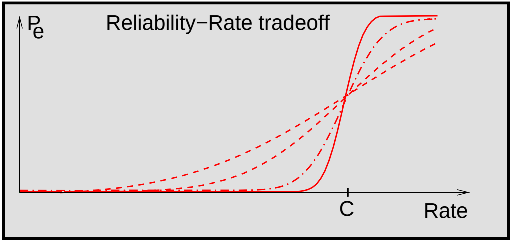
How to describe evolution of the boundary?



Classical results:

- **Vertical asymptotics:** fixed rate, reliability function
  Elias, Dobrushin, Fano, Shannon-Gallager-Berlekamp
- **Horizontal asymptotics:** fixed ϵ, strong converse, $\sqrt{n}$ terms
  Wolfowitz, Weiss, Dobrushin, Strassen, Kemperman

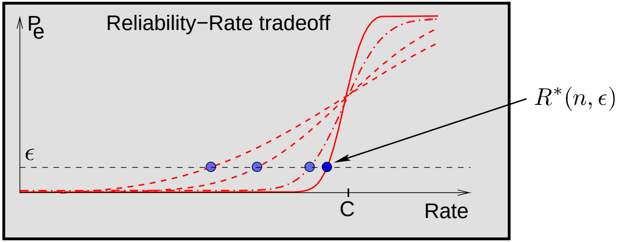
How to describe evolution of the boundary?



XXI century:

- Tight non-asymptotic bounds
- Remarkable precision of normal approximation
- Extended results on *horizontal* asymptotics
  AWGN, $O(\log n)$, cost constraints, feedback, *etc.*

Finite blocklength fundamental limit



Definition

$$R^*(n, \epsilon) = \max \left\{ \frac{1}{n} \log M : \exists (n, M, \epsilon)\text{-code} \right\}$$

(max. achievable rate for blocklength n and prob. of error ϵ)

Note: Exact value unknown (search is doubly exponential in n).

Minimal delay and error-exponents

Fix $R < C$. What is the smallest blocklength n^* needed to achieve

$$R^*(n, \epsilon) \geq R \quad ?$$

Minimal delay and error-exponents

Fix $R < C$. What is the smallest blocklength n^* needed to achieve

$$R^*(n, \epsilon) \geq R \quad ?$$

Classical answer: Approximation via reliability function [Shannon-Gallager-Berlekamp'67]:

$$n^* \approx \frac{1}{E(R)} \log \frac{1}{\epsilon}$$

E.g., take $BSC(0.11)$ and $R = 0.9C$, prob. of error $\epsilon = 10^{-3}$:

$$n^* \approx 5000 \qquad \text{(channel uses)}$$

Difficulty: How to verify accuracy of this estimate?

Bounds

- Bounds are implicit in Shannon's theorem

$$\lim_{n\to\infty} R^*(n,\epsilon) = C \quad \Longleftrightarrow \begin{cases} R^*(n,\epsilon) \le C + o(1)\,, \\ R^*(n,\epsilon) \ge C + o(1)\,. \end{cases}$$

(Feinstein'54, Shannon'57, Wolfowitz'57, Fano)

- Reliability function: even better bounds (Elias'55, Shannon'59, Gallager'65, SGB'67)
- Problems: derived for asymptotics (need "de-asymptotization")
  unexpected sensitivity:

$$\epsilon \le e^{-nE_r(R)} \qquad \text{[Gallager'65]}$$

$$\epsilon \le e^{-nE_r(R-o(1))+O(\log n)} \qquad \text{[Csiszár-Körner'81]}$$

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For BSC($n = 10^3, 0.11$): $o(1) \approx 0.1$, $e^{O(\log n)} \approx 10^{24}$ (!)

Bounds

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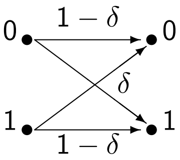
Strassen'62: Take $n > \frac{19600}{\epsilon^{16}}$ . . . (!)

- Solution: Derive bounds from scratch.

New achievability bound



BSC

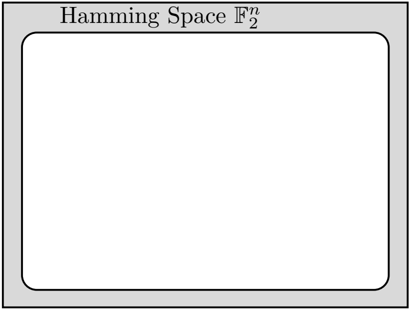
Theorem (RCU)

For a BSC(δ) there exists a code with rate R, blocklength n and

$$\epsilon \le \sum_{t=0}^{n} \binom{n}{t} \delta^t (1-\delta)^{n-t} \min\left\{1, \ \sum_{k=0}^{t} \binom{n}{k} 2^{-n+nR}\right\} .$$

Proof of RCU bound for the BSC

- Input space: $A = \{0,1\}^n$



Proof of RCU bound for the BSC

- Input space: $A = \{0,1\}^n$
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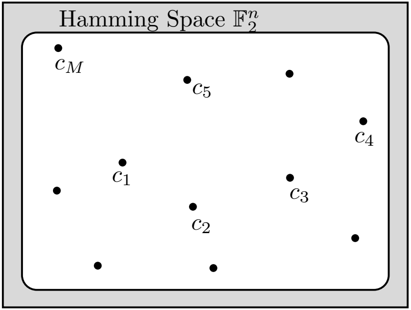

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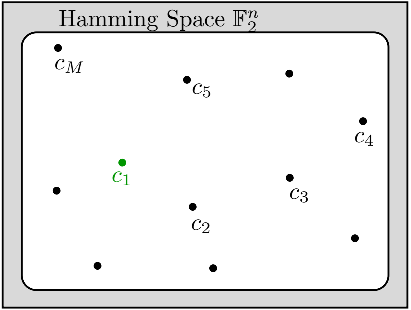

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- Input space: $A = \{0, 1\}^n$
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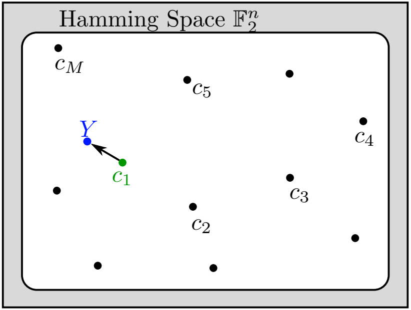
- Noise displaces $c_1 \rightarrow Y$

$Y = c_1 + Z, \quad Z \sim Bern(\delta)^n$



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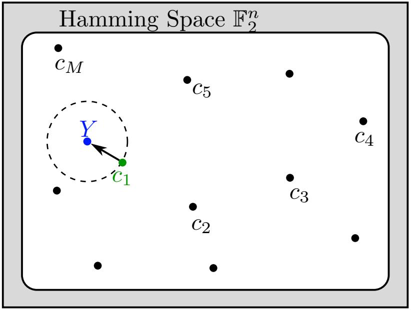

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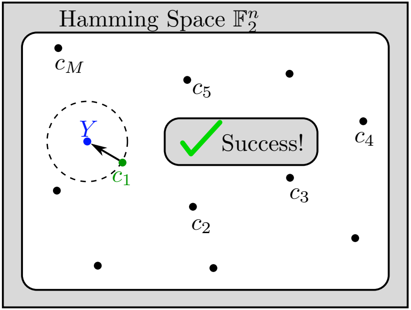

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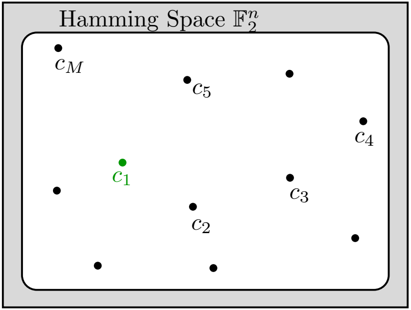

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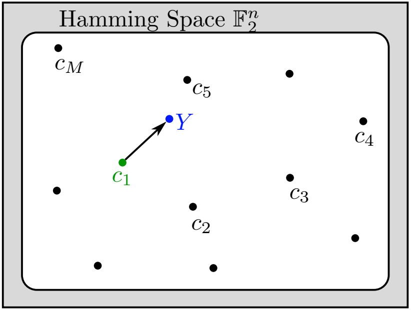

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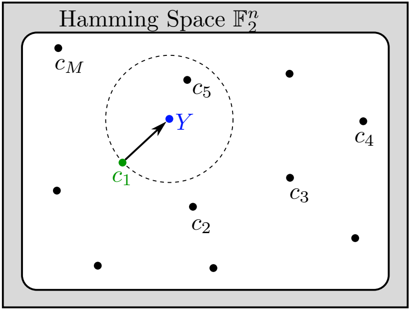

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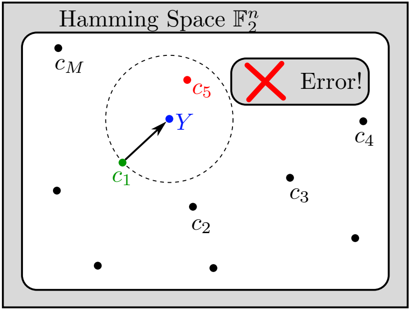

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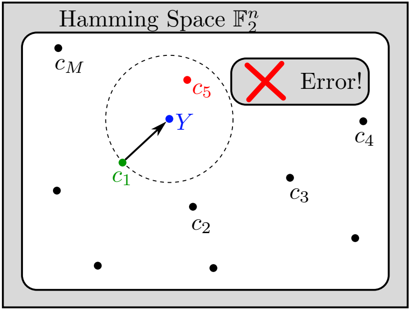

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- Decoder: find closest codeword to Y
- Probability of error analysis:

$$\begin{aligned}\mathbb{P}[\text{error}|Y, \mathrm{wt}(Z) = t] &= \mathbb{P}[\exists j > 1 : c_j \in \mathrm{Ball}(Y,t)] \\ &\le \sum_{j=2}^{M} \mathbb{P}[c_j \in \mathrm{Ball}(Y,t)] \\ &\le 2^{nR} \sum_{k=0}^{t} \binom{n}{k} 2^{-n}\end{aligned}$$

. . . cont'd . . .

- Conditional probability of error:

$$\mathbb{P}[\text{error}|Y, \mathrm{wt}(Z) = t] \le \sum_{k=0}^{t} \binom{n}{k} 2^{-n+nR}$$

- Key observation: For large noise t RHS is > 1. Tighten:

$$\mathbb{P}[\text{error}|Y, \mathrm{wt}(Z) = t] \le \min\left\{1, \sum_{k=0}^{t} \binom{n}{k} 2^{-n+nR}\right\} \qquad (*)$$

- Average $\mathrm{wt}(Z) \sim \mathit{Binomial}(n, \delta) \Longrightarrow$ **Q.E.D.**

Note: Step $(*)$ tightens Gallager's ρ-trick:

$$\mathbb{P}\left[\bigcup_j A_j\right] \le \left(\sum_j \mathbb{P}[A_j]\right)^{\rho}$$

Sphere-packing converse (BSC variation)

Theorem (Elias'55)

For any (n, M, ϵ) code over the BSC(δ):

$$\epsilon \geq f\left(\frac{2^n}{M}\right),$$

where $f(\cdot)$ is a piecewise-linear decreasing convex function:

$$f\left(\sum_{j=0}^{t}\binom{n}{j}\right) = \sum_{j=t+1}^{n}\binom{n}{j}\delta^j(1-\delta)^{n-j} \qquad t = 0, \ldots, n$$

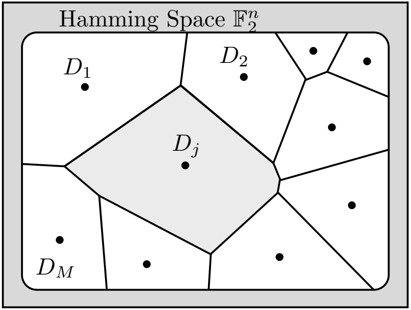
Note: Convexity of f follows from general properties of β_α (below)

Sphere-packing converse (BSC variation)

Proof:

- Denote decoding regions D_j: $\coprod D_j = \{0,1\}^n$
- Probability of error is:

$$\epsilon = \frac{1}{M} \sum_j \mathbb{P}[c_j + Z \notin D_j]$$



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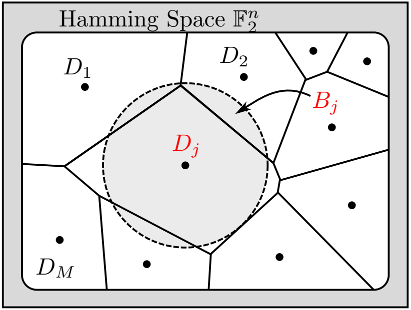
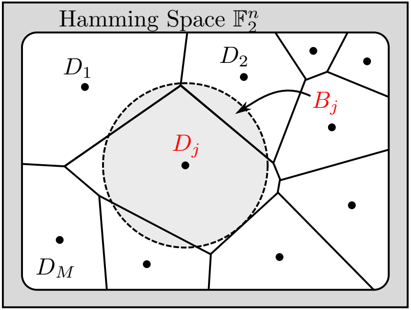


- $B_j =$ ball centered at 0 s.t. $\mathrm{Vol}(B_j) = \mathrm{Vol}(D_j)$

Sphere-packing converse (BSC variation)



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$$\geq \frac{1}{M}\sum_j \mathbb{P}[Z \notin B_j]$$

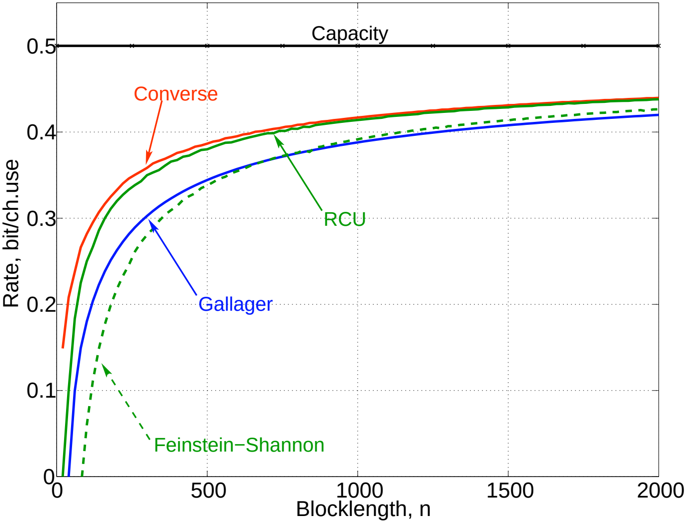
- $B_j =$ ball centered at 0 s.t. $\mathrm{Vol}(B_j) = \mathrm{Vol}(D_j)$
- Simple calculation:

$$\mathbb{P}[Z \notin B_j] = f(\mathrm{Vol}(B_j))$$

- f – convex, apply Jensen:

$$\epsilon \geq f\left(\frac{1}{M}\sum_{j=1}^{M}\mathrm{Vol}(D_j)\right) = f\left(\frac{2^n}{M}\right)$$

Bounds: example $BSC(0.11)$, $\epsilon = 10^{-3}$



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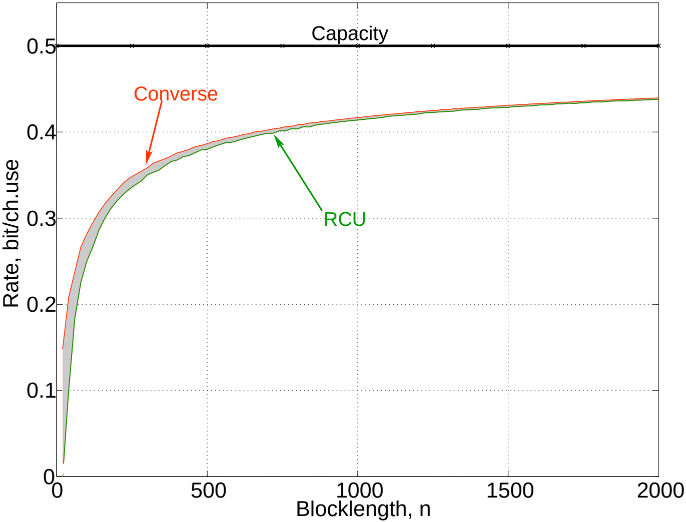

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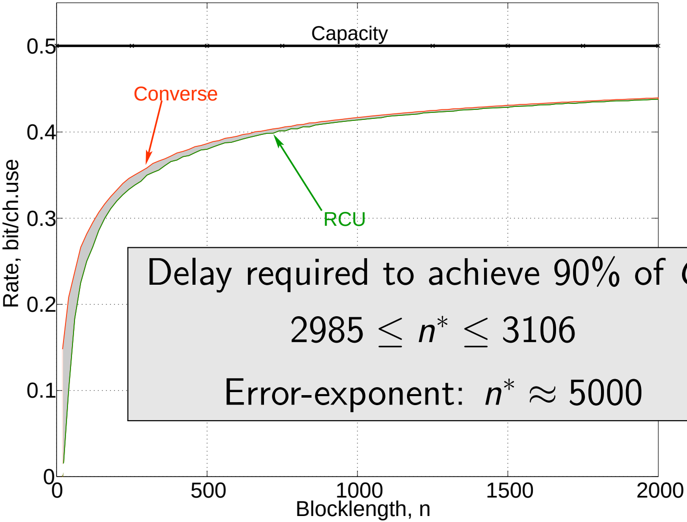


Delay required to achieve 90% of C:

$$2985 \leq n^* \leq 3106$$

Error-exponent: $n^* \approx 5000$

Normal approximation

Theorem

For the BSC(δ) and $0 < \epsilon < 1$,

$$R^*(n,\epsilon) = C - \sqrt{\frac{V}{n}}Q^{-1}(\epsilon) + \frac{1}{2}\frac{\log n}{n} + O\left(\frac{1}{n}\right)$$

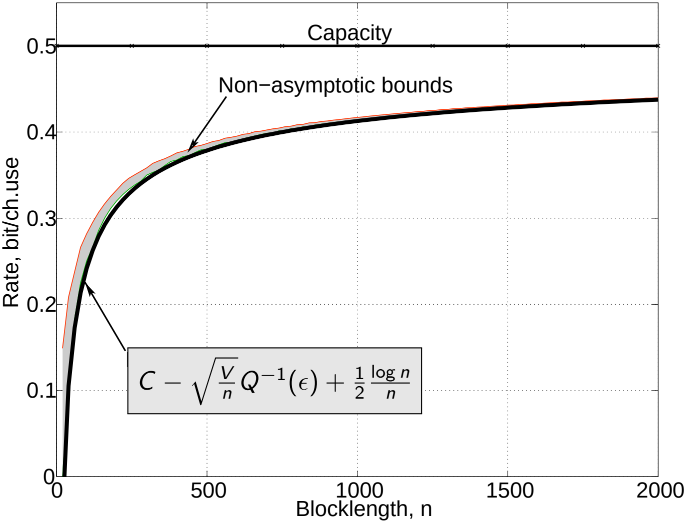
where

$$\begin{aligned} C(\delta) &= \log 2 + \delta \log \delta + (1-\delta)\log(1-\delta) \\ V &= \delta(1-\delta)\log^2 \frac{1-\delta}{\delta} \\ Q(x) &= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \end{aligned}$$

Proof: Bounds + Stirling's formula

Note: Now we see the explicit dependence on ϵ!

Normal approximation: $BSC(0.11)$; $\epsilon = 10^{-3}$



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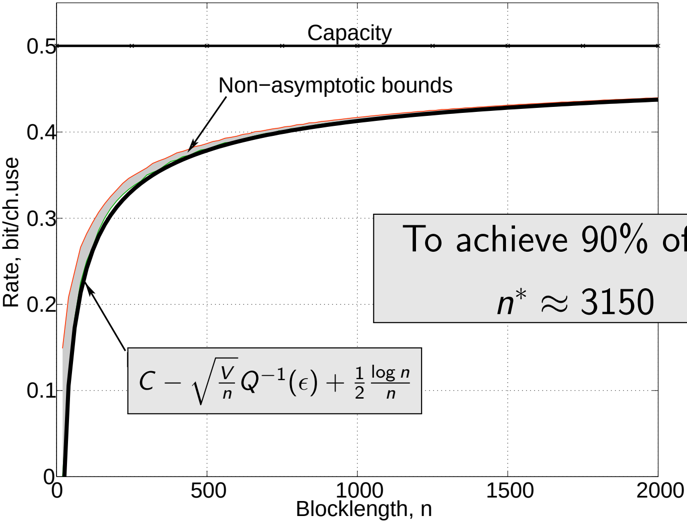


To achieve 90% of C:

$n^* \approx 3150$

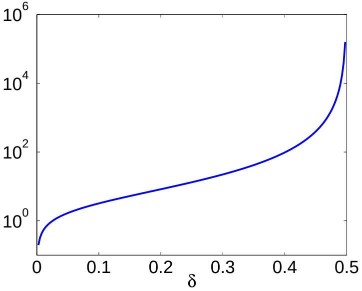
Dispersion and minimal required delay

Delay needed to achieve $R = \eta C$:

$$n^* \gtrsim \left(\frac{Q^{-1}(\epsilon)}{1-\eta} \right)^2 \cdot \frac{V}{C^2}$$

Note: $\frac{V}{C^2}$ is "coding horizon".

Behavior of $\frac{V}{C^2}$ (BSC)



Less noise More noise

BSC summary

Delay required to achieve 90 % of capacity:

- Error-exponents:

$$n^* \approx 5000$$

- True value:

$$2985 \le n^* \le 3106$$

- Channel dispersion:

$$n^* \approx 3150$$

Converse Bounds

Notation

- Take a random transformation $\mathsf{A} \xrightarrow{P_{Y|X}} \mathsf{B}$
  (think $\mathsf{A} = \mathcal{A}^n$, $\mathsf{B} = \mathcal{B}^n$, $P_{Y|X} = P_{Y^n|X^n}$)
- Input distribution P_X induces $P_Y = P_{Y|X} \circ P_X$
  $P_{XY} = P_X P_{Y|X}$
- Fix code:

$$W \xrightarrow{encoder} X \to Y \xrightarrow{decoder} \hat{W}$$

  $W \sim Unif[M]$ and $M = \#$ of codewords
  Input distribution P_X associated to a code:

$$P_X[\cdot] \triangleq \frac{\#\text{ of codewords } \in (\cdot)}{M}.$$

- Goal: Upper bounds on $\log M$ in terms of $\epsilon \triangleq \mathbb{P}[error]$
  As by-product: $R^*(n,\epsilon) \lesssim C - \sqrt{\frac{V}{n}} Q^{-1}(\epsilon)$

Fano's inequality

Theorem (Fano)

For any code

$$W \xrightarrow{\text{encoder}} X \xrightarrow{P_{Y|X}} Y \xrightarrow{\text{decoder}} \hat{W}$$

with $W \sim Unif\{1, \ldots, M\}$:

$$\log M \leq \frac{\sup_{P_X} I(X;Y) + h(\epsilon)}{1-\epsilon}, \quad \epsilon = \mathbb{P}[W \neq \hat{W}]$$

Implies *weak converse*:

$$R^*(n, \epsilon) \leq \frac{C}{1-\epsilon} + o(1).$$

Proof: ϵ-small $\implies$ $H(W|\hat{W})$-small $\implies$ $I(X;Y) \approx H(W) = \log M$

A (very long) proof of Fano via *channel substitution*

Consider two distributions on $(W, X, Y, \hat{W})$:

$$\mathbb{P}: \quad P_{WXY\hat{W}} = P_W \times P_{X|W} \times P_{Y|X} \times P_{\hat{W}|Y}$$

DAG: $W \to X \to Y \to \hat{W}$

$$\mathbb{Q}: \quad Q_{WXY\hat{W}} = P_W \times P_{X|W} \times Q_Y \times P_{\hat{W}|Y}$$

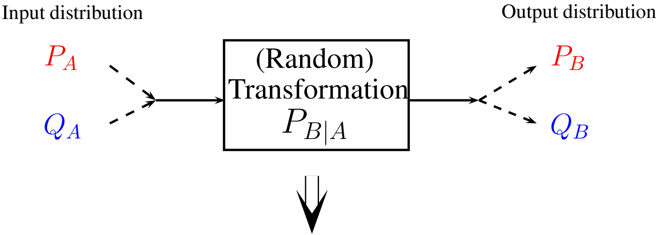
DAG: $W \to X \quad Y \to \hat{W}$

Under $\mathbb{Q}$ the channel is useless:

$$\mathbb{Q}[W = \hat{W}] = \sum_{m=1}^{M} P_W(m) Q_{\hat{W}}(m) = \frac{1}{M} \sum_{m=1}^{M} Q_{\hat{W}}(m) = \frac{1}{M}$$

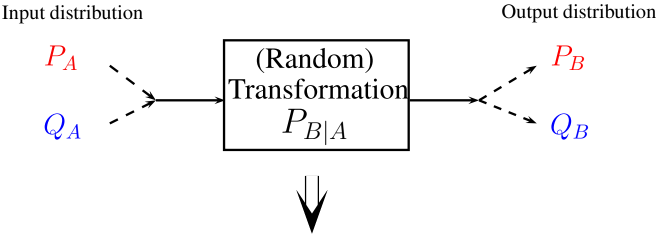
Next step: data-processing for relative entropy $D(\cdot||\cdot)$

Data-processing for $D(\cdot||\cdot)$



$$D(P_A \| Q_A) \qquad \geq \qquad D(P_B \| Q_B)$$

Data-processing for $D(\cdot||\cdot)$



$$D(P_A \| Q_A) \qquad \geq \qquad D(P_B \| Q_B)$$

Apply to transform: $(W, X, Y, \hat{W}) \mapsto 1\{W \neq \hat{W}\}$:

$$\begin{aligned} D(P_{WXY\hat{W}} \| Q_{WXY\hat{W}}) &\geq d(\, \mathbb{P}[W = \hat{W}] \,\|\, \mathbb{Q}[W = \hat{W}]\,) \\ &= d(1 - \epsilon || \tfrac{1}{M}) \end{aligned}$$

where $d(x||y) = x \log \frac{x}{y} + (1 - x) \log \frac{1-x}{1-y}$.

A proof of Fano via *channel substitution*

So far:

$$D(P_{WXY\hat{W}} \| Q_{WXY\hat{W}}) \geq d(1-\epsilon \| \tfrac{1}{M})$$

Lower-bound RHS:

$$d(1-\epsilon \| \tfrac{1}{M}) \geq (1-\epsilon)\log M - h(\epsilon)$$

Analyze LHS:

$$\begin{aligned} D(P_{WXY\hat{W}} \| Q_{WXY\hat{W}}) &= D(P_{XY} \| Q_{XY}) \\ &= D(P_X P_{Y|X} \| P_X Q_Y) \\ &= D(P_{Y|X} \| Q_Y | P_X) \end{aligned}$$

(Recall: $D(P_{Y|X} \| Q_Y | P_X) = \mathbb{E}_{x \sim P_X}[D(P_{Y|X=x} \| Q_Y)]$)

A proof of Fano via *channel substitution*: last step

Putting it all together:

$$\boxed{(1-\epsilon)\log M \le D(P_{Y|X}||Q_Y|P_X) + h(\epsilon) \quad \forall Q_Y \quad \forall \text{code}}$$

Two methods:

1. Compute $\sup_{P_X} \inf_{Q_Y}$ and recall

$$\inf_{Q_Y} D(P_{Y|X}\|Q_Y|P_X) = I(X;Y)$$

2. Take $Q_Y = P_Y^* =$ the caod (capacity achieving output dist.) and recall

$$D(P_{Y|X}\|P_Y^*|P_X) \le \sup_{P_X} I(X;Y) \qquad \forall P_X$$

Conclude:

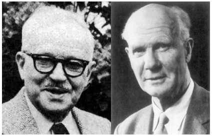
$$(1-\epsilon)\log M \le \sup_{P_X} I(X;Y) + h(\epsilon)$$

Important: Second method is particularly useful for FBL!

Tightening: from $D(\cdot||\cdot)$ to $\beta_\alpha(\cdot,\cdot)$

Question: How about replacing $D(\cdot||\cdot)$ with other divergences?

Answer:

	$D(\cdot\|\|\cdot)$	relative entropy (KL divergence)	weak converse
	$D_\lambda(\cdot\|\|\cdot)$	Rényi divergence	strong converse
	$\beta_\alpha(\cdot,\cdot)$	Neyman-Pearson ROC curve	FBL bounds

Next: What is β_α?

Neyman-Pearson's β_α

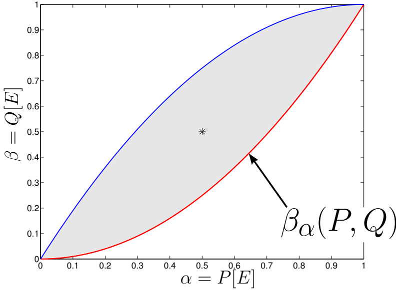
Definition

For every pair of measures P, Q

$$\beta_\alpha(P,Q) \triangleq \inf_{E:P[E]\geq\alpha} Q[E]\,.$$

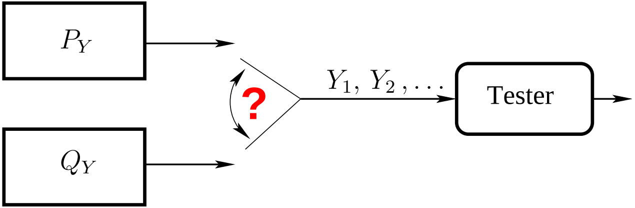
iterate over
all "sets" E
plot pairs $(P[E], Q[E])$

$\Longrightarrow$



Important: Like relative entropy β_α satisfies data-processing.

β_α = binary hypothesis testing



Two types of errors:

$$\begin{aligned} \mathbb{P}[\text{Tester says "}Q_Y\text{"}] &\le \epsilon \\ \mathbb{Q}[\text{Tester says "}P_Y\text{"}] &\to \min \end{aligned}$$

Hence: Solve binary HT $\iff$ compute $\beta_\alpha(P_Y^n, Q_Y^n)$

Stein's Lemma: For many i.i.d. observations

$$\log \beta_{1-\epsilon}(P_Y^n, Q_Y^n) = -nD(P_Y || Q_Y) + o(n)\,.$$

But in fact $\log \beta_\alpha(P_Y^n, Q_Y^n)$ can also be computed exactly!

How to compute β_α?

Theorem (Neyman-Pearson)

β_α is given parametrically by $-\infty \le \gamma \le +\infty$:

$$\mathbb{P}\left[\log \frac{P(X)}{Q(X)} \ge \gamma\right] = \alpha$$

$$\mathbb{Q}\left[\log \frac{P(X)}{Q(X)} \ge \gamma\right] = \beta_\alpha(P, Q)$$

For product measures $\log \frac{P^n(X)}{Q^n(X)} =$ sum of i.i.d. $\Longrightarrow$ from CLT:

$$\log \beta_\alpha(P^n, Q^n) = -nD(P||Q) + \sqrt{nV(P||Q)}Q^{-1}(\alpha) + o(\sqrt{n}),$$

where

$$V(P||Q) = \mathrm{Var}_P\left[\log \frac{P(X)}{Q(X)}\right]$$

Back to proving converse

Recall two measures:

$$\mathbb{P}: \quad P_{WXY\hat{W}} = P_W \times P_{X|W} \times P_{Y|X} \times P_{\hat{W}|Y}$$

DAG: $W \to X \to Y \to \hat{W}$

$$\mathbb{P}[W = \hat{W}] = 1-\epsilon$$

$$\mathbb{Q}: \quad Q_{WXY\hat{W}} = P_W \times P_{X|W} \times Q_Y \times P_{\hat{W}|Y}$$

DAG: $W \to X \quad Y \to \hat{W}$

$$\mathbb{Q}[W = \hat{W}] = \frac{1}{M}$$

Then by definition of β_α:

$$\beta_{1-\epsilon}(P_{WXY\hat{W}}, Q_{WXY\hat{W}}) \le \frac{1}{M}$$

But $\log \frac{P_{WXY\hat{W}}}{Q_{WXY\hat{W}}} = \log \frac{P_X P_{Y|X}}{P_X Q_Y} \quad \Longrightarrow$

$$\log M \le -\log \beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_Y) \qquad \forall Q_Y \quad \forall \text{code}$$

Meta-converse: minimax version

Theorem

Every (M, ϵ)-code for channel $P_{Y|X}$ satisfies

$$\log M \le -\log\left\{\inf_{P_X}\sup_{Q_Y}\beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_Y)\right\}.$$

Meta-converse: minimax version

Theorem

Every (M, ϵ)-code for channel $P_{Y|X}$ satisfies

$$\log M \le -\log\left\{\inf_{P_X}\sup_{Q_Y}\beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_Y)\right\}.$$

- Finding good Q_Y for every P_X is not needed:

$$\inf_{P_X}\sup_{Q_Y}\beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_Y) = \sup_{Q_Y}\inf_{P_X}\beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_Y) \quad (*)$$

- Saddle-point property of β_α is similar to $D(\cdot||\cdot)$:

$$\inf_{P_X}\sup_{Q_Y} D(P_X P_{Y|X}\|P_X Q_Y) = \sup_{Q_Y}\inf_{P_X} D(P_X P_{Y|X}\|P_X Q_Y) = C$$

Meta-converse: minimax version

Theorem

Every (M, ϵ)-code for channel $P_{Y|X}$ satisfies

$$\log M \leq -\log \left\{ \inf_{P_X} \sup_{Q_Y} \beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_Y) \right\} .$$

Bound is tight in two senses:

- There exist *non-signalling assisted* (NSA) codes attaining the upper-bound. [Matthews, Trans. IT'2012]
- **ISIT'2013:** For any (M, ϵ)-code with ML decoder

$$\log M = -\log \left\{ \sup_{Q_Y} \beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_Y) \right\}$$

Vazquez-Vilar *et al* [WeB4]

Meta-converse: minimax version

Theorem

Every (M, ϵ)*-code for channel* $P_{Y|X}$ *satisfies*

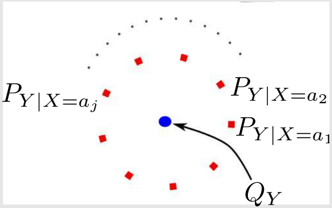
$$\log M \le -\log \left\{ \inf_{P_X} \sup_{Q_Y} \beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_Y) \right\} .$$

In practice: evaluate with a luckily guessed (suboptimal) Q_Y.

How to guess good Q_Y?

- Try caod P_Y^*
- Analyze channel symmetries
- Use geometric intuition. $\longrightarrow$
  good $Q_Y \approx$ "center" of $P_{Y|X}$
- *Exercise:* Redo BSC.

Space of distributions on $\mathcal{Y}$



Example: Converse for AWGN

The AWGN Channel

$$
\begin{array}{ccccc}
 & & Z \sim \mathcal{N}(0, \sigma^2) & & \\
 & & \downarrow & & \\
X & \longrightarrow & \bigoplus & \longrightarrow & Y
\end{array}
$$

Codewords $X^n \in \mathbb{R}^n$ satisfy power-constraint:

$$\sum_{j=1}^{n} |X_j|^2 \leq nP$$

Goal: Upper-bound # of codewords decodable with $P_e \leq \epsilon$.

Example: Converse for AWGN

- Given $\{c_1, \ldots, c_M\} \in \mathbb{R}^n$ with $\mathbb{P}[W \neq \hat{W}] \leq \epsilon$ on $AWGN(1)$.
- Yaglom-map trick: replacing $n \to n+1$ equalize powers:

$$\|c_j\|^2 = nP \qquad \forall j \in \{1, \ldots, M\}$$

Example: Converse for AWGN

- Given $\{c_1, \ldots, c_M\} \in \mathbb{R}^n$ with $\mathbb{P}[W \neq \hat{W}] \leq \epsilon$ on $AWGN(1)$.
- Yaglom-map trick: replacing $n \to n+1$ equalize powers:

$$\|c_j\|^2 = nP \qquad \forall j \in \{1, \ldots, M\}$$

- Take $Q_{Y^n} = \mathcal{N}(0, 1+P)^n$ (the caod!)
- Optimal test "$P_{X^n Y^n}$ vs. $P_{X^n} Q_{Y^n}$" (Neyman-Pearson):

$$\log \frac{P_{Y^n|X^n}}{Q_{Y^n}} = nC + \frac{\log e}{2} \cdot \left(\frac{\|Y^n\|^2}{1+P} - \|Y^n - X^n\|^2 \right)$$

where $C = \frac{1}{2}\log(1+P)$.

- Under $\mathbb{P}$: $Y^n = X^n + \mathcal{N}(0, \mathbf{I}_n)$
  $\Longrightarrow$ distribution of LLR (CLT approx.)

$$\approx nC + \sqrt{nV} Z \,, \qquad Z \sim \mathcal{N}(0,1)$$

Simple algebra: $V = \frac{\log^2 e}{2} \left(1 - \frac{1}{(1+P)^2} \right)$

...cont'd ...

- Under $\mathbb{P}$: distribution of LLR (CLT approx.)

$$\approx nC + \sqrt{nV}Z\,, \qquad Z \sim \mathcal{N}(0,1)$$

- Take $\gamma = nC - \sqrt{nV}Q^{-1}(\epsilon) \implies$

$$\mathbb{P}\left[\log \frac{d\mathbb{P}}{d\mathbb{Q}} \geq \gamma\right] \approx 1 - \epsilon\,.$$

- Under $\mathbb{Q}$: standard change-of-measure shows

$$\mathbb{Q}\left[\log \frac{d\mathbb{P}}{d\mathbb{Q}} \geq \gamma\right] \approx \exp\{-\gamma\}\,.$$

...cont'd ...

- Under $\mathbb{P}$: distribution of LLR (CLT approx.)

$$\approx nC + \sqrt{nV}Z\,, \qquad Z \sim \mathcal{N}(0,1)$$

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- Under $\mathbb{Q}$: standard change-of-measure shows

$$\mathbb{Q}\left[\log\frac{d\mathbb{P}}{d\mathbb{Q}} \geq \gamma\right] \approx \exp\{-\gamma\}\,.$$

- By Neyman-Pearson

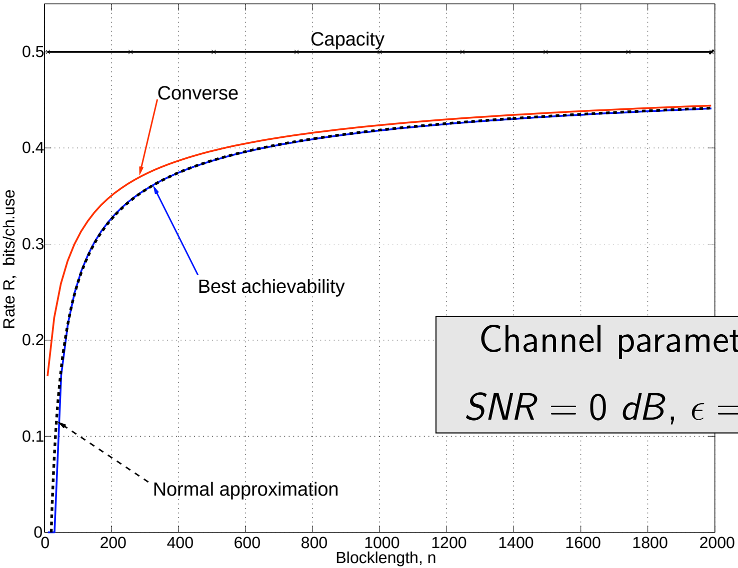
$$\log\beta_{1-\epsilon}(P_{Y^n|X^n=c}, Q_{Y^n}) \approx -nC + \sqrt{nV}Q^{-1}(\epsilon)$$

- Punchline: $\forall(n, M, \epsilon)$-code

$$\log M \lesssim nC - \sqrt{nV}Q^{-1}(\epsilon)$$

N.B.! RHS can be exactly expressed via *non-central* χ^2 *dist.*
... and computed in MATLAB (w/o any CLT approx).

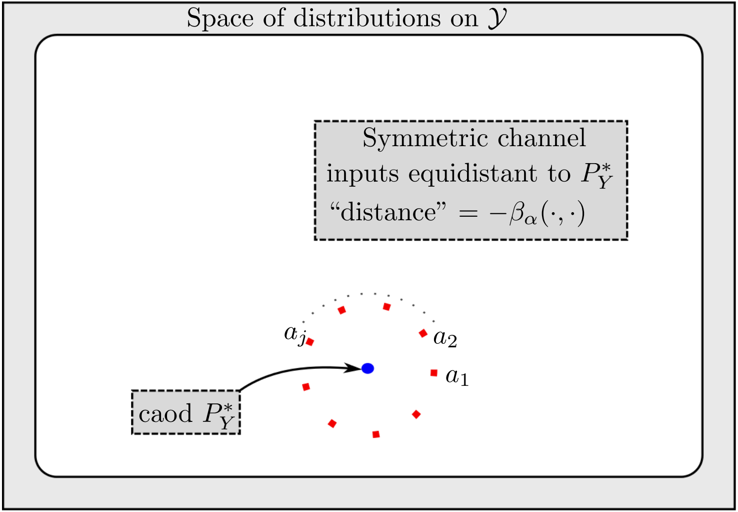
AWGN: Converse from $\beta_\alpha(P, Q)$ with $Q_Y = \mathcal{N}(0,1)^n$



Channel parameters:

$SNR = 0\ dB$, $\epsilon = 10^{-3}$

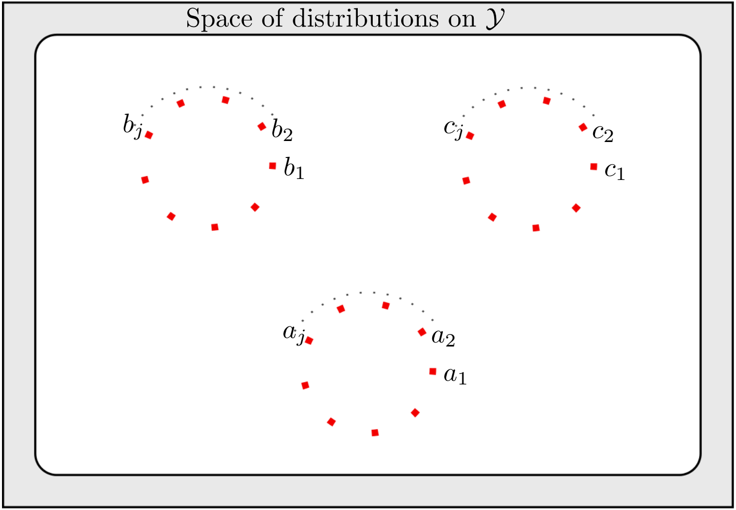
From one Q_Y to many



points
$\Updownarrow$
ch. inputs
(a_j, b_j, c_j)

Symmetric channel: choice of Q_Y is clear

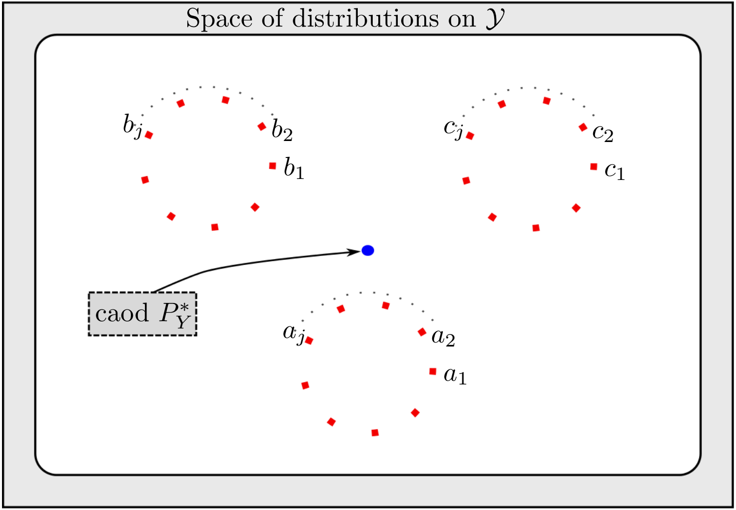
From one Q_Y to many



points
$\Updownarrow$
ch. inputs
(a_j, b_j, c_j)

General channels: Inputs cluster (by composition, power-allocation, ...)
(Clusters $\iff$ orbits of channel symmetry gp.)

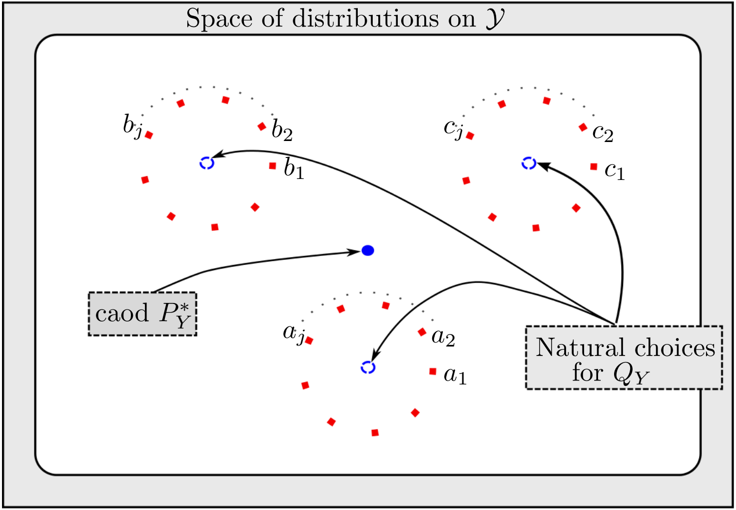
From one Q_Y to many



points
$\Updownarrow$
ch. inputs
(a_j, b_j, c_j)

General channels: Caod is no longer equidistant to all inputs (read: analysis horrible!)

From one Q_Y to many



points $\Updownarrow$ ch. inputs (a_j, b_j, c_j)

Solution: Take Q_Y different for each cluster!

I.e. think of $Q_{Y|X}$

General meta-converse principle

Steps:

- Select auxiliary channel $Q_{Y|X}$ (art)

  E.g.: $Q_{Y|X=x}$ = center of a cluster of x
- Prove converse bound for channel $Q_{Y|X}$

  E.g.: $\mathbb{Q}[W = \hat{W}] \lesssim \frac{\text{\# of clusters}}{M}$
- Find $\beta_\alpha(\mathbb{P}, \mathbb{Q})$, i.e. compare:

$$\mathbb{P} : \ P_{WXY\hat{W}} = P_W \times P_{X|W} \times P_{Y|X} \times P_{\hat{W}|Y}$$

vs.

$$\mathbb{Q} : \ P_{WXY\hat{W}} = P_W \times P_{X|W} \times Q_{Y|X} \times P_{\hat{W}|Y}$$

- Amplify converse for $Q_{Y|X}$ to a converse for $P_{Y|X}$:

$$\beta_{1-P_e(P_{Y|X})} \le 1 - P_e(Q_{Y|X}) \qquad \forall \text{code}$$

Meta-converse theorem: point-to-point channels

Theorem

For any code $\epsilon \triangleq \mathbb{P}[\text{error}]$ *and* $\epsilon' \triangleq \mathbb{Q}[\text{error}]$ *satisfy*

$$\beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_{Y|X}) \le 1 - \epsilon'$$

Advanced examples of $Q_{Y|X}$:

- General DMC: $Q_{Y|X=x} = P_{Y|X} \circ \hat{P}_x$
  Why? To reduce DMC to symmetric DMC
- Parallel AWGN: $Q_{Y|X=x} = f(\text{power-allocation})$
  Why? Since water-filling is not FBL-optimal
- Feedback: $\mathbb{Q}[Y \in \cdot | W = w] = \mathbb{P}[Y \in \cdot | W \neq w]$
  Why? To get bounds in terms of Burnashev's C_1
- PAPR of codes: $Q_{Y^n|X^n=x^n} = f(\text{peak power of } x)$
  Why? To show peaky codewords waste power

Meta-converse generalizes many classical methods

Theorem

For any code $\epsilon \triangleq \mathbb{P}[\mathit{error}]$ *and* $\epsilon' \triangleq \mathbb{Q}[\mathit{error}]$ *satisfy*

$$\beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_{Y|X}) \le 1 - \epsilon'$$

Corollaries:

- Fano's inequality
- Wolfowitz strong converse
- Shannon-Gallager-Berlekamp's sphere-packing
  + improvements: [Valembois-Fossorier'04], [Wiechman-Sason'08]
- Haroutounian's sphere-packing
- list-decoding converses
- Berlekamp's low-rate converse
- Verdú-Han and Poor-Verdú information spectrum converses
- Arimoto's converse (+ extension to feedback)

Meta-converse generalizes many classical methods

Theorem

For any code $\epsilon \triangleq \mathbb{P}[\mathit{error}]$ *and* $\epsilon' \triangleq \mathbb{Q}[\mathit{error}]$ *satisfy*

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Corollaries:

- Fano's inequality
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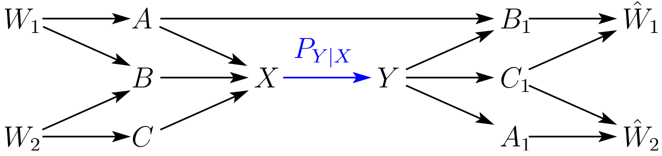
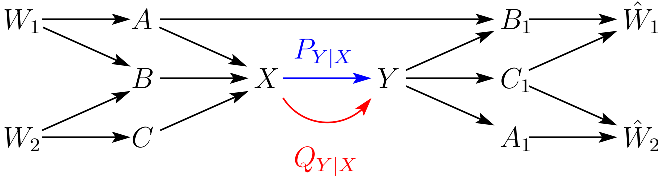
  $+$ improvements: [Valembois-Fossorier'04], [Wiechman-Sason'08]
- Haroutounian's sphere-packing
- list-decoding converses E.g.: $\mathbb{Q}[W \in \{\text{list}\}] = \frac{|\{\text{list}\}|}{M}$
- Berlekamp's low-rate converse
- Verdú-Han and Poor-Verdú information spectrum converses
- Arimoto's converse ($+$ extension to feedback)

Meta-converse in networks



$$\{\text{error}\} = \{W_1 \neq \hat{W}_1\} \cup \{W_2 \neq \hat{W}_2\}$$

Meta-converse in networks



$$\{\text{error}\} = \{W_1 \neq \hat{W}_1\} \cup \{W_2 \neq \hat{W}_2\}$$

- Probability of error depends on channel:

$$\begin{aligned} \mathbb{P}[\text{error}] &= \epsilon\,, \\ \mathbb{Q}[\text{error}] &= \epsilon'\,. \end{aligned}$$

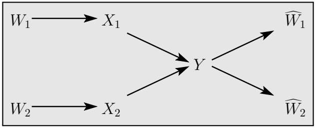
- Same idea: use code as a suboptimal binary HT: $P_{Y|X}$ vs. $Q_{Y|X}$
- ... and compare to the best possible test:

$$D(P_{XY} \| Q_{XY}) \geq d(1-\epsilon \| 1-\epsilon')$$

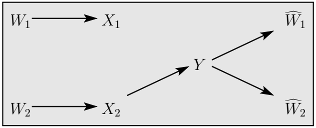
$$\beta_{1-\epsilon}(P_X P_{Y|X}, P_X Q_{Y|X}) \leq 1-\epsilon'$$

Example: MAC (weak-converse)

$\mathbb{P}$:



$\mathbb{Q}$:



$$\mathbb{P}[\hat{W}_{1,2} = W_{1,2}] = 1 - \epsilon \qquad \mathbb{Q}[\hat{W}_{1,2} = W_{1,2}] = \frac{1}{M_1}$$

... apply data processing of $D(\cdot||\cdot)$ ...

$$\Downarrow$$

$$d(1-\epsilon \| \tfrac{1}{M_1}) \le D(P_{Y|X_1X_2} \| Q_{Y|X_1} | P_{X_1} P_{X_2})$$

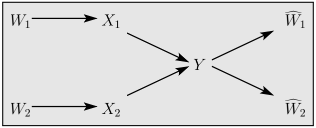
Optimizing $Q_{Y|X_1}$:

$$\log M_1 \le \frac{I(X_1; Y|X_2) + h(\epsilon)}{1-\epsilon}$$

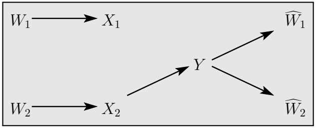
Also with $X_1 \leftrightarrow X_2 \implies$ weak converse (usual pentagon)

Example: MAC (FBL?)

$\mathbb{P}$:



$\mathbb{Q}$:



$$\mathbb{P}[\hat{W}_{1,2} = W_{1,2}] = 1 - \epsilon \qquad \mathbb{Q}[\hat{W}_{1,2} = W_{1,2}] = \frac{1}{M_1}$$

$$\ldots \text{ use } \beta_\alpha(\cdot,\cdot) \ldots$$
$$\Downarrow$$
$$\beta_{1-\epsilon}(P_{X_1 X_2 Y}, P_{X_1} Q_{X_2 Y}) \le \frac{1}{M_1}$$

On-going work: This β_α is highly non-trivial to compute.
[Huang-Moulin, MolavianJazi-Laneman, Yagi-Oohama]

Achievability Bounds

Notation

- A random transformation $\mathsf{A} \xrightarrow{P_{Y|X}} \mathsf{B}$
- (M, ϵ) codes:

$$W \to \mathsf{A} \to \mathsf{B} \to \hat{W} \qquad W \sim Unif\{1, \ldots, M\}$$

$$\mathbb{P}[W \neq \hat{W}] \leq \epsilon$$

- For every $P_{XY} = P_X P_{Y|X}$ define **information density**:

$$\imath_{X;Y}(x;y) \triangleq \log \frac{dP_{Y|X=x}}{dP_Y}(y)$$

  - $\mathbb{E}\left[\imath_{X;Y}(X;Y)\right] = I(X;Y)$
  - $\mathrm{Var}[\imath_{X;Y}(X;Y)|X] = V$
  - Memoryless channels: $\imath_{A^n;B^n}(A^n;B^n) =$ sum of iid.

$$\imath_{A^n;B^n}(A^n;B^n) \stackrel{d}{\approx} nI(A;B) + \sqrt{nV}Z, \qquad Z \sim \mathcal{N}(0,1)$$

- Goal: Prove FBL bounds.
  As by-product: $R^*(n,\epsilon) \gtrsim C - \sqrt{\frac{V}{n}}Q^{-1}(\epsilon)$

Achievability bounds: classical ideas

Goal: select codewords $C_1, \ldots, C_M$ in the input space A.

Two principal approaches:

- Random coding: generate $C_1, \ldots, C_M$ – iid with P_X and compute average probability of error [Shannon'48, Erdös'47].
- Maximal coding: choose C_j one by one until the output space is exhausted [Gilbert'52, Feinstein'54, Varshamov'57].

Complication: Many inequivalent ways to apply these ideas! Which ones are the best for FBL?

Classical bounds

- Feinstein'55 bound: $\exists(M,\epsilon)$-code:

$$M \geq \sup_{\gamma \geq 0} \left\{ \gamma(\epsilon - \mathbb{P}[\imath_{X;Y}(X;Y) \leq \log \gamma]) \right\}$$

- Shannon'57 bound: $\exists(M,\epsilon)$-code:

$$\epsilon \leq \inf_{\gamma \geq 0} \left\{ \mathbb{P}[\imath_{X;Y}(X;Y) \leq \log \gamma] + \frac{M-1}{\gamma} \right\} .$$

- Gallager'65 bound: $\exists(n,M,\epsilon)$-code over memoryless channel:

$$\epsilon \leq \exp\left\{ -nE_r\left(\frac{\log M}{n}\right) \right\} .$$

- Up to $M \leftrightarrow (M-1)$ Feinstein and Shannon are equivalent.

New bounds: RCU

Theorem (Random Coding Union Bound)

For any P_X there exists a code with M codewords and

$$\epsilon \le \mathbb{E}\left[\min\left\{1, (M-1)\pi(X,Y)\right\}\right]$$

$$\pi(a,b) = \mathbb{P}[\imath_{X;Y}(\bar{X};Y) \ge \imath_{X;Y}(X;Y) \,|\, X=a, Y=b]$$

where $P_{XY\bar{X}}(a,b,c) = P_X(a)P_{Y|X}(b|a)P_X(c)$

Proof:

- Reason as in RCU for BSC with $d_{Ham}(\cdot,\cdot) \leftrightarrow -\imath_{X;Y}(\cdot,\cdot)$
- For example ML decoder: $\hat{W} = \mathrm{argmax}_j\, \imath_{X;Y}(C_j;Y)$
- Conditional prob. of error:

$$\mathbb{P}[\text{error} \,|\, X,Y] \le (M-1)\mathbb{P}[\imath_{X;Y}(\bar{X};Y) \ge \imath_{X;Y}(X;Y) \,|\, X,Y]$$

- Same idea: take $\min\{\cdot, 1\}$ before averaging over (X,Y).

New bounds: RCU

Theorem (Random Coding Union Bound)

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Highlights:

- Strictly stronger than Feinstein-Shannon and Gallager
- Not easy to analyze asymptotics
- Computational complexity $O(n^{2(|X|-1)|Y|})$

New bounds: DT

Theorem (Dependence Testing Bound)

For any P_X there exists a code with M codewords and

$$\epsilon \le \mathbb{E}\left[\exp\left\{-\left|\imath_{X;Y}(X;Y) - \log\frac{M-1}{2}\right|^+\right\}\right].$$

Highlights:

- Strictly stronger than Feinstein-Shannon
- … and no optimization over γ!
- Easier to compute than RCU
- Easier asymptotics: $\epsilon \le \mathbb{E}\left[e^{-n\left|\frac{1}{n}\imath(X^n;Y^n)-R\right|^+}\right]$
  $\approx Q\left(\sqrt{\frac{n}{V}}\{I(X;Y)-R\}\right)$
- Has a form of f-divergence: $1-\epsilon \ge D_f(P_{XY}\|P_X P_Y)$

DT bound: Proof

- Codebook: random $C_1, \dots C_M \sim P_X$ iid
- Feinstein decoder:

$$\hat{W} = \text{smallest } j \text{ s.t. } \imath_{X;Y}(C_j; Y) > \gamma$$

- j-th codeword's probability of error:

$$\mathbb{P}[\text{error} \mid W = j] \le \underbrace{\mathbb{P}[\imath_{X;Y}(X;Y) \le \gamma]}_{a} + (j-1)\underbrace{\mathbb{P}[\imath_{X;Y}(\bar{X};Y) > \gamma]}_{b}$$

In a: C_j too far from Y
In b: C_k with $k < j$ is too close to Y

- Average over W:

$$\mathbb{P}[\text{error}] \le \mathbb{P}\left[\imath_{X;Y}(X;Y) \le \gamma\right] + \frac{M-1}{2}\mathbb{P}\left[\imath_{X;Y}(\bar{X};Y) > \gamma\right]$$

DT bound: Proof

- Recap: for every γ there exists a code with

$$\epsilon \le \mathbb{P}\left[\imath_{X;Y}(X;Y) \le \gamma\right] + \frac{M-1}{2}\mathbb{P}\left[\imath_{X;Y}(\bar{X};Y) > \gamma\right] .$$

- Key step: closed-form optimization of γ.
  Note: $\imath_{X;Y} = \log \frac{dP_{XY}}{dP_{\bar{X}Y}}$

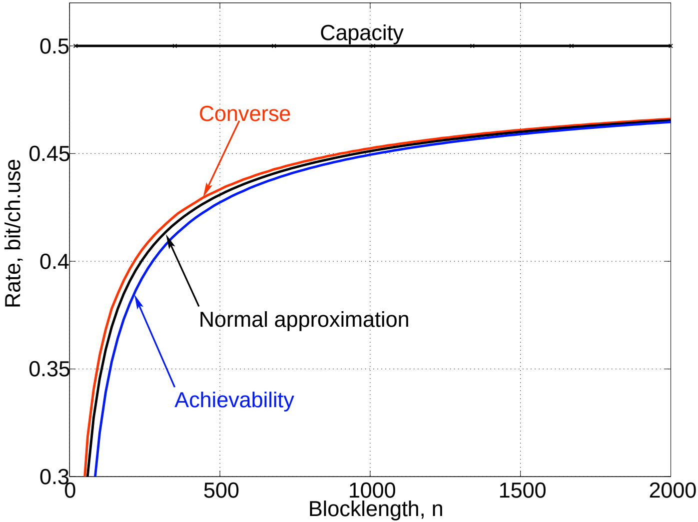
$$\frac{M+1}{2}\left(\frac{2}{M+1}P_{XY}\left[\frac{dP_{XY}}{dP_{\bar{X}Y}} \le e^{\gamma}\right] + \frac{M-1}{M+1}P_{\bar{X}Y}\left[\frac{dP_{XY}}{dP_{\bar{X}Y}} > e^{\gamma}\right]\right)$$

  Bayesian dependence testing!
  Optimum threshold: Ratio of priors $\Longrightarrow$ $\boxed{\gamma^* = \log \frac{M-1}{2}}$

- Change of measure argument:

$$P\left[\frac{dP}{dQ} \le \tau\right] + \tau Q\left[\frac{dP}{dQ} > \tau\right] = \mathbb{E}_P\left[\exp\left\{-\left|\log\frac{dP}{dQ} - \log\tau\right|^{+}\right\}\right] .$$

Example: Binary Erasure Channel $BEC(0.5)$, $\epsilon = 10^{-3}$



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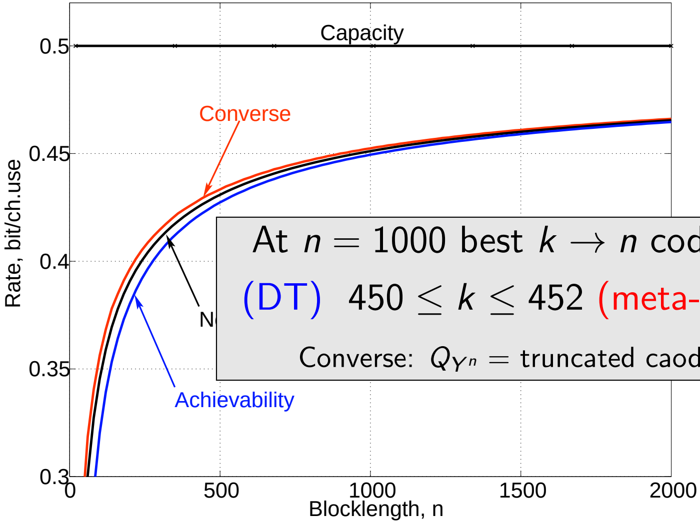


At $n = 1000$ best $k \to n$ code:

(DT) $450 \leq k \leq 452$ (meta-c.)

Converse: $Q_{Y^n} =$ truncated caod

Input constraints: $\kappa\beta$ bound

Theorem

For all Q_Y and τ there exists an (M, ϵ)-code inside $\mathsf{F} \subset \mathsf{A}$

$$M \geq \frac{\kappa_\tau}{\sup_x \beta_{1-\epsilon+\tau}(P_{Y|X=x}, Q_Y)}$$

where

$$\kappa_\tau = \inf_{\{E\colon P_{Y|X}[E|x] \geq \tau \ \forall x \in \mathsf{F}\}} Q_Y[E]$$

Highlights:

- Key for channels with cost constraints (e.g. AWGN).
- Bound parameterized by the output distribution.
- Reduces coding to binary HT.

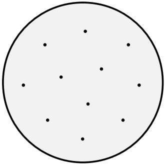
$\kappa\beta$ bound: idea

Decoder:

- Take received $\mathbf{y}$.
- Test $\mathbf{y}$ against *each* codeword $\mathbf{c}_i$, $i = 1, \ldots M$:

  Run optimal binary HT for :

$$\begin{array}{ccc} \mathcal{H}_0 & : & P_{Y|X=c_i} \\ \mathcal{H}_1 & : & Q_Y \end{array}$$



$$\begin{array}{rcl} \mathbb{P}[\text{detect } \mathcal{H}_0] & = & 1 - \epsilon + \tau \\ \mathbb{Q}[\text{detect } \mathcal{H}_0] & = & \beta_{1-\epsilon+\tau}(P_{Y|X=x}, Q_Y) \end{array}$$

- First test that returns $\mathcal{H}_0$ becomes **the decoded codeword**.
- If all $\mathcal{H}_1$ – **declare error**.

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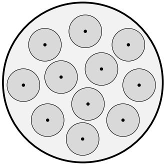

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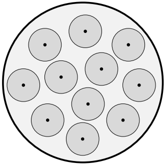
- If all $\mathcal{H}_1$ – **declare error**.

$\kappa\beta$ bound: idea

Codebook:

- Pick codewords s.t. "balls" are τ-disjoint: $\mathbb{P}[Y \in B_x \cap \text{others} | x] \le \tau$
- Key step: Cannot pick more codewords $\Longrightarrow$ $\bigcup_{j=1}^{M} \{j\text{-th decoding region}\}$ is a composite HT:



$$\begin{array}{lcll} \mathcal{H}_0 & : & P_{Y|X=x} & x \in \mathsf{F} \\ \mathcal{H}_1 & : & Q_Y & \end{array}$$

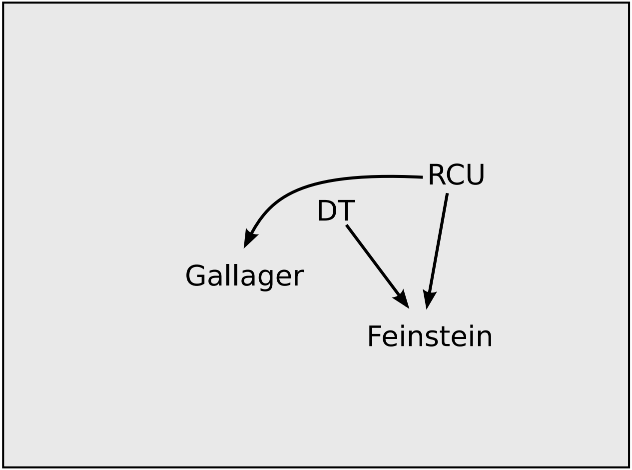
- Performance of the best test:

$$\kappa_\tau = \inf_{\{E:\, P_{Y|X}[E|x] \ge \tau \ \forall x \in \mathsf{F}\}} Q_Y[E]\,.$$

- Thus:

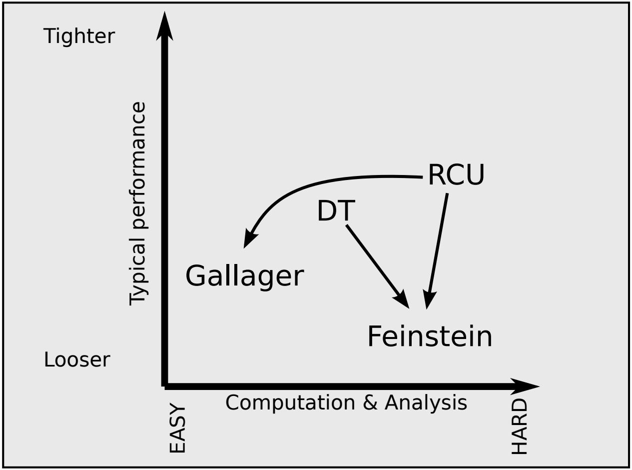
$$\begin{aligned} \kappa_\tau &\le \mathbb{Q}[\text{all } M \text{ "balls"}] \\ &\le M \sup_x \beta_{1-\epsilon+\tau}(P_{Y|X=x}, Q_Y) \end{aligned}$$

Hierarchy of achievability bounds (no cost constr.)



- Arrows show logical implication

Hierarchy of achievability bounds (no cost constr.)



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- Performance $\leftrightarrow$ computation **rule of thumb**.

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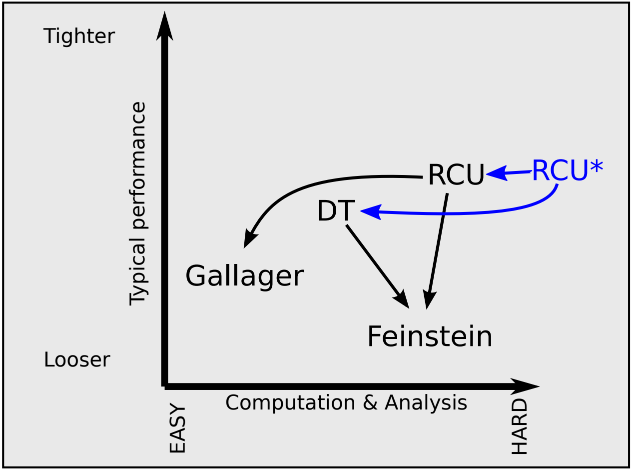

- Arrows show logical implication
- Performance $\leftrightarrow$ computation **rule of thumb**.
- **ISIT'2013**: Haim-Kochman-Erez [WeB4]

Channel Dispersion

Connection to CLT

Recap:

- Let $P_{Y^n|X^n} = P^n_{Y|X}$ be memoryless. FBL fundamental limit:

$$R^*(n,\epsilon) = \max\left\{\frac{1}{n}\log M : \exists (n, M, \epsilon)\text{-code}\right\}$$

- Converse bounds (roughly):

$$R^*(n,\epsilon) \lesssim \epsilon\text{-th quantile of } \frac{1}{n}\log\frac{dP_{Y^n|X^n}}{dQ_{Y^n}}$$

- Achievability bounds (roughly):

$$R^*(n,\epsilon) \gtrsim \epsilon\text{-th quantile of } \frac{1}{n}\imath_{X^n;Y^n}(X^n;Y^n)$$

- Both random variables have form: $\frac{1}{n}\cdot$(sum of iid) $\Longrightarrow$ by CLT

$$R^*(n,\epsilon) = C + \theta\left(\frac{1}{\sqrt{n}}\right)$$

This section: Study $\sqrt{n}$-term.

General definition of channel dispersion

Definition

For any channel we define channel dispersion as

$$V = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{n\left(C - R^*(n,\epsilon)\right)^2}{2 \ln \frac{1}{\epsilon}}$$

Rationale is the expansion (see below)

$$R^*(n,\epsilon) = C - \sqrt{\frac{V}{n}} Q^{-1}(\epsilon) + o\left(\frac{1}{\sqrt{n}}\right) \quad (*)$$

and the fact $Q^{-1}(\epsilon) \sim 2 \ln \frac{1}{\epsilon}$ for $\epsilon \to 0$

Recall: Approximation via $(*)$ is remarkably tight

General definition of channel dispersion

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Heuristic connection to error exponents $E(R)$:

$$E(R) = \frac{(R-C)^2}{2} \cdot \frac{\partial^2 E(R)}{\partial R^2} + o((R-C)^2)$$

and thus

$$V = \left(\frac{\partial^2 E(R)}{\partial R^2}\right)^{-1}$$

Dispersion of memoryless channels

- DMC [Dobrushin'61,Strassen'62]:

$$V = \mathrm{Var}[\imath_{X;Y}(X;Y)] \qquad X \sim \text{capacity-achieving}$$

- AWGN channel [PPV'08]:

$$V = \frac{\log^2 e}{2}\left[1 - \left(\frac{1}{1+\mathrm{SNR}}\right)^2\right]$$

- Parallel AWGN [PPV'09]:

$$V = \sum_{j=1}^{L} V_{AWGN}\left(\frac{W_j}{\sigma_j^2}\right) \qquad \{W_j\}\text{–waterfilling powers}$$

- DMC with input constraints [Hayashi'09,P'10]:

$$V = \mathrm{Var}[\imath_{X;Y}(X;Y)|X] \quad X \sim \text{capacity-achieving}$$

Dispersion of channels with memory

From [PPV'09, PPV'10, PV'11]:

- Non-white Gaussian noise with PSD $N(f)$:

$$V = \frac{\log^2 e}{2} \int_{-1/2}^{1/2} \left[1 - \frac{|N(f)|^4}{P^2 \xi^2}\right]^+ df, \qquad \int_{-1/2}^{1/2} \left[\xi - \frac{|N(f)|^2}{P}\right]^+ df = 1$$

- AWGN subject to stationary fading process H_i (CSI at receiver):

$$V = \mathrm{PSD}_{\frac{1}{2}\log(1+PH_i^2)}(0) + \frac{\log^2 e}{2}\left(1 - \mathbb{E}^2\left[\frac{1}{1+PH_0^2}\right]\right)$$

- State-dependent discrete additive noise (CSI at receiver):

$$V = \mathrm{PSD}_{C(S_i)}(0) + \mathbb{E}\left[V(S)\right]$$

- **ISIT'13**: Quasi-static fading channels: $V = 0$ (!)
  Yang-Durisi-Koch-P. in [WeA4]

Dispersion: product vs generic channels

- Relation to alphabet size:

$$V \le 2\log^2 \min(|\mathcal{A}|, |\mathcal{B}|) - C^2 \,.$$

- Dispersion is additive:

$$\left\{ \begin{array}{ccccc} \mathcal{A}_1 & \to & \boxed{DMC_1} & \to & \mathcal{B}_1 \\ \mathcal{A}_2 & \to & \boxed{DMC_2} & \to & \mathcal{B}_2 \end{array} \right\} = \mathcal{A}_1 \times \mathcal{A}_2 \to \boxed{DMC} \to \mathcal{B}_1 \times \mathcal{B}_2$$

$$C = C_1 + C_2\,, \qquad V_\epsilon = V_{1,\epsilon} + V_{2,\epsilon}$$

- $\Longrightarrow$ product DMCs have atypically low dispersion.

Dispersion and normal approximation

Let $P_{Y|X}$ be DMC and

$$V_\epsilon \triangleq \begin{cases} \max_{P_X} \mathrm{Var}[i(X,Y)|X]\,, & \epsilon < 1/2\,, \\ \min_{P_X} \mathrm{Var}[i(X,Y)|X]\,, & \epsilon > 1/2 \end{cases}$$

where optimization is over all P_X s.t. $I(X;Y) = C$.

Theorem (Strassen'62)

$$R^*(n,\epsilon) = C - \sqrt{\frac{V_\epsilon}{n}} Q^{-1}(\epsilon) + O\left(\frac{\log n}{n}\right)$$

But [PPV'10]: a counter-example with

$$R^*(n,\epsilon) = C + \Theta\left(n^{-\frac{2}{3}}\right)$$

Dispersion and normal approximation

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Theorem (Strassen'62, PPV'10)

$$R^*(n,\epsilon) = C - \sqrt{\frac{V_\epsilon}{n}} Q^{-1}(\epsilon) + O\left(\frac{\log n}{n}\right)$$

unless DMC is exotic in which case $O\left(\frac{\log n}{n}\right)$ becomes $O(n^{-\frac{2}{3}})$.

Further results on $O\left(\frac{\log n}{n}\right)$

- For **BEC** we have:

$$R^*(n,\epsilon) = C - \sqrt{\frac{V}{n}}Q^{-1}(\epsilon) + 0 \cdot \frac{\log n}{n} + O\left(\frac{1}{n}\right)$$

- For most other symmetric channels (incl. **BSC** and **AWGN***):

$$R^*(n,\epsilon) = C - \sqrt{\frac{V}{n}}Q^{-1}(\epsilon) + \frac{1}{2}\frac{\log n}{n} + O\left(\frac{1}{n}\right)$$

- For **most DMC** (under mild conditions):

$$R^*(n,\epsilon) \geq C - \sqrt{\frac{V_\epsilon}{n}}Q^{-1}(\epsilon) + \frac{1}{2}\frac{\log n}{n} + O\left(\frac{1}{n}\right)$$

- ISIT'13: For **all DMC**

$$R^*(n,\epsilon) \leq C - \sqrt{\frac{V_\epsilon}{n}}Q^{-1}(\epsilon) + \frac{1}{2}\frac{\log n}{n} + O\left(\frac{1}{n}\right)$$

Tomamichel-Tan, Moulin in [WeA4]

Applications

Evaluating performance of real-world codes

- Comparing codes: usual method – waterfall plots P_e vs. SNR
- **Problem:** Not fair for different rates.

  $\Longrightarrow$ define rate-invariant metric:

Evaluating performance of real-world codes

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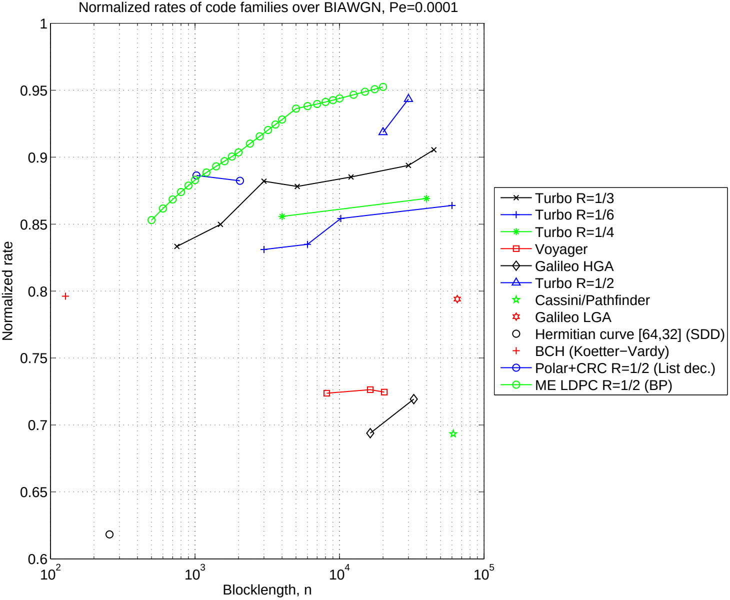
Definition (Normalized rate)

Given rate R code find SNR at which $P_e = \epsilon$.

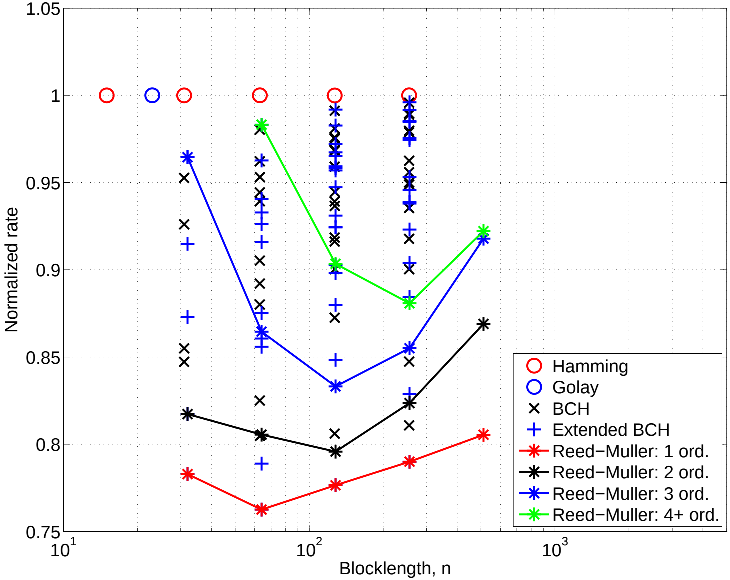
$$R_{norm} = \frac{R}{R^*(n, \epsilon, SNR)}$$

- Agreement: $\epsilon = 10^{-3}$ or 10^{-4}
- Take $R^*(n, \epsilon, SNR) \approx C - \sqrt{\frac{V}{n}} Q^{-1}(\epsilon)$
- A family of channels needed (e.g. AWGN or BSC)

Codes vs. fundamental limits (from 1970's to 2012)



Performance of short algebraic codes (BSC, $\epsilon = 10^{-3}$)



Optimizing ARQ systems

- End-user wants $P_e = 0$
- Usual method: automatic repeat request (ARQ)

$$\text{average throughput} = \text{Rate} \times (1 - \mathbb{P}[\text{error}])$$

- **Question:** Given k bits what rate (equiv. ϵ) maximizes throughput?

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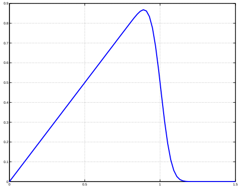
$$\text{average throughput} = \text{Rate} \times (1 - \mathbb{P}[\text{error}])$$

- **Question:** Given k bits what rate (equiv. ϵ) maximizes throughput?
- Assume (C, V) is known. Then approximately

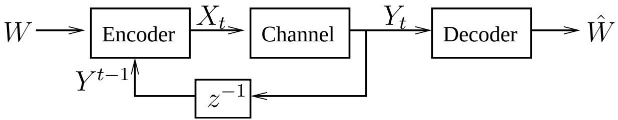
$$T^*(k) \approx \max_R R \cdot \left(1 - Q\left(\sqrt{\frac{kR}{V}}\left\{\frac{C}{R} - 1\right\}\right)\right)$$



- Solution: $\epsilon^*(k) \sim \frac{1}{\sqrt{kt \log kt}}, \quad t = \frac{C}{V}$
- Punchline: For $k \sim 1000$ bit and reasonable channels

$$\boxed{\epsilon \approx 10^{-3}..10^{-2}}$$

Benefits of feedback: From ARQ to Hybrid ARQ



- Memoryless channels: feedback does not improve C [Shannon'56]
- Question: What about higher order terms?

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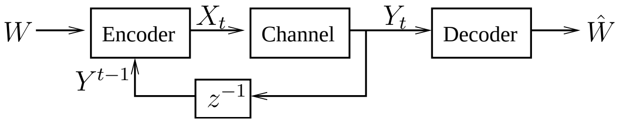


- Memoryless channels: feedback does not improve C [Shannon'56]
- Question: What about higher order terms?

Theorem

For any DMC with capacity C and $0 < \epsilon < 1$ we have for codes with feedback and variable length:

$$R_f^*(n, \epsilon) = \frac{C}{1-\epsilon} + O\left(\frac{\log n}{n}\right) .$$

Note: dispersion is zero!

Stop feedback bound (BSC version)

Theorem

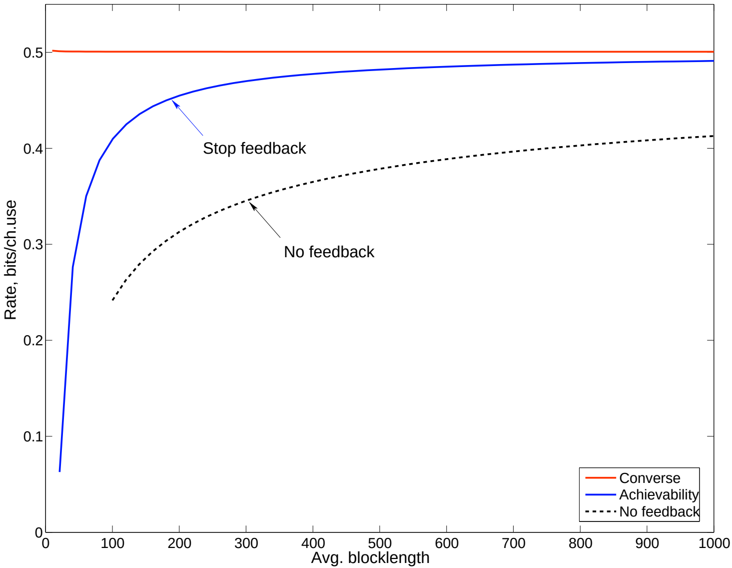
For any $\gamma > 0$ there exists a stop feedback code of rate R, average length $\ell = \mathbb{E}[\tau]$ and probability of error over BSC(δ)

$$\epsilon \le \mathbb{E}\left[f(\tau)\right],$$

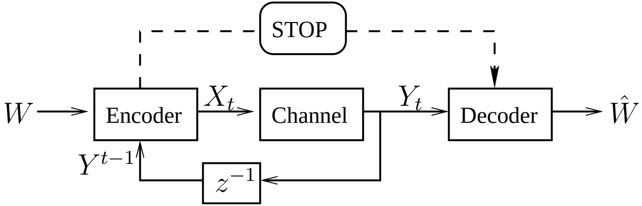
where

$$\begin{aligned} f(n) &\triangleq \mathbb{E}\left[1\{\tau \le n\}2^{\ell R - S_\tau}\right] \\ \tau &\triangleq \inf\{n \ge 0 : S_n \ge \gamma\} \\ S_n &\triangleq n\log(2-2\delta) + \log\frac{\delta}{1-\delta}\cdot\sum_{k=1}^{n} Z_k \\ Z_k &\sim \textit{i.i.d. Bernoulli}(\delta)\,. \end{aligned}$$

Feedback codes for BSC(0.11), $\epsilon = 10^{-3}$



Effects of flow control



- Modeling of packet termination
- Often: reliability of start/end $\gg$ reliability of payload

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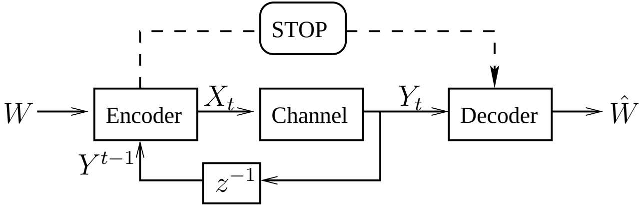


- Modeling of packet termination
- Often: reliability of start/end $\gg$ reliability of payload

Theorem

If reliable termination is available, then there exist codes with variable length and feedback achieving

$$R_t^*(n, 0) \geq C + O\left(\frac{1}{n}\right) .$$

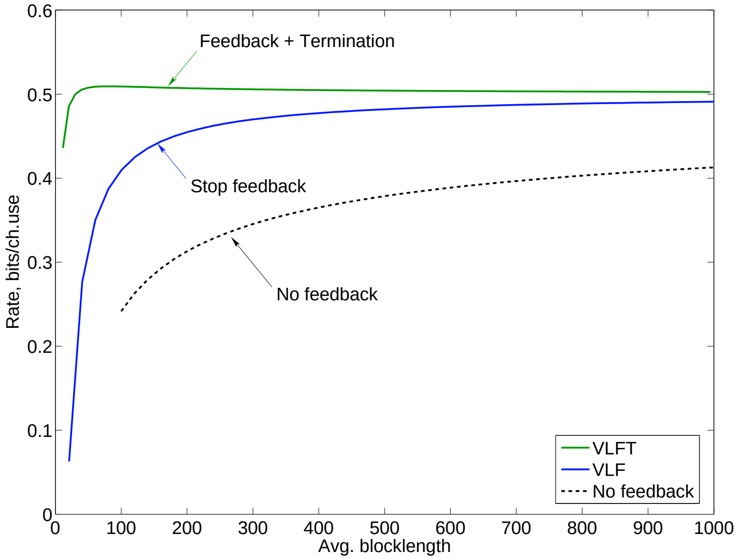
Codes employing feedback + termination (BSC version)

Theorem

Consider a BSC(δ) with feedback and reliable termination. There exists a code sending k bits with zero error and average length

$$\ell \leq \sum_{n=0}^{\infty} \sum_{t=0}^{n} \binom{n}{t} \delta^t (1-\delta)^{n-t} \min\left\{1, \sum_{k=0}^{t} \binom{n}{k} 2^{k-n}\right\} .$$

Feedback + termination for the BSC(0.11)



Benefit of feedback

Delay to achieve 90% of the capacity of the BSC(0.11):

- No feedback:

$$n \approx 3100$$

- Stop feedback + variable-length:

$$n \lesssim 200$$

- Feedback + variable-length + termination:

$$n \lesssim 20$$

Benefit of feedback

Delay to achieve 90% of the capacity of the BSC(0.11):

- No feedback:

$$n \approx 3100 \qquad \text{penalty term:} \quad O\left(\frac{1}{\sqrt{n}}\right)$$

- Stop feedback + variable-length:

$$n \lesssim 200 \qquad \text{penalty term:} \quad O\left(\frac{\log n}{n}\right)$$

- Feedback + variable-length + termination:

$$n \lesssim 20 \qquad \text{penalty term:} \quad O\left(\frac{1}{n}\right)$$

Gaussian channel: Energy per bit

$$
\begin{array}{ccccc}
 & & Z_i \sim \mathcal{N}\left(0, \frac{N_0}{2}\right) & & \\
 & & \downarrow & & \\
X_i & \longrightarrow & \bigoplus & \longrightarrow & Y_i
\end{array}
$$

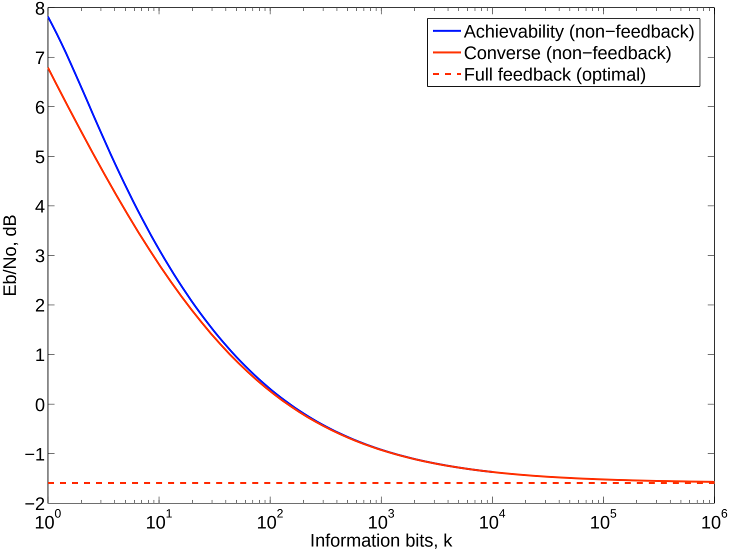
Problem: minimal energy-per-bit E_b *vs.* payload size k:

$$\mathbb{E}\left[\sum_{i=1}^{n} |X_i|^2\right] \le kE_b \,.$$

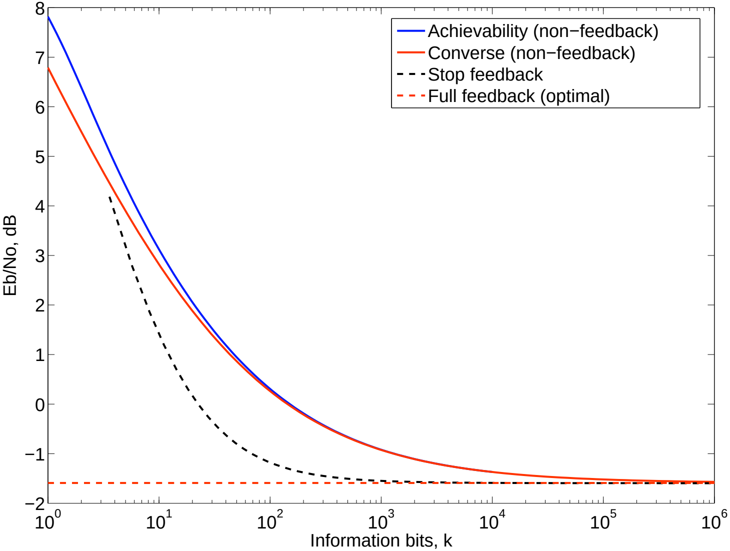
Asymptotically: [Shannon'49]

$$\min\left(\frac{E_b}{N_0}\right) \to \log 2 = -1.6 \text{ dB} \qquad , k \to \infty \,.$$

Energy per bit vs. # of information bits ($\epsilon = 10^{-3}$)

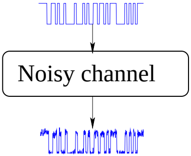


Energy per bit vs. # of information bits ($\epsilon = 10^{-3}$)

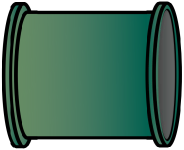


Summary

Classical: $(n \to \infty)$

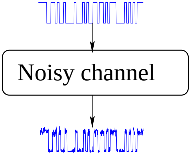


Optimal coding $\Longrightarrow$

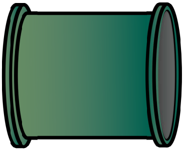


C

Finite blocklength: $(n - \text{finite})$



Optimal coding $\Longrightarrow$



$\mathcal{N}\left(C, \frac{V}{n}\right)$

What we had to skip

- **Hypothesis testing methods in Quantum IT:** [Wang-Colbeck-Renner'09], [Matthews-Wehner'12], [Tomamichel'12],[Kumagai-Hayashi'13]
- **Channels with state:** [Ingber-Feder'10], [Tomamichel-Tan'13], [Yang-Durisi-Koch-P.'12]
- **FBL theory of lattice codes:** [Ingber-Zamir-Feder'12]
- **Feedback codes:** [Naghshvar-Javidi'12], [Williamson-Chen-Wesel'12]
- **Random coding bounds and approximations:** [Martinez-Guillen i Fabregas'11], [Kosut-Tan'12]
- **Other FBL questions:** [Riedl-Coleman-Singer'11], [Varshney-Mitter-Goyal'12], [Asoodeh-Lapidoth-Wang'12], [P.-Wu'13]

...and many more (apologies!) ... (cf: References)

New results at ISIT'2013: two terminals

- Universal lossless compression: Kosut-Sankar [MoD5]
- Random number generation: Kumagai-Hayashi [WeA3]
- Quasi-static SIMO: Yang-Durisi-Koch-P. [WeA4]
- $O(\log n) = \frac{1}{2} \log n$: Tomamichel-Tan, Moulin [WeA4]
- Meta-converse is tight: Vazquez-Vilar et al [WeB4]
- Meta-converse for unequal error protection: Shkel-Tan-Draper [WeB4]
- RCU* bound: Haim-Kochman-Erez [WeB4]
- Cost constraints: Kostina-Verdú [WeB4]
- Lossless compression: Kontoyiannis-Verdú [WeB5]
- Feedback: Chen-Williamson-Wesel [ThD6]

New results at ISIT'2013: multi-terminal

- achievability bounds: Yassae-Aref-Gohari [TuD1]
- random binning: Yassae-Aref-Gohari [ThA1]
- interference channel: Le-Tan-Motani [ThA1]
- Gaussian line network: Subramanian-Vellambi-Land [ThA1]
- Slepian-Wolf for mixed sources: Nomura-Han [ThA7]
- one-help-one and Wyner-Ziv: Watanabe-Kuzuoka-Tan [FrC5]

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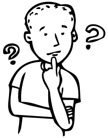
Thank you!



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