# Broadcasting on Two-Dimensional Regular Grids 

Anuran Makur, Elchanan Mossel, and Yury Polyanskiy


#### Abstract

We study an important specialization of the general problem of broadcasting on directed acyclic graphs, namely, that of broadcasting on two-dimensional (2D) regular grids. Consider an infinite directed acyclic graph with the form of a 2D regular grid, which has a single source vertex $X$ at layer 0 , and $k+1$ vertices at layer $k \geq 1$, which are at a distance of $k$ from $X$. Every vertex of the 2D regular grid has outdegree 2, the vertices at the boundary have indegree 1 , and all other non-source vertices have indegree 2. At time $0, X$ is given a uniform random bit. At time $k \geq 1$, each vertex in layer $k$ receives transmitted bits from its parents in layer $k-1$, where the bits pass through independent binary symmetric channels with common crossover probability $\delta \in\left(0, \frac{1}{2}\right)$ during the process of transmission. Then, each vertex at layer $k$ with indegree 2 combines its two input bits using a common deterministic Boolean processing function to produce a single output bit at the vertex. The objective is to recover $X$ with probability of error better than $\frac{1}{2}$ from all vertices at layer $k$ as $k \rightarrow \infty$. Besides their natural interpretation in the context of communication networks, such broadcasting processes can be construed as one-dimensional (1D) probabilistic cellular automata, or discrete-time statistical mechanical spin-flip system on 1D lattices, with boundary conditions that limit the number of sites at each time $k$ to $k+1$. Inspired by the literature surrounding the "positive rates conjecture" for 1D probabilistic cellular automata, we conjecture that it is impossible to propagate information in a 2 D regular grid regardless of the noise level $\delta$ and the choice of common Boolean processing function. In this paper, we make considerable progress towards establishing this conjecture, and prove using ideas from percolation and coding theory that recovery of $X$ is impossible for any $\delta \in\left(0, \frac{1}{2}\right)$ provided that all vertices with indegree 2 use either AND or XOR for their processing functions. Furthermore, we propose a detailed and general martingale-based approach that establishes the impossibility of recovering $X$ for any $\delta \in\left(0, \frac{1}{2}\right)$ when all NAND processing functions are used if certain structured supermartingales can be rigorously constructed. We also provide strong numerical evidence for the existence of these supermartingales by computing several explicit examples for different values of $\delta$ via linear programming.


## Index Terms

Broadcasting, probabilistic cellular automata, oriented bond percolation, linear code, supermartingale, potential function, linear programming.

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A. Makur and Y. Polyanskiy are with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139, USA (e-mail: a_makur@mit.edu; yp@mit.edu).

This research was supported in part by the Center for Science of Information (CSoI), a National Science Foundation (NSF) Science and Technology Center, under grant agreement CCF-09-39370, in part by the NSF grant CCF-17-17842, and in part by the MIT-IBM Watson AI Lab.
E. Mossel is with the Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA (e-mail: elmos@mit.edu).

This research was supported in part by the Vannevar Bush Faculty Fellowship N00014-20-1-2826, in part by the NSF award CCF-1918421, in part by the Simons Investigator award (622132), in part by the MURI grant W911NF1910217, and in part by the NSF award DMS-1737944.
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## I. Introduction

The problem of broadcasting on two-dimensional (2D) regular grids is an important specialization of the broader question of broadcasting on bounded indegree directed acyclic graphs (DAGs), which was introduced in [1]-[3] to generalize the classical problem of broadcasting on trees and Ising models, cf. [4]. In contrast to the canonical study of reliable communication through broadcast channels in network information theory [5, Chapters 5 and 8], the broadcasting problem, as studied in this paper, analyzes whether or not the "wavefront of information" emitted (or broadcasted) by a single transmitter decays irrecoverably as it propagates through a large, and typically structured, Bayesian network. For example, in the broadcasting on trees setting, we are given a Bayesian network whose underlying DAG is a rooted tree $T$, vertices are Bernoulli random variables, and edges are independent binary symmetric channels (BSCs) with common crossover probability $\delta \in\left(0, \frac{1}{2}\right)$. The root contains a uniform random bit that it transmits through $T$, and our goal is to decode this bit from the received values of the vertices at an arbitrarily deep layer $k$ (i.e. at distance $k$ from the root) of $T$. It is proved in a sequence of papers [4], [6], [7] that the root bit can be decoded with minimum probability of error bounded away from $\frac{1}{2}$ as $k \rightarrow \infty$ if and only if $(1-2 \delta)^{2} \operatorname{br}(T)>1$, i.e. the noise level $\delta$ is strictly less than the critical Kesten-Stigum threshold $\frac{1}{2}\left(1-(1 / \operatorname{br}(T))^{1 / 2}\right)$, which depends on the branching number $\operatorname{br}(T)$ (see [8, Chapter 1.2]). This key result and its generalizations, cf. [9]-[17], precisely characterize when information about the root bit contained in the vertices at arbitrarily deep layers of a Bayesian network with a tree-structured topology vanishes completely.

Although broadcasting on trees is amenable to various kinds of tractable analysis, the general problem of broadcasting on bounded indegree DAGs arguably better models real-world communication or social networks, where each vertex or agent usually receives multiple noisy input signals and has to judiciously consolidate these signals using simple rules. In the broadcasting on DAGs setting, we are given a Bayesian network with a single unbiased Bernoulli source vertex such that all other Bernoulli vertices have bounded indegree, and all edges are independent BSCs with noise level $\delta \in\left(0, \frac{1}{2}\right)$. Moreover, the vertices with indegree larger than 1 compute their values by applying Boolean processing functions to their noisy inputs. Determining the precise conditions on the graph topology (e.g. the bound on the indegrees), the noise level $\delta$, and the choices of Boolean processing functions that permit successful reconstruction of the source bit is quite challenging. As a result, we usually characterize when reconstruction is possible for specific classes of DAGs and choices of processing functions.

For instance, our earlier work [1], [2] studies randomly constructed DAGs with indegrees $d$ and layer sizes $L_{k}$, where $L_{k}$ denotes the number of vertices at layer $k$ (i.e. at distance $k$ from the source vertex). It is established in [2, Theorem 1] that if $d \geq 3$ and all majority processing functions are used, then reconstruction of the source bit is possible using the majority decoder when $\delta<\delta_{\text {maj }}(d)$ and $L_{k} \geq C(\delta, d) \log (k)$ for all sufficiently large $k$, where $\delta_{\text {maj }}(d)$ is a known critical threshold (cf. [2, Equation (11)]) and $C(\delta, d)>0$ is some fixed constant. Furthermore, [2, Theorem 2] shows a similar result when $d=2$ and all NAND processing functions are used. (Partial converse results to [2, Theorems 1 and 2] are also developed in [2].) The aforementioned results demonstrate, after employing the probabilistic method, the existence of deterministic DAGs with bounded indegree and $L_{k}=\Theta(\log (k))$ such that
reconstruction is possible for sufficiently small noise levels $\delta$. In fact, [2, Theorem 3, Proposition 2] also presents an explicit quasi-polynomial time construction of such deterministic DAGs using regular bipartite lossless expander graphs. In a nutshell, the results in [1], [2] illustrate that while $L_{k}$ must be exponential in $k$ for reconstruction to be possible in trees, logarithmic $L_{k}$ suffices for reconstruction in bounded indegree DAGs, because DAGs enable information fusion (or local "error correction") at the vertices.

As opposed to the randomly constructed and expander-based DAGs analyzed in [1], [2], in this paper, we consider the problem of broadcasting on another simple and important class of deterministic DAGs, namely, 2D regular grids. 2D regular grids correspond to DAGs with $L_{k}=k+1$ such that every vertex has outdegree 2 , the vertices at the boundary have indegree 1 , and all other non-source vertices have indegree 2 . For simplicity, we study the setting where the Boolean processing functions at all vertices with indegree 2 are the same. As we will explain shortly, the literature on one-dimensional (1D) probabilistic cellular automata suggests that reconstruction of the source bit is impossible for such 2 D regular grids regardless of the noise level $\delta$ and the choice of Boolean processing function. In this vein, the main contributions of this paper include two impossibility results that partially justify this intuition. In particular, we prove that recovery of the source bit is impossible on a 2 D regular grid if all intermediate vertices with indegree 2 use logical AND as the processing function, or all use logical XOR as the processing function. These proofs leverage ideas from percolation and coding theory. This leaves only NAND as the remaining symmetric processing function where the impossibility of reconstruction is unknown. Although we do not provide a complete proof of the impossibility of broadcasting with NAND processing functions, another main contribution of this paper is a careful elaboration of a martingale-based technique that yields the desired impossibility result if a certain family of superharmonic potential functions can be rigorously constructed. While such a (theoretical) construction currently remains open, we present some strong numerical evidence for it via linear programming. Furthermore, this martingale-based approach can easily be modified to obtain potential proofs for the impossibility of broadcasting on 2 D regular grids for other choices of Boolean processing functions. Thus, we believe that it is instructive for future work in this area.

## A. Motivation

As discussed in our earlier work [2], the general problem of broadcasting on bounded indegree DAGs is closely related to a myriad of problems in the literature. Besides its canonical broadcast interpretation in the context of communication networks, broadcasting on DAGs is a natural model of reliable computation and storage, cf. [18][23]. Indeed, the model can be construed as a noisy circuit that has been constructed to remember (or store) a bit, where the edges are wires that independently make errors, and the Boolean processing functions at the vertices are perfect logic gates. Special cases of the broadcasting model on certain families of DAGs also correspond to wellknown models in statistical physics. For example, broadcasting on trees corresponds to the extremality of certain Gibbs measures of ferromagnetic Ising models [4, Section 2.2], and broadcasting on regular grids is closely related to the ergodicity of discrete-time statistical mechanical spin-flip systems (such as probabilistic cellular automata) on lattices [24]-[26]. Furthermore, other special cases of the broadcasting model, such as on trees, represent information flow in biological networks, cf. [27]-[30], play a crucial role in random constraint satisfaction problems, cf. [31][34], and are useful in proving converse results for community detection in stochastic block models, cf. [35, Section 5.1].

The main motivation for this work, i.e. the problem of understanding whether it is possible to propagate information starting from the source specifically in regular grids (see Figure 1 for a 2D example), stems from the theory of probabilistic cellular automata (PCA). Our conjecture is that such propagation is possible for sufficiently low noise levels $\delta$ in 3 or more dimensions, and impossible for 2 D regular grids regardless of the noise level and the choice of Boolean processing function (which is the same for every vertex). This conjecture resembles and is inspired by the literature on the positive rates conjecture for 1D PCA, cf. [26, Section 1], and the existence of nonergodic 2D PCA such as that defined by Toom's North-East-Center (NEC) rule, cf. [36]. Notice that broadcasting processes on 2D regular grids can be perceived as 1D PCA with boundary conditions that limit the layer sizes to be $L_{k}=k+1$, and the impossibility of reconstruction on 2D regular grids intuitively corresponds to the ergodicity of 1D PCA (along with sufficiently fast convergence rate to stationarity). Therefore, the existence of a 2D regular grid with a choice of Boolean processing function which remembers its initial state bit for infinite time would suggest the existence of non-ergodic infinite 1D PCA consisting of 2-input binary-state cells. However, the positive rates conjecture states (in essence) that "relatively simple" 1D PCA with local interactions and strictly positive noise probabilities are ergodic, and known counter-example constructions to this conjecture either require a lot more states [37], or are non-uniform in time and space [38]. This intuition gives credence to our conjecture that


Fig. 1. Illustration of a 2 D regular grid where each vertex is a Bernoulli random variable and each edge is a BSC with parameter $\delta \in\left(0, \frac{1}{2}\right)$. Moreover, each vertex with indegree 2 uses a common Boolean processing function to combine its noisy input bits.
broadcasting is impossible for 2 D regular grids. Furthermore, much like 2D regular grids, broadcasting on threedimensional (3D) regular grids can be perceived as 2D PCA with boundary conditions (as shown in subsection II-D). Hence, the existence of non-ergodic 2D PCA, such as that in [36], suggests the existence of 3D regular grids where broadcasting is possible, thereby lending further credence to our conjecture. In this paper, we take some first steps towards establishing the 2D part of our larger conjecture.

Since the thrust of this paper partly hinges on the positive rates conjecture, we close this subsection by briefly expounding the underlying intuition behind it. Borrowing terminology from statistical physics, where a spin-flip system is said to experience a phase transition if it has more than one invariant measure, the positive rates conjecture informally states that there are "no phase transitions in one dimension" [26, Section 1]. In fact, this statement is true for the special (and better understood) case of Ising models, where there is a phase transition in 2D when the temperature is sufficiently small, but not in 1D [26, Section 2]. Building intuition off of Ising models, it is explained in [26, Section 2] that when a binary-state 2D PCA is non-ergodic, its multiple invariant measures are "close to" the various stable ground state configurations, e.g. "all 0's" and "all 1's." Furthermore, for a ground state configuration such as "all 0 's" to be deemed stable, we require that finite "islands" of 1 's that are randomly formed by the noise process are killed by the transition (or Boolean processing) functions of the automaton, e.g. Toom's NEC rule kills such finite islands starting with their corners. However, for 1D PCA, a transition function with finite interaction neighborhood that is at the boundary of a large finite island cannot easily distinguish the island from the true ground state. Therefore, we cannot expect stable ground state configurations to exist in such simple 1D PCA, and hence, it is conjectured that such 1D PCA are ergodic.

## B. Outline

We now delineate the organization of the remainder of this paper. The next subsection I-C formally defines the 2D regular grid model in order to present our results in future sections. Section II presents our main impossibility results for the AND and XOR cases, describes our partial impossibility result for the NAND case and provides some accompanying numerical evidence, discusses some other related impossibility results, and elucidates the connection between 3D regular grid models and a variant of Toom's PCA. Sections III and IV contain the proofs of our main impossibility results for AND processing functions and XOR processing functions, respectively. Then, we derive our partial impossibility result in section V by carefully expounding a promising approach for proving impossibility results for 2 D regular grids with NAND, and possibly other, processing functions via the construction of pertinent supermartingales. Finally, we conclude our discussion and propose future research directions in section VI.

## C. 2D Regular Grid Model

A 2 D regular grid model consists of an infinite DAG with vertices that are Bernoulli random variables (with range $\{0,1\})$ and edges that are independent BSCs. The root or source random variable of the grid is $X_{0,0} \sim \operatorname{Bernoulli}\left(\frac{1}{2}\right)$, and we let $X_{k}=\left(X_{k, 0}, \ldots, X_{k, k}\right)$ be the vector of vertex random variables at distance (i.e. length of shortest path) $k \in \mathbb{N} \triangleq\{0,1,2, \ldots\}$ from the root. So, $X_{0}=\left(X_{0,0}\right)$ and there are $k+1$ vertices at distance $k$. Furthermore, the 2 D regular grid contains the directed edges $\left(X_{k, j}, X_{k+1, j}\right)$ and ( $X_{k, j}, X_{k+1, j+1}$ ) for every $k \in \mathbb{N}$ and every
$j \in[k+1] \triangleq\{0, \ldots, k\}$, where the notation $(v, w)$ denotes an edge from source vertex $v$ to destination vertex $w$. The underlying DAG of such a 2D regular grid is shown in Figure 1.

To construct a Bayesian network (or directed graphical model) on this 2D regular grid, we fix some crossover probability parameter $\delta \in\left(0, \frac{1}{2}\right),{ }^{1}$ and two Boolean processing functions $f_{1}:\{0,1\} \rightarrow\{0,1\}$ and $f_{2}:\{0,1\}^{2} \rightarrow$ $\{0,1\}$. Then, for any $k \in \mathbb{N} \backslash\{0,1\}$ and $j \in\{1, \ldots, k-1\}$, we define: ${ }^{2}$

$$
\begin{equation*}
X_{k, j}=f_{2}\left(X_{k-1, j-1} \oplus Z_{k, j, 1}, X_{k-1, j} \oplus Z_{k, j, 2}\right) \tag{1}
\end{equation*}
$$

and for any $k \in \mathbb{N} \backslash\{0\}$, we define:

$$
\begin{equation*}
X_{k, 0}=f_{1}\left(X_{k-1,0} \oplus Z_{k, 0,2}\right) \text { and } X_{k, k}=f_{1}\left(X_{k-1, k-1} \oplus Z_{k, k, 1}\right) \tag{2}
\end{equation*}
$$

where $\oplus$ denotes addition modulo 2 , and $\left\{Z_{k, j, i}: k \in \mathbb{N} \backslash\{0\}, j \in[k+1], i \in\{1,2\}\right\}$ are i.i.d Bernoulli $(\delta)$ random variables that are independent of everything else. This implies that each edge is a $\operatorname{BSC}(\delta)$, i.e. a BSC with parameter $\delta$. Together, (1) and (2) characterize the conditional distribution of any $X_{k, j}$ given its parents.

Observe that the sequence $\left\{X_{k}: k \in \mathbb{N}\right\}$ forms a Markov chain, and our goal is to determine whether or not the value at the root $X_{0}$ can be decoded from the observations $X_{k}$ as $k \rightarrow \infty$. Given $X_{k}$ for any fixed $k \in \mathbb{N} \backslash\{0\}$, inferring the value of $X_{0}$ is a binary hypothesis testing problem with minimum achievable probability of error:

$$
\begin{equation*}
\mathbb{P}\left(h_{\mathrm{ML}}^{k}\left(X_{k}\right) \neq X_{0}\right)=\frac{1}{2}\left(1-\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}}\right) \tag{3}
\end{equation*}
$$

where $h_{\mathrm{ML}}^{k}:\{0,1\}^{k+1} \rightarrow\{0,1\}$ is the maximum likelihood (ML) decision rule based on $X_{k}$ at level $k$ (with knowledge of the 2D regular grid), $P_{X_{k}}^{+}$and $P_{X_{k}}^{-}$are the conditional distributions of $X_{k}$ given $X_{0}=1$ and $X_{0}=0$, respectively, and for any two probability measures $P$ and $Q$ on the same measurable space $(\Omega, \mathcal{F})$, their total variation (TV) distance is defined as:

$$
\begin{equation*}
\|P-Q\|_{\mathrm{TV}} \triangleq \sup _{A \in \mathcal{F}}|P(A)-Q(A)| \tag{4}
\end{equation*}
$$

It is straightforward to verify that the sequence $\left\{\mathbb{P}\left(h_{\mathrm{ML}}^{k}\left(X_{k}\right) \neq X_{0}\right): k \in \mathbb{N} \backslash\{0\}\right\}$ is non-decreasing and bounded above by $\frac{1}{2}$. Indeed, the monotonicity of this sequence is a consequence of (3) and the data processing inequality for TV distance, and the upper bound on the sequence trivially holds because a randomly generated bit cannot beat the ML decoder. Therefore, the limit of this sequence exists, and we say that reconstruction of the root bit $X_{0}$ is impossible, or "broadcasting is impossible," when:

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \mathbb{P}\left(h_{\mathrm{ML}}^{k}\left(X_{k}\right) \neq X_{0}\right)=\frac{1}{2} \quad \Leftrightarrow \quad \lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}}=0 \tag{5}
\end{equation*}
$$

where the equivalence follows from (3). ${ }^{3}$ In every impossibility result in this paper, we will prove that reconstruction of $X_{0}$ is impossible in the sense of (5).

## II. Main Results and Discussion

In this section, we state our main results, briefly delineate the main techniques or intuition used in the proofs, and discuss related results and models in the literature.

## A. Impossibility Results for AND and XOR 2D Regular Grids

In contrast to our work in [2] where we analyze broadcasting on random DAGs, the deterministic 2D regular grids we now study are much harder to analyze due to the dependence between adjacent vertices in a given layer. So, as we will explain later, we have to employ seemingly ad hoc proof techniques from percolation theory, coding theory, and martingale theory in this paper instead of the simple and elegant fixed point iteration intuition exploited in [2]. As mentioned earlier, we analyze the setting where all Boolean processing functions in the 2D regular grid

[^0]with two inputs are the same, and all Boolean processing functions in the 2 D regular grid with one input are the identity rule.

Our first main result shows that reconstruction is impossible for all $\delta \in\left(0, \frac{1}{2}\right)$ when AND processing functions are used.

Theorem 1 (AND 2D Regular Grid). If $\delta \in\left(0, \frac{1}{2}\right)$, and all Boolean processing functions with two inputs in the $2 D$ regular grid are the AND rule, then broadcasting is impossible in the sense of (5):

$$
\lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}}=0
$$

Theorem 1 is proved in section III. The proof couples the 2 D regular grid starting at $X_{0,0}=0$ with the 2D regular grid starting at $X_{0,0}=1$, and "runs" them together. Using a phase transition result concerning bond percolation on 2D lattices, we show that we eventually reach a layer where the values of all vertices in the first grid equal the values of the corresponding vertices in the second grid. So, the two 2 D regular grids "couple" almost surely regardless of their starting state. This implies that we cannot decode the starting state by looking at vertices in layer $k$ as $k \rightarrow \infty$. We note that in order to prove that the two 2 D regular grids "couple," we have to consider two different regimes of $\delta$ and provide separate arguments for each. The details of these arguments are presented in section III.

Our second main result shows that reconstruction is impossible for all $\delta \in\left(0, \frac{1}{2}\right)$ when XOR processing functions are used.

Theorem 2 (XOR 2D Regular Grid). If $\delta \in\left(0, \frac{1}{2}\right)$, and all Boolean processing functions with two inputs in the $2 D$ regular grid are the XOR rule, then broadcasting is impossible in the sense of (5):

$$
\lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}}=0
$$

Theorem 2 is proved in section IV. In the XOR 2D regular grid, every vertex at level $k$ can be written as a (binary) linear combination of the source bit and all the BSC noise random variables in the grid up to level $k$. This linear relationship can be captured by a binary matrix. The main idea of the proof is to perceive this matrix as a parity check matrix of a linear code. The problem of inferring $X_{0,0}$ from $X_{k}$ turns out to be equivalent to the problem of decoding the first bit of a codeword drawn uniformly from this code after observing a noisy version of the codeword. Basic facts from coding theory can then be used to complete the proof.

We remark that at first glance, Theorems 1 and 2 seem intuitively obvious from the random DAG model perspective of [2, Section I-C]. For example, consider a random DAG model where the number of vertices at level $k \in \mathbb{N}$ is $L_{k}=k+1$, two incoming edges for each vertex in level $k \in \mathbb{N} \backslash\{0\}$ are chosen randomly, uniformly, and independently (with replacement) from the vertices in level $k-1$, and all Boolean processing functions are the AND rule. Then, letting $\sigma_{k} \triangleq \sum_{j=0}^{k} X_{k, j} /(k+1) \in[0,1]$ be the proportion of 1 's in level $k \in \mathbb{N}$, the conditional expectation function $g(\sigma)=\mathbb{E}\left[\sigma_{k} \mid \sigma_{k-1}=\sigma\right]$ has only one fixed point regardless of the value of $\delta \in\left(0, \frac{1}{2}\right)$, and we intuitively expect $\sigma_{k}$ to tend to this fixed point (which roughly captures the equilibrium between AND gates killing 1 's and $\operatorname{BSC}(\delta)$ 's producing new 1 's) as $k \rightarrow \infty$. So, reconstruction is impossible in this random DAG model, which suggests that reconstruction is also impossible in the AND 2D regular grid. However, although Theorems 1 and 2 seem intuitively easy to understand in this way, we emphasize that this random DAG intuition does not capture the subtleties engendered by the "regularity" of the 2 D grid. In fact, the intuition derived from the random DAG model can even be somewhat misleading. Consider the random DAG model described above with all NAND processing functions (instead of AND processing functions). This model was analyzed in [2, Theorem 2], because using alternating layers of AND and OR processing functions is equivalent to using all NAND processing functions (see [2, Footnote 10]). [2, Theorem 2] portrays that reconstruction of the source bit is possible for $\delta<(3-\sqrt{7}) / 4$. Yet, evidence from [39, Theorem 1], which establishes the ergodicity of 1D PCA with NAND gates, and the detailed discussion, results, and numerical simulations in subsection II-B and section V , suggest that reconstruction is actually impossible for the 2 D regular grid with NAND processing functions. Therefore, the 2 D regular grid setting of Theorems 1 and 2 should be intuitively understood using random DAG models with caution. Indeed, as sections III and IV illustrate, the proofs of these theorems are nontrivial.

The impossibility of broadcasting in Theorems 1 and 2 also seems intuitively plausible due to the ergodicity results for numerous 1D PCA-see e.g. [24] and the references therein. (Indeed, as we delineated in subsection I-A, our conjecture was inspired by the positive rates conjecture for 1D PCA.) However, there are four key differences between broadcasting on 2D regular grids and 1D PCA. Firstly, the main question in the study of 1D PCA is
whether a given automaton is ergodic, i.e. whether the Markov process defined by it converges to a unique invariant probability measure on the configuration space for all initial configurations. This question of ergodicity is typically addressed by considering the convergence of finite-dimensional distributions over the sites (i.e. weak convergence). Hence, for many 1D PCA that have special characteristics, such as translation invariance, finite interaction range, positivity, and attractiveness (or monotonicity), cf. [24], it suffices to consider the convergence of distributions on finite intervals, e.g. marginal distribution at a given site. In contrast to this setting, we are concerned with the stronger notion of convergence in TV distance. Indeed, Theorems 1 and 2 show that the TV distance between $P_{X_{k}}^{+}$ and $P_{X_{k}}^{-}$vanishes as $k \rightarrow \infty$.

Secondly, since a 1D PCA has infinitely many sites, the problem of remembering or storing a bit in a 1D PCA (with binary state space) corresponds to distinguishing between the "all 0's" and "all 1's" initial configurations. On the other hand, as mentioned in subsection I-A, a 2D regular grid can be construed as a 1D PCA with boundary conditions; each level $k \in \mathbb{N}$ corresponds to an instance in discrete-time, and there are $L_{k}=k+1$ sites at time $k$. Moreover, its initial configuration has only one copy of the initial bit as opposed to infinitely many copies. As a result, compared a 2D regular grid, a 1D PCA (without boundary conditions) intuitively appears to have a stronger separation between the two initial states as time progresses. The aforementioned boundary conditions form another barrier to translating results from the 1D PCA literature to 2D regular grids.

Thirdly, in our broadcasting model in subsection I-C, the independent BSCs are situated on the edges of the 2D regular grid. On the other hand, a corresponding 1D PCA (which removes the boundary conditions of the 2D regular grid) first uses the unadulterated bits from the previous layer while computing its processing functions in the current layer, and then applies independent BSC noise to the outputs of the functions. Equivalently, a 1D PCA with BSC noise behaves like a 2D regular grid where the edges between levels 0 and 1 are noise-free, and for any site at any level $k \geq 1$, the BSCs of its two outgoing edges are coupled so that they flip their input bit simultaneously (almost surely). It is due this difference between our broadcasting model and the canonical 1D PCA model that we cannot, for example, easily translate the ergodicity result for 1D PCA with NAND gates and BSC noise in [39, Theorem 1] to an impossibility result for broadcasting on 2D regular grids with NAND processing functions. We additionally remark, for completeness, that there are some known connections between vertex noise and edge noise, cf. [40], but these results do not help with our problem.

Fourthly, it is also worth mentioning that most results on 1D PCA pertain to the continuous-time setting-see e.g. [24], [41] and the references therein. This is because sites are (almost surely) updated one by one in a continuoustime automaton, but they are updated in parallel in a discrete-time automaton. So, the discrete-time setting is often harder to analyze. Indeed, some of the only known discrete-time 1D PCA ergodicity results are in [39, Theorem $1]$ and [25, Section 3], where the latter outlines the proof of ergodicity of the 3 -input majority vote model (i.e. 1D PCA with 3 -input majority gates) for sufficiently small noise levels. ${ }^{4}$ This is another reason why results from the 1D PCA literature cannot be easily transferred to our model.

## B. Partial Impossibility Result for NAND 2D Regular Grid

We next present our final main result for the 2D regular grid with all NAND processing functions. Based on our broader conjecture in subsection I-A, we first state the following conjecture in analogy with Theorems 1 and 2.

Conjecture 1 (NAND 2D Regular Grid). If $\delta \in\left(0, \frac{1}{2}\right)$, and all Boolean processing functions with two inputs in the 2 D regular grid are the NAND rule, then broadcasting is impossible in the sense of (5):

$$
\lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}}=0
$$

As noted in section I, we will present a detailed "program" for proving this conjecture in section V. This program is inspired by the potential function technique employed in the proof of ergodicity of 1D PCA with NAND gates in [39, Theorem 1], and can be construed as a more general approach to proving impossibility of broadcasting on 2 D regular grids with other processing functions or ergodicity of the corresponding 1D PCA. In this subsection, we delineate a sufficient condition for proving Conjecture 1 that follows from the arguments in section V, and provide accompanying numerical evidence that this sufficient condition is actually true.

[^1]To this end, we begin with some necessary setup, notation, and definitions that are relevant to the NAND 2D regular grid. Let $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ denote versions of the Markov chain $\left\{X_{k}: k \in \mathbb{N}\right\}$ initialized at $X_{0}^{+}=1$ and $X_{0}^{-}=0$, respectively. Define the coupled 2D grid variables $\left\{Y_{k, j}=\left(X_{k, j}^{-}, X_{k, j}^{+}\right): k \in \mathbb{N}, j \in[k+1]\right\}$, which yield the Markovian coupling $\left\{Y_{k}=\left(Y_{k, 0}, \ldots, Y_{k, k}\right): k \in \mathbb{N}\right\}$. The coupled Markov chain $\left\{Y_{k}: k \in \mathbb{N}\right\}$ "runs" its "marginal" Markov chains $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ on a common underlying 2D regular grid so that along any edge BSC, either both inputs are copied with probability $1-2 \delta$, or a shared independent Bernoulli $\left(\frac{1}{2}\right)$ output bit is produced with probability $2 \delta$. Moreover, we assume that each $Y_{k, j}$ for $k \in \mathbb{N}$ and $j \in[k+1]$ takes values in the extended alphabet set $\mathcal{Y} \triangleq\{0,1, u\}$, where $Y_{k, j}=0$ if and only if $X_{k, j}^{-}=X_{k, j}^{+}=0, Y_{k, j}=1$ if and only if $X_{k, j}^{-}=X_{k, j}^{+}=1$, and $Y_{k, j}=u$ if and only if $X_{k, j}^{-} \neq X_{k, j}^{+}$. Our Markovian coupling, along with (44) and (45) from subsection V-A, completely determines the transition kernels of $\left\{Y_{k}: k \in \mathbb{N}\right\}$, i.e. the conditional distributions of $Y_{k+1} \in \mathcal{Y}^{k+2}$ given $Y_{k}$ for all $k \in \mathbb{N}$, and this coupled Markov chain starts at $Y_{0}=u$ almost surely. As we will show in subsection V-A, it suffices to analyze the coupled Markov chain $\left\{Y_{k}: k \in \mathbb{N}\right\}$ to deduce the impossibility of broadcasting. (A more detailed explanation of our Markovian coupling can also be found in subsection V-A.) Lastly, in order to present our final main result, we introduce the class of cyclic potential functions (inspired by [39]), a partial order over these potential functions, and a pertinent linear operator on the space of potential functions in the next definition.

Definition 1 (Cyclic Potential Functions and Related Notions). Given any finite set of strings $v_{1}, \ldots, v_{m} \in \mathcal{Y}^{*}=$ $\cup_{k \in \mathbb{N} \backslash\{0\}} \mathcal{Y}^{k}$ and any associated coefficients $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R}$ (with $m \in \mathbb{N} \backslash\{0\}$ ), we may define a corresponding cyclic potential function $w: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ via the formal sum:

$$
w=\sum_{j=1}^{m} \alpha_{j}\left\{v_{j}\right\}
$$

where curly braces are used to distinguish a string $v \in \mathcal{Y}^{*}$ from its associated potential function $\{v\}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$. In particular, for every $k \in \mathbb{N} \backslash\{0\}$ and every string $y=\left(y_{0} \cdots y_{k-1}\right) \in \mathcal{Y}^{k}$ of length $k$, the cyclic potential function $w$ is evaluated as follows:

$$
w[y] \triangleq \sum_{j=1}^{m} \alpha_{j} \mathbb{1}\left\{\text { length of } v_{j} \leq k\right\} \sum_{i=0}^{k-1} \mathbb{1}\left\{\left(y_{(i)_{k}} \cdots y_{(i+s-1)_{k}}\right)=v_{j}\right\}
$$

where $(i)_{k} \equiv i(\bmod k)$ for every $i \in \mathbb{N}$. Furthermore, we say that $w$ is $u$-only if the strings $v_{1}, \ldots, v_{m}$ all contain a $u$. For any fixed $r \in \mathbb{N} \backslash\{0\}$, we may also define a partial order $\succeq_{c}$ over the set of all cyclic potential functions for which the lengths of the underlying strings (with non-zero coefficients) defining their formal sums are bounded by $r$. Specifically, for any pair of such cyclic potential functions $w_{1}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ and $w_{2}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$, we have:

$$
w_{1} \succeq_{\mathrm{c}} w_{2} \quad \Leftrightarrow \quad \forall y \in \bigcup_{k \geq r} \mathcal{Y}^{k}, \quad w_{1}[y] \geq w_{2}[y]
$$

Finally, we define the conditional expectation operator $\mathcal{E}$ on the space of cyclic potential functions based on the coupled NAND 2D regular grid as follows. For any input cyclic potential function $w$ (defined by the formal sum above), $\mathcal{E}$ outputs the cyclic potential function with formal sum:

$$
\mathcal{E}(w) \triangleq \sum_{j=1}^{m} \alpha_{j} \sum_{z \in \mathcal{Y}^{s_{j}+1}} \mathbb{P}\left(\left(Y_{s_{j}+1,1}, \ldots, Y_{s_{j}+1, s_{j}}\right)=v_{j} \mid Y_{s_{j}}=z\right)\{z\}
$$

where $s_{j}$ denotes the length of $v_{j}$ for all $j \in\{1, \ldots, m\}$, and the probabilities are determined by the Markovian coupling $\left\{Y_{k}: k \in \mathbb{N}\right\}$.

We note that Definition 1 collects several smaller definitions interspersed throughout section V, where they are each carefully explained in greater detail, and many other closely related ideas and terminology are also introduced. Using the concepts in Definition 1, the ensuing main theorem presents a sufficient condition for the impossibility of broadcasting on the NAND 2D regular grid.

Theorem 3 (Sufficient Condition for NAND 2D Regular Grid). For any noise level $\delta \in\left(0, \frac{1}{2}\right)$, suppose that there exists $r \in \mathbb{N} \backslash\{0,1\}$, and a cyclic potential function $w_{\delta}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ whose formal sum is constructed with strings (with non-zero coefficients) of length at most $r-1$, such that:

1) $w_{\delta}$ is u-only,

TABLE I
Feasible Solutions $\alpha^{*}(\delta)$ of The LP (82) For Different $\delta \in\left(0, \frac{1}{2}\right)$ when $r=4$

| $\delta$ |  | 0.0001 | 0.001 | 0.01 | 0.02 | 0.05 | 0.1 | 0.2 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\alpha^{*}(\delta)$ | (000) | 07 |  |  |  | [ 0 | 0 |  |
|  | (001) | 0 | 0 | 0 | 0 | 0 | 0 | $\left[\begin{array}{l}0 \\ 0\end{array}\right.$ |
|  | (00u) | 0.0000 | 0.0000 | 0.0171 | 0.0624 | 0.1151 | 0.1368 |  |
|  | (010) | 0 | 0 | 0 | 0 | 0 | 00 | $\begin{gathered} 0.1402 \\ 0 \end{gathered}$ |
|  | (011) | 0 | 0 | 0 | 0 |  |  | 0 |
|  | (01u) | 1.9991 | 1.9908 | 1.9223 | 1.7777 | 1.3064 | 0.7631 | 0.3943 |
|  | (0u0) | 0.0004 | 0.0040 | 0.0421 | 0.0783 | 0.1477 | 0.15990.7283 | 0.1852 |
|  | (0u1) | 1.9990 | 1.9904 | 1.9164 | 1.7509 | 1.2696 |  | 0.4138 |
|  | (0uu) | 1.9986 | 1.9864 | 1.9528 | 1.8429 | 1.3672 | 0.7283 0.8458 | 0.5363 |
|  | (100) | 0 | 0 | 0 | 0 | 0 | 0.8458 0 | 0 |
|  | (101) | 0 | 0 | 0 | 0 | 0 | 0 0 | 0 |
|  | (10u) | 0.0002 | 0.0020 | 0.0358 | 0.0948 | 0.1516 | 0.1589 | 0.1428 |
|  | (110) | 0 | 0 | 0 | 0 | 0 | 00 | 00 |
|  | (111) | 0 | 0 | 0 | 0 | 0 |  |  |
|  | (11u) | 0.9988 | 0.9884 | 1.0010 | 0.9906 | 0.7999 | 0.53220.7787 | 0.3189 |
|  | (1u0) | 1.0005 | 1.0047 | 1.3343 | 1.6963 | 1.3745 |  | 0.5149 |
|  | (1u1) | 1.9996 | 1.9958 | 1.9526 | 1.8252 | 1.4142 | 0.9025 | 0.6328 |
|  | (1uu) | 1.9995 | 1.9948 | 1.9610 | 1.8482 | 1.4430 | 0.9489 | 0.7120 |
|  | (u00) | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
|  | (u01) | 1.0002 | 1.0020 | 1.0057 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
|  | ( $u 0 u$ ) | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.00001.1221 | 1.0000 |
|  | (u10) | 2.0003 | 2.0027 | 2.2894 | 2.5746 | 2.0237 |  | 1.0000 |
|  | (u11) | 1.0006 | 1.0058 | 1.3689 | 1.7854 | 1.5929 | 1.1221 1.0445 | 1.0000 |
|  | (u1u) | 1.9988 | 1.9881 | 1.9396 | 1.7970 | 1.4229 | 1.0445 1.0000 | 1.00001.0000 |
|  | (uu0) | 1.0000 | 1.0000 | 1.3358 | 1.6983 | 1.4511 | 1.0000 |  |
|  | (uu1) | 1.9991 | 1.9910 | 1.9535 | 1.8211 | 1.4682 | 1.0000 | 1.0000 |
|  | (uuu) | 1.9990 | 1.9897 ] | 1.9499 | 1.8119 | 1.4577 | 1.0000 | 1.0000 |

2) $w_{\delta} \succeq_{c} \mathcal{E}\left(w_{\delta}\right)$,
3) There exists a constant $C=C(\delta)>0$ (which may depend on $\delta$ ) such that $w_{\delta} \succeq_{c} C\{u\}$,
where $\succeq_{c}$ is defined using $r$ (as in Definition 1), and $C\{u\}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ is the cyclic potential function consisting of a single string $(u) \in \mathcal{Y}$ with coefficient $C$. Then, Conjecture 1 is true, i.e. broadcasting on the $2 D$ regular grid where all Boolean processing functions with two inputs are the NAND rule is impossible in the sense of (5).

Theorem 3 is proved in subsection V-D. Intuitively, the first two conditions in the theorem statement ensure that $\left\{w_{\delta}\left(Y_{k}\right): k \in \mathbb{N}\right\}$ is a supermartingale, and the third condition ensures that this supermartingale upper bounds the total number of uncoupled grid variables at successive levels. Then, using a martingale convergence argument along with some careful analysis of the stochastic dynamics of the coupled 2 D regular grid, we can deduce that the number of uncoupled grid variables converges to zero almost surely. Akin to the proof of Theorem 1, this implies that broadcasting is impossible on the NAND 2D regular grid. As noted earlier, the details of the entire argument can be found in section V .

While Theorem 3 describes specific cyclic potential functions that can be used to prove Conjecture 1, we do not rigorously prove their existence for all $\delta \in\left(0, \frac{1}{2}\right)$ in this paper. However, much of the development in section V aims to carefully explain these desired potential functions and derive results that enable us to computationally construct them. In particular, we demonstrate in Proposition 10 that for fixed values of $\delta, r$, and $C$, the problem of finding an appropriate cyclic potential function $w_{\delta}$ satisfying the three conditions of Theorem 3 can be posed as a linear program (LP). This connection stems from a graph theoretic characterization of $\succeq_{c}$ (the details of which are expounded in subsection V-C).

Armed with this connection, we present some illustrative LP simulation results (computed using MATLAB with CVX optimization packages) in Table I which numerically construct cyclic potential functions $w_{\delta}^{*}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ that satisfy the three conditions of Theorem 3 with the fixed constants $r=4$ and $C=C(\delta)=1$. Specifically, we consider different representative values of $\delta \in\left(0, \frac{1}{2}\right)$ in the first row of Table I. For any such fixed $\delta$ value, the second row of Table I displays a corresponding vector of coefficients $\alpha^{*}(\delta) \in \mathbb{R}^{27}$ (which has been rounded to 4 decimal places) that defines a cyclic potential function $w_{\delta}^{*}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ satisfying the three conditions of Theorem 3 and consisting of a formal sum over all strings of length 3 . Indeed, for the readers' convenience, we
index these vectors of coefficients $\alpha^{*}(\delta)$ with $\mathcal{Y}^{3}$ in Table I, so that each $\alpha^{*}(\delta)$ defines $w_{\delta}^{*}$ via the formal sum constructed by scaling each index in $\mathcal{Y}^{3}$ with the associated value in $\alpha^{*}(\delta)$ and adding them up (see Definition 1). For example, the first column of Table I states that when $\delta=0.0001$, the cyclic potential function, $w_{\delta}^{*}=$ $1.9991\{01 u\}+0.0004\{0 u 0\}+\cdots+1.9991\{u u 1\}+1.9990\{u u u\}$, satisfies the conditions of Theorem 3 with $r=4$ and $C=1$. Furthermore, it is straightforward to see that the coefficients corresponding to strings in $\mathcal{Y}^{3}$ with no $u$ 's are always zero in Table I, and the coefficients corresponding to the last 9 strings in $\mathcal{Y}^{3}$ that begin with a $u$ are all lower bounded by 1 . The former observation immediately confirms that the cyclic potential functions represented in Table I satisfy the first condition of Theorem 3, and the latter observation shows that they satisfy the third condition of Theorem 3 with $C=1$ (using the idea of "purification" from section V). We refer readers to subsection V-E for further details and discussion regarding how to compute the vectors $\alpha^{*}(\delta)$ by solving LPs.

We close this subsection with three further remarks. Firstly, Table I only presents a small subset of our simulation results for brevity. We have solved LPs for various other values of $\delta \in\left(0, \frac{1}{2}\right)$ and always obtained associated feasible vectors of coefficients $\alpha^{*}(\delta)$. We do not present simulation results for $\delta>0.146446 \ldots$ (other than $\delta=0.2$ ), because part 2 of Proposition 2 implies that broadcasting is impossible in the sense of (5) in this case-see (9) and the discussion below in subsection II-C. Moreover, the impossibility of broadcasting is intuitively more surprising for smaller values of noise $\delta$, so we emphasized the LP solutions for smaller values of $\delta$ in Table I since they are more compelling. (Note, however, that we do not know any rigorous monotonicity result which shows that broadcasting is impossible on the NAND 2D regular grid for $\delta^{\prime} \in\left(0, \frac{1}{2}\right)$ if it is impossible for a smaller $\delta \in\left(0, \frac{1}{2}\right)$ value, i.e. $\delta<\delta^{\prime}$.)

Secondly, it is worth mentioning that Theorem 3 and associated LP simulation results (as in Table I) yield nonrigorous computer-assisted proofs of the impossibility of broadcasting on NAND 2D regular grids for individual values of $\delta \in\left(0, \frac{1}{2}\right)$. Hence, Table I provides strong evidence that broadcasting is indeed impossible on the NAND 2D regular grid for all $\delta \in\left(0, \frac{1}{2}\right)$ in the sense of (5). We elaborate on possible approaches to rigorize our LP-based argument at the end of subsection V-E.

Finally, we emphasize that by fully establishing the AND and XOR cases in Theorems 1 and 2, and partially establishing the NAND case in Theorem 3 and our simulations, we have made substantial progress towards proving the 2 D aspect our conjecture in subsection I-A that broadcasting is impossible for 2 D regular grids for all choices of common Boolean processing functions. Indeed, there are 16 possible 2 -input logic gates that can serve as our processing function. It turns out that to prove our conjecture, we only need to establish the impossibility of broadcasting for four nontrivial cases out of the 16 gates. To elaborate further, notice that the two constant Boolean processing functions that always output 0 or 1 engender 2 D regular grids where only the vertices at the boundary carry any useful information about the source bit. However, since the boundaries of the 2 D regular grid are ergodic binary-state Markov chains (since $\delta \in\left(0, \frac{1}{2}\right)$ ), broadcasting is clearly impossible for such constant processing functions. The four 2 -input Boolean processing functions that are actually 1 -input functions, namely, the identity maps for the first or second input and the inverters (or NOT gates) for the first or second input, beget 2D regular grids that are actually trees. Moreover, these trees have branching number 1, and hence, the results of [4] (outlined in section I) imply that broadcasting is impossible for such 1-input Boolean processing functions. The six remaining symmetric Boolean processing functions are AND, NAND, OR, NOR, XOR, and XNOR. Due to the symmetry of 0 's and 1's in our model (see subsection I-C), we need only prove the impossibility of broadcasting for the three logic gates AND, XOR, and NAND (which are equivalent to OR, XNOR, and NOR, respectively). This leaves four asymmetric 2 -input Boolean processing functions out of the original 16. Once again, due to the symmetry of 0's and 1's in our model, we need only consider two of these logic gates: the first gate outputs 1 when its two inputs are equal, and outputs its first input when its two inputs are different, and the second gate outputs 1 when its two inputs are equal, and outputs its second input when its two inputs are different. However, due to the symmetry of the edge configuration in our 2D regular grid construction (see subsection I-C), it suffices to analyze only the second gate. Since this second function is not commonly viewed as a logic gate, we write down its truth table for the readers' convenience:

| $x_{1}$ | $x_{2}$ | $x_{1} \Rightarrow x_{2}$ |
| :---: | :---: | :---: |
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

and recognize it as the implication relation, denoted as IMP. Therefore, to prove the 2D aspect of our conjecture that broadcasting on 2 D regular grids is impossible, we only have to analyze four nontrivial Boolean processing
functions:

1) AND ,
2) XOR ,
3) NAND
4) IMP .

Clearly, Theorems 1 and 2 completely tackle the first two cases, and Theorem 3, the convincing computer simulations for its sufficient condition in Table I, and the discussion in section V partially address the third case. So, only the fourth case remains entirely open.

## C. Related Results in the Literature

In this subsection, we present and discuss two complementary impossibility results from the literature regarding broadcasting on general DAGs (not just 2D regular grids). To this end, consider any infinite DAG $\mathcal{G}$ with a single source vertex at level $k=0$ and $L_{k} \in \mathbb{N} \backslash\{0\}$ vertices at level $k \in \mathbb{N}$ (where $L_{0}=1$ ), and assume that $\mathcal{G}$ is in topological ordering so that all its edges are directed from lower levels to higher levels. Suppose further that for any level $k \in \mathbb{N} \backslash\{0\}$, each vertex of $\mathcal{G}$ at level $k$ has indegree $d \in \mathbb{N} \backslash\{0\}$ and the $d$ incoming edges originate from vertices at level $k-1 .{ }^{5}$ Much like subsection I-C, we define a Bayesian network on this infinite DAG by letting each vertex be a Bernoulli random variable: For $k \in \mathbb{N}$ and $j \in\left[L_{k}\right]$, let $X_{k, j} \in\{0,1\}$ be the random variable corresponding to the $j$ th vertex at level $k$, and let $X_{k}=\left(X_{k, 0}, \ldots, X_{k, L_{k}-1}\right)$. As before, the edges of $\mathcal{G}$ are independent $\operatorname{BSC}(\delta)$ 's with common parameter $\delta \in\left(0, \frac{1}{2}\right)$, and the vertices of $\mathcal{G}$ combine their inputs using Boolean processing functions. We say that broadcasting is impossible on this DAG if and only if (5) holds.

The first impossibility result, which we proved in [2, Proposition 3], illustrates that if $L_{k}$ is sub-logarithmic for every sufficiently large $k \in \mathbb{N}$, then broadcasting is impossible.

Proposition 1 (Slow Growth of Layers [2, Proposition 3]). For any $\delta \in\left(0, \frac{1}{2}\right)$ and any $d \in \mathbb{N} \backslash\{0\}$, if $L_{k} \leq \log (k) /$ $(d \log (1 /(2 \delta)))$ for all sufficiently large $k \in \mathbb{N}$, then for all choices of Boolean processing functions (which may vary between vertices), broadcasting is impossible on $\mathcal{G}$ in the sense of (5):

$$
\lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}}=0
$$

where $\log (\cdot)$ denotes the natural logarithm, and $P_{X_{k}}^{+}$and $P_{X_{k}}^{-}$denote the conditional distributions of $X_{k}$ given $X_{0,0}=1$ and $X_{0,0}=0$, respectively.

For the 2 D regular grid models that we consider, the underlying DAGs have $L_{k}=k+1$. Hence, the impossibility of broadcasting in our models is not trivial, because this result does not apply to them. We further remark that an analogous impossibility result to Proposition 1 for DAGs without the bounded indegree assumption is also proved in [2, Proposition 4]: When each vertex at level $k \in \mathbb{N} \backslash\{0\}$ of our DAG is connected to all $L_{k-1}$ vertices at level $k-1$, for any $\delta \in\left(0, \frac{1}{2}\right)$, if $L_{k} \leq \sqrt{\log (k) / \log (1 /(2 \delta))}$ for all sufficiently large $k \in \mathbb{N}$, then broadcasting is impossible in the sense of (5) for all choices of Boolean processing functions (which may vary between vertices).

The second impossibility result, which specializes a more general result proved by Evans and Schulman in the context of von Neumann's model of fault-tolerant computation using noisy circuits [21, Lemma 2 and p.2373], portrays an upper bound on the mutual information between $X_{0}$ and $X_{k}$ for every $k \in \mathbb{N}$ which decays exponentially under appropriate conditions (also see [2, Proposition 5]).

Proposition 2 (Decay of Mutual Information [21, Lemma 2]). The following are true:

1) For the Bayesian network defined on $\mathcal{G}$, given any choices of Boolean processing functions (which may vary between vertices), the mutual information (in natural units) between $X_{0}$ and $X_{k}$, denoted $I\left(X_{0} ; X_{k}\right)$, satisfies:

$$
I\left(X_{0} ; X_{k}\right) \leq \log (2) L_{k}\left((1-2 \delta)^{2} d\right)^{k}
$$

where $L_{k} d^{k}$ bounds the number of paths from the source $X_{0}$ to layer $X_{k}$, and $(1-2 \delta)^{2 k}$ represents the "aggregate contraction" of mutual information along each path.
2) For the Bayesian network defined on the 2D regular grid in subsection I-C, given any choices of Boolean processing functions (which may vary between vertices), the mutual information between $X_{0}$ and $X_{k}$ satisfies:

$$
I\left(X_{0} ; X_{k}\right) \leq \log (2)\left(2(1-2 \delta)^{2}\right)^{k}
$$

[^2]where $2^{k}$ denotes the number of paths from the source $X_{0}$ to layer $X_{k}$, and $(1-2 \delta)^{2 k}$ represents the "aggregate contraction" of mutual information along each path.

Proposition 2 is discussed in far greater detail in [2, Section II-C], where several related references are also provided. ${ }^{6}$ Note that for the Bayesian network on $\mathcal{G}$, part 1 of Proposition 2 shows that if $(1-2 \delta)^{2} d<1$ (or equivalently, $\delta>\frac{1}{2}-\frac{1}{2 \sqrt{d}}$ ) and $L_{k}=o\left(1 /\left((1-2 \delta)^{2} d\right)^{k}\right)$, then for all choices of Boolean processing functions, we have:

$$
\begin{equation*}
\lim _{k \rightarrow \infty} I\left(X_{0} ; X_{k}\right)=0 \tag{8}
\end{equation*}
$$

which implies that broadcasting is impossible in the sense of (5) (due to Pinsker's inequality). Likewise, for the Bayesian network on the 2D regular grid, part 2 of Proposition 2 shows that if the "Evans-Schulman condition" holds:

$$
\begin{equation*}
2(1-2 \delta)^{2}<1 \quad \Leftrightarrow \quad \delta>\frac{1}{2}-\frac{1}{2 \sqrt{2}}=0.146446 \ldots \tag{9}
\end{equation*}
$$

then for all choices of Boolean processing functions, we get (8) and broadcasting is impossible in the sense of (5).

## D. 3D Regular Grid Model and Toom's NEC Rule

In view of our broader conjecture in subsection I-A that broadcasting should be possible in 3D regular grids, in this subsection, we elucidate a connection between a 3D regular grid with all majority processing functions and a 2D PCA with noise on the edges and boundary conditions that uses Toom's NEC rule [36]. We first define the 3 D regular grid model (akin to subsection I-C). A 3 D regular grid is an infinite DAG whose vertex set $\mathbb{N}^{3}$ is the intersection of the 3 D integer lattice and the 3 D non-negative orthant, and corresponding to each vertex $v \in \mathbb{N}^{3}$, we associate a Bernoulli random variable $X_{v} \in\{0,1\}$. Furthermore, a 3D regular grid contains the directed edges $\left(X_{v}, X_{v+e_{1}}\right)$, $\left(X_{v}, X_{v+e_{2}}\right)$, and $\left(X_{v}, X_{v+e_{3}}\right)$ for every $v \in \mathbb{N}^{3}$, which are independent BSCs with crossover probability $\delta \in\left(0, \frac{1}{2}\right)$, where $e_{i}$ denotes the $i$ th standard basis vector of appropriate dimension (which has 1 in the $i$ th position and 0 elsewhere). In this subsection, we set all the Boolean processing functions at the vertices to be the majority rule. This implies that:

1) For any vertex $v=\left(v_{1}, v_{2}, v_{3}\right) \in \mathbb{N}^{3} \backslash\{\mathbf{0}\}$ on an axis of $\mathbb{R}^{3}$, i.e. there exists a unique $i \in\{1,2,3\}$ such that $v_{i}>0$, we have:

$$
\begin{equation*}
X_{v}=X_{v-e_{i}} \oplus Z_{v, i} \tag{10}
\end{equation*}
$$

where $\mathbf{0}$ denotes the zero vector of appropriate dimension,
2) For any vertex $v=\left(v_{1}, v_{2}, v_{3}\right) \in \mathbb{N}^{3} \backslash\{\boldsymbol{0}\}$ in a plane spanned by two axes of $\mathbb{R}^{3}$, i.e. there exist distinct $i, j \in\{1,2,3\}$ such that $v_{i}, v_{j}>0$ and $v_{k}=0$ for $k \in\{1,2,3\} \backslash\{i, j\}$, we have:

$$
X_{v}= \begin{cases}X_{v-e_{i}} \oplus Z_{v, i}, & \text { with probability } \frac{1}{2}  \tag{11}\\ X_{v-e_{j}} \oplus Z_{v, j}, & \text { with probability } \frac{1}{2}\end{cases}
$$

where one of the two noisy inputs of $X_{v}$ is chosen randomly and independently of everything else,
3) For any vertex $v=\left(v_{1}, v_{2}, v_{3}\right) \in \mathbb{N}^{3}$ in the interior of the 3 D non-negative orthant, i.e. $v_{1}, v_{2}, v_{3}>0$, we have:

$$
\begin{equation*}
X_{v}=\operatorname{majority}\left(X_{v-e_{1}} \oplus Z_{v, 1}, X_{v-e_{2}} \oplus Z_{v, 2}, X_{v-e_{3}} \oplus Z_{v, 3}\right) \tag{12}
\end{equation*}
$$

where $\left\{Z_{v, i}: v \in \mathbb{N}^{3} \backslash\{\mathbf{0}\}, i \in\{1,2,3\}\right\}$ are i.i.d. Bernoulli $(\delta)$ noise random variables that are independent of everything else, and majority: $\{0,1\}^{3} \rightarrow\{0,1\}$ is the 3 -input majority Boolean function. This defines the Bayesian network corresponding to the majority $3 D$ regular grid. In particular, for any $k \in \mathbb{N}$, let the discretized 2 -simplex $\mathcal{S}_{k} \triangleq\left\{v=\left(v_{1}, v_{2}, v_{3}\right) \in \mathbb{N}^{3}: v_{1}+v_{2}+v_{3}=k\right\}$ denote the $k$ th layer of vertices in the 3D regular grid at distance $k$ from the source (so that $\mathbb{N}^{3}=\cup_{k \in \mathbb{N}} \mathcal{S}_{k}$ ), and $X_{\mathcal{S}_{k}} \triangleq\left(X_{v}: v \in \mathcal{S}_{k}\right)$ denote the corresponding collection of random variables. Then, the majority 3 D regular grid is completely characterized by the set of Markov transition kernels $\left\{P_{X_{\mathcal{S}_{k}} \mid X_{0}}: k \in \mathbb{N}\right\}$ (which are defined by (10), (11), and (12)). (While the definitions of these Markov kernels suffice for our purposes here, if we were to fully analyze the broadcasting question for majority 3D regular grids, then we would also assume that the source bit is $X_{\mathbf{0}} \sim \operatorname{Bernoulli}\left(\frac{1}{2}\right)$.)

We next recall a version of Toom's 2D PCA which has noise on the edges rather than the vertices, cf. [36]. Consider the 2 D integer lattice $\mathbb{Z}^{2}$ of sites, the binary state space $\{0,1\}$, and the configuration space $\{0,1\}^{\mathbb{Z}^{2}}$

[^3]of functions $\xi: \mathbb{Z}^{2} \rightarrow\{0,1\}$ (which map sites to their bit values). Moreover, fix the PCA's transition function to be majority, and let its interaction neighborhood be $\mathcal{N}=\left\{-e_{1}, \mathbf{0},-e_{2}\right\} \subseteq \mathbb{Z}^{2}$, which together define Toom's NEC rule [36]. ${ }^{7}$ With these building blocks in place, we now describe the dynamics of Toom's 2D PCA with edge noise. Suppose at time $k \in \mathbb{N}$, the automaton has configuration $\xi_{k} \in\{0,1\}^{\mathbb{Z}^{2}}$. Then, its configuration $\xi_{k+1} \in$ $\{0,1\}^{\mathbb{Z}^{2}}$ at time $k+1$ is determined as follows: Each site $x \in \mathbb{Z}^{2}$ simultaneously receives three noisy input bits $\left(\xi_{k}(x+v) \oplus Z_{k, x, v}: v \in \mathcal{N}\right)$ from its interaction neighborhood, and then computes its value at time $k+1$ using Toom's NEC rule, i.e. $\xi_{k+1}(x)=\operatorname{majority}\left(\xi_{k}(x+v) \oplus Z_{k, x, v}: v \in \mathcal{N}\right)$, where $\left\{Z_{k, x, v}: k \in \mathbb{N}, x \in \mathbb{Z}^{2}, v \in \mathcal{N}\right\}$ are i.i.d. Bernoulli $(\delta)$ noise random variables that are independent of everything else. This defines the transition kernel of the Markov process corresponding to Toom's 2D PCA with edge noise. (Note that if we seek to analyze the ergodicity of this PCA, then we would have to additionally fix different initial configurations $\xi_{0} \in\{0,1\}^{\mathbb{Z}^{2}}$ at time $k=0$.)

By modifying Toom's 2D PCA with edge noise outlined above, we now introduce Toom's 2D PCA with boundary conditions. For any $k \in \mathbb{N}$, consider the projected discretized 2-simplices $\hat{\mathcal{S}}_{k} \triangleq\left\{x=\left(x_{1}, x_{2}\right) \in \mathbb{N}^{2}: x_{1}+x_{2} \leq k\right\}$. Using the notation introduced above, define the jointly distributed binary random variables $\left\{\xi_{k}(x): x \in \hat{\mathcal{S}}_{k}, k \in \mathbb{N}\right\}$, and let $\xi_{k} \triangleq\left(\xi_{k}(x): x \in \hat{\mathcal{S}}_{k}\right)$ be the values of the bits of this automaton at time $k \in \mathbb{N}$. The dynamics of Toom's 2D PCA with boundary conditions follows the dynamics of Toom's 2D PCA explained above (mutatis mutandis). Specifically, conditioned on $\xi_{k}$ at time $k \in \mathbb{N}$, the probability distribution of $\xi_{k+1}$ at time $k+1$ is characterized by:

1) For the site $\mathbf{0} \in \hat{\mathcal{S}}_{k+1}$, we have:

$$
\begin{equation*}
\xi_{k+1}(\mathbf{0})=\xi_{k}(\mathbf{0}) \oplus Z_{k, \mathbf{0}, \mathbf{0}} \tag{13}
\end{equation*}
$$

and for the sites $(k+1) e_{i} \in \hat{\mathcal{S}}_{k+1}$ with $i \in\{1,2\}$, we have:

$$
\begin{equation*}
\xi_{k+1}\left((k+1) e_{i}\right)=\xi_{k}\left(k e_{i}\right) \oplus Z_{k,(k+1) e_{i},-e_{i}} \tag{14}
\end{equation*}
$$

2) For any site $x=\left(x_{1}, x_{2}\right) \in \hat{\mathcal{S}}_{k+1} \backslash\left\{\mathbf{0},(k+1) e_{1},(k+1) e_{2}\right\}$ such that $x_{i}>0$ and $x_{j}=0$ for $i, j \in\{1,2\}$, we have:

$$
\xi_{k+1}(x)= \begin{cases}\xi_{k}(x) \oplus Z_{k, x, \mathbf{0}}, & \text { with probability } \frac{1}{2}  \tag{15}\\ \xi_{k}\left(x-e_{i}\right) \oplus Z_{k, x,-e_{i}}, & \text { with probability } \frac{1}{2}\end{cases}
$$

where one of the two noisy inputs of $\xi_{k+1}(x)$ is chosen randomly and independently of everything else, and for any site $x=\left(x_{1}, x_{2}\right) \in \hat{\mathcal{S}}_{k+1} \backslash\left\{(k+1) e_{1},(k+1) e_{2}\right\}$ such that $x_{1}+x_{2}=k+1$, we have:

$$
\xi_{k+1}(x)= \begin{cases}\xi_{k}\left(x-e_{1}\right) \oplus Z_{k, x,-e_{1}}, & \text { with probability } \frac{1}{2}  \tag{16}\\ \xi_{k}\left(x-e_{2}\right) \oplus Z_{k, x,-e_{2}}, & \text { with probability } \frac{1}{2}\end{cases}
$$

where, once again, one of the two noisy inputs of $\xi_{k+1}(x)$ is chosen randomly and independently of everything else,
3) For any site $x=\left(x_{1}, x_{2}\right) \in \hat{\mathcal{S}}_{k+1}$ such that $x_{1}, x_{2}>0$, we have:

$$
\begin{equation*}
\xi_{k+1}(x)=\operatorname{majority}\left(\xi_{k}\left(x-e_{1}\right) \oplus Z_{k, x,-e_{1}}, \xi_{k}(x) \oplus Z_{k, x, \mathbf{0}}, \xi_{k}\left(x-e_{2}\right) \oplus Z_{k, x,-e_{2}}\right) \tag{17}
\end{equation*}
$$

where, as before, $\left\{Z_{k, x, v}: k \in \mathbb{N}, x \in \hat{\mathcal{S}}_{k+1}, v \in \mathcal{N}\right\}$ are i.i.d. Bernoulli( $\delta$ ) noise random variables that are independent of everything else. This completely specifies the transition kernels $\left\{P_{\xi_{k} \mid \xi_{0}(\mathbf{0})}: k \in \mathbb{N}\right\}$ of the Markov process corresponding to Toom's 2D PCA with boundary conditions (which are defined by (13), (14), (15), (16), and (17)). (To define a valid joint probability distribution of $\left\{\xi_{k}(x): x \in \hat{\mathcal{S}}_{k}, k \in \mathbb{N}\right\}$, we can additionally impose the initial condition $\xi_{0}(\mathbf{0}) \sim \operatorname{Bernoulli}\left(\frac{1}{2}\right)$ akin to the uniform source distribution for the majority 3D regular grid.)

It turns out that the majority 3D regular grid and Toom's 2D PCA with boundary conditions are statistically equivalent, i.e. their BSC noise random variables can be coupled so that the Markov processes $\left\{X_{\mathcal{S}_{k}}: k \in \mathbb{N}\right\}$ and $\left\{\xi_{k}: k \in \mathbb{N}\right\}$ are equal almost surely. This equivalence is presented in the ensuing proposition.

Proposition 3 (Majority 3D Regular Grid Equivalence). If $X_{0}=\xi_{0}(\mathbf{0})$ almost surely with any initial source distribution, then we can couple the majority $3 D$ regular grid and Toom's $2 D$ PCA with boundary conditions so that almost surely:

$$
\forall v=\left(v_{1}, v_{2}, v_{3}\right) \in \mathbb{N}^{3}, \quad X_{v}=\xi_{v_{1}+v_{2}+v_{3}}\left(\left(v_{1}, v_{2}\right)\right),
$$

[^4]or equivalently:
$$
\forall k \in \mathbb{N}, \forall x=\left(x_{1}, x_{2}\right) \in \hat{\mathcal{S}}_{k}, \quad \xi_{k}(x)=X_{\left(x_{1}, x_{2}, k-x_{1}-x_{2}\right)} .
$$

Proposition 3 is proved in appendix A via a projection argument. The result implies that broadcasting is impossible in the majority 3D regular grid if and only if reconstruction of $\xi_{0}(\mathbf{0})$ is impossible in Toom's 2D PCA with boundary conditions. Since the standard 2D PCA (with noise on the vertices) that uses Toom's NEC rule is non-ergodic [36], we believe that reconstruction should be possible in our variant of Toom's 2D PCA with boundary conditions. Toom's proof of non-ergodicity in [36] is quite sophisticated, and a simpler combinatorial version of it has been proposed in [44]. We feel that it is an interesting open problem to establish the feasibility of broadcasting in the 3D regular grid with majority processing functions by possibly modifying the simple version of Toom's proof in [44].

## III. Percolation Analysis of 2D Regular Grid with AND Processing Functions

In this section, we prove Theorem 1. Recall that we are given a 2D regular grid where all Boolean processing functions with two inputs are the AND rule, and all Boolean processing functions with one input are the identity rule, i.e. $f_{2}\left(x_{1}, x_{2}\right)=x_{1} \wedge x_{2}$ and $f_{1}(x)=x$, where $\wedge$ denotes the logical AND operation.

As in our proof of [2, Theorem 1], we begin by constructing a useful "monotone Markovian coupling" (see [45, Chapter 5] for basic definitions of Markovian couplings). Let $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ denote versions of the Markov chain $\left\{X_{k}: k \in \mathbb{N}\right\}$ (i.e. with the same transition kernels) initialized at $X_{0}^{+}=1$ and $X_{0}^{-}=0$, respectively. Note that the marginal distributions of $X_{k}^{+}$and $X_{k}^{-}$are $P_{X_{k}}^{+}$and $P_{X_{k}}^{-}$, respectively. Furthermore, define the coupled 2D grid variables $\left\{Y_{k, j}=\left(X_{k, j}^{-}, X_{k, j}^{+}\right): k \in \mathbb{N}, j \in[k+1]\right\}$, so that our Markovian coupling of the Markov chains $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ is the Markov chain $\left\{Y_{k}=\left(Y_{k, 0}, \ldots, Y_{k, k}\right): k \in \mathbb{N}\right\}$. We will couple $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ to "run" on a common underlying 2D regular grid with shared edge BSCs.

Recall that each edge $\operatorname{BSC}(\delta)$ either copies its input bit with probability $1-2 \delta$, or generates an independent Bernoulli $\left(\frac{1}{2}\right)$ output bit with probability $2 \delta .{ }^{8}$ This follows from appropriately interpreting the following decomposition of the BSC transition matrix:

$$
\underbrace{\left[\begin{array}{cc}
1-\delta & \delta  \tag{18}\\
\delta & 1-\delta
\end{array}\right]}_{\text {BSC matrix }}=(1-2 \delta) \underbrace{\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right]}_{\text {copy matrix }}+(2 \delta) \underbrace{\left[\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{array}\right]}_{\text {random bit }} .
$$

Since the underlying 2D regular grid is fixed, we couple $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ so that along any edge BSC of the grid, say $\left(X_{k, j}, X_{k+1, j}\right), X_{k, j}^{+}$and $X_{k, j}^{-}$are either both copied with probability $1-2 \delta$, or a shared independent Bernoulli $\left(\frac{1}{2}\right)$ bit is produced with probability $2 \delta$ that becomes the value of both $X_{k+1, j}^{+}$and $X_{k+1, j}^{-}$. The Markovian coupling $\left\{Y_{k}: k \in \mathbb{N}\right\}$ exhibits the following properties:

1) The "marginal" Markov chains are $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$.
2) For every $k \in \mathbb{N}, X_{k+1}^{+}$is conditionally independent of $X_{k}^{-}$given $X_{k}^{+}$, and $X_{k+1}^{-}$is conditionally independent of $X_{k}^{+}$given $X_{k}^{-}$.
3) For every $k \in \mathbb{N}$ and every $j \in[k+1], X_{k, j}^{+} \geq X_{k, j}^{-}$almost surely.

Here, the third (monotonicity) property of our coupling holds because $1=X_{0,0}^{+} \geq X_{0,0}^{-}=0$ is true by assumption, each edge BSC preserves monotonicity, and AND processing functions are symmetric and monotone non-decreasing. In this section, probabilities of events that depend on the coupled 2 D grid variables $\left\{Y_{k, j}: k \in \mathbb{N}, j \in[k+1]\right\}$ are defined with respect to this Markovian coupling.

Since the marginal Markov chains $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ run on the same 2D regular grid with common BSCs, we keep track of the Markov chain $\left\{Y_{k}: k \in \mathbb{N}\right\}$ in a single coupled 2D regular grid. This 2D regular grid has the same underlying graph as the 2 D regular grid described in subsection I-C. Its vertices are the coupled 2D grid variables $\left\{Y_{k, j}=\left(X_{k, j}^{-}, X_{k, j}^{+}\right): k \in \mathbb{N}, j \in[k+1]\right\}$, and we relabel the alphabet of these variables for simplicity. So, each $Y_{k, j}=\left(X_{k, j}^{-}, X_{k, j}^{+}\right) \in \mathcal{Y}$ with:

$$
\begin{equation*}
\mathcal{Y} \triangleq\left\{0_{\mathrm{c}}, 1_{\mathrm{u}}, 1_{\mathrm{c}}\right\} \tag{19}
\end{equation*}
$$

[^5]where $0_{\mathrm{c}}=(0,0), 1_{\mathrm{u}}=(0,1)$, and $1_{\mathrm{c}}=(1,1)$. (Note that we do not require a letter $0_{\mathrm{u}}=(1,0)$ in this alphabet due to the monotonicity in the coupling.) Furthermore, each edge of the coupled 2 D regular grid is a channel (conditional distribution) $W$ between the alphabets $\mathcal{Y}$ and $\mathcal{Y}$ that captures the action of a shared $\mathrm{BSC}(\delta)$-we describe $W$ using the following row stochastic matrix:
\[

\left.W=$$
\begin{array}{c} 
 \tag{20}\\
0_{\mathrm{c}} \\
1_{\mathrm{u}} \\
1_{\mathrm{c}}
\end{array}
$$ $$
\begin{array}{ccc}
0_{\mathrm{c}} & 1_{\mathrm{u}} & 1_{\mathrm{c}} \\
1-\delta & 0 & \delta \\
\delta & 1-2 \delta & \delta \\
\delta & 0 & 1-\delta
\end{array}
$$\right]
\]

where the $(i, j)$ th entry gives the probability of output $j$ given input $i$. It is straightforward to verify that $W$ describes the aforementioned Markovian coupling. Finally, the AND rule can be equivalently described on the alphabet $\mathcal{Y}$ as:

| $y_{1}$ | $y_{2}$ | $y_{1} \wedge y_{2}$ |
| :---: | :---: | :---: |
| $0_{\mathrm{c}}$ | $\star$ | $0_{\mathrm{c}}$ |
| $1_{\mathrm{u}}$ | $1_{\mathrm{u}}$ | $1_{\mathrm{u}}$ |
| $1_{\mathrm{u}}$ | $1_{\mathrm{c}}$ | $1_{\mathrm{u}}$ |
| $1_{\mathrm{c}}$ | $1_{\mathrm{c}}$ | $1_{\mathrm{c}}$ |

where $\star$ denotes any letter in $\mathcal{Y}$, and the symmetry of the AND rule covers all other possible input combinations. This coupled 2D regular grid model completely characterizes the Markov chain $\left\{Y_{k}: k \in \mathbb{N}\right\}$, which starts at $Y_{0}=1_{\mathrm{u}}$ almost surely. We next prove Theorem 1 by further analyzing this model.

Proof of Theorem 1. We first bound the TV distance between $P_{X_{k}}^{+}$and $P_{X_{k}}^{-}$using Dobrushin's maximal coupling characterization of TV distance, cf. [45, Chapter 4.2]:

$$
\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}} \leq \mathbb{P}\left(X_{k}^{+} \neq X_{k}^{-}\right)=1-\mathbb{P}\left(X_{k}^{+}=X_{k}^{-}\right) .
$$

The events $\left\{X_{k}^{+}=X_{k}^{-}\right\}$are non-decreasing in $k$, i.e. $\left\{X_{k}^{+}=X_{k}^{-}\right\} \subseteq\left\{X_{k+1}^{+}=X_{k+1}^{-}\right\}$for all $k \in \mathbb{N}$. Indeed, suppose for any $k \in \mathbb{N}$, the event $\left\{X_{k}^{+}=X_{k}^{-}\right\}$occurs. Since we have:

$$
\begin{aligned}
\left\{X_{k}^{+}=X_{k}^{-}\right\} & =\left\{Y_{k} \in\left\{0_{\mathrm{c}}, 1_{\mathrm{c}}\right\}^{k+1}\right\} \\
& =\left\{\text { there are no } 1_{\mathrm{u}} \text { 's in level } k \text { of the coupled 2D regular grid }\right\}
\end{aligned}
$$

the channel (20) and the rule (21) imply that there are no $1_{u}$ 's in level $k+1$. Hence, the event $\left\{X_{k+1}^{+}=X_{k+1}^{-}\right\}$ occurs as well. Letting $k \rightarrow \infty$, we can use the continuity of $\mathbb{P}$ with the events $\left\{X_{k}^{+}=X_{k}^{-}\right\}$to get:

$$
\lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}} \leq 1-\lim _{k \rightarrow \infty} \mathbb{P}\left(X_{k}^{+}=X_{k}^{-}\right)=1-\mathbb{P}(A)
$$

where we define:
$A \triangleq\left\{\exists k \in \mathbb{N}\right.$, there are no $1_{\mathrm{u}}$ 's in level $k$ of the coupled 2D regular grid $\}$.
Therefore, it suffices to prove that $\mathbb{P}(A)=1$.
To prove this, we recall a well-known result from [47, Section 3] on oriented bond percolation in 2D lattices. Given the underlying DAG of our 2D regular grid from subsection I-C, suppose we independently keep each edge "open" with some probability $p \in[0,1]$, and delete it ("closed") with probability $1-p$. Define the event:

$$
\Omega_{\infty} \triangleq\{\text { there is an infinite open path starting at the root }\}
$$

and the quantities:

$$
\begin{aligned}
& R_{k} \triangleq \sup \{j \in[k+1]: \text { there is an open path from the root to the vertex }(k, j)\} \\
& L_{k} \triangleq \inf \{j \in[k+1]: \text { there is an open path from the root to the vertex }(k, j)\}
\end{aligned}
$$

which are the rightmost and leftmost vertices at level $k \in \mathbb{N}$, respectively, that are connected to the root. (Here, we refer to the vertex $X_{k, j}$ using $(k, j)$ as we do not associate a random variable to it.) It is proved in [47, Section 3] that the occurrence of $\Omega_{\infty}$ experiences a phase transition phenomenon as the open probability parameter $p$ varies from 0 to 1 .

Lemma 1 (Oriented Bond Percolation [47, Section 3]). For the aforementioned bond percolation process on the 2D regular grid, there exists a critical threshold $\delta_{\mathrm{perc}} \in\left(\frac{1}{2}, 1\right)$ around which we observe the following phase transition phenomenon:

1) If $p>\delta_{\text {perc }}$, then $\mathbb{P}_{p}\left(\Omega_{\infty}\right)>0$ and:

$$
\begin{equation*}
\mathbb{P}_{p}\left(\lim _{k \rightarrow \infty} \frac{R_{k}}{k}=\frac{1+\alpha(p)}{2} \text { and } \left.\lim _{k \rightarrow \infty} \frac{L_{k}}{k}=\frac{1-\alpha(p)}{2} \right\rvert\, \Omega_{\infty}\right)=1 \tag{22}
\end{equation*}
$$

for some constant $\alpha(p)>0$, where $\alpha(p)$ is defined in [47, Section 3, Equation (6)], and $\mathbb{P}_{p}$ is the probability measure defined by the bond percolation process.
2) If $p<\delta_{\text {perc }}$, then $\mathbb{P}_{p}\left(\Omega_{\infty}\right)=0$.

We will use Lemma 1 to prove $\mathbb{P}(A)=1$ by considering two cases.
Case 1: Suppose $1-2 \delta<\delta_{\text {perc }}\left(\right.$ i.e. $\left.\delta>\left(1-\delta_{\text {perc }}\right) / 2\right)$ in our coupled 2D grid. The root of the coupled 2D regular grid is $Y_{0,0}=1_{\mathrm{u}}$ almost surely, and we consider an oriented bond percolation process (as described above) with $p=1-2 \delta$. In particular, we say that each edge of the grid is open if and only if the corresponding BSC copies its input (with probability $1-2 \delta$ ). In this context, $\Omega_{\infty}^{c}$ is the event that there exists $k \in \mathbb{N}$ such that none of the vertices at level $k$ are connected to the root via a sequence of BSCs that are copies. Suppose the event $\Omega_{\infty}^{c}$ occurs. Since (20) and (21) portray that a $1_{\mathrm{u}}$ moves from level $k$ to level $k+1$ only if one of its outgoing edges is open (and the corresponding BSC is a copy), there exists $k \in \mathbb{N}$ such that none of the vertices at level $k$ are $1_{\mathrm{u}}$ 's. This proves that $\Omega_{\infty}^{c} \subseteq A$. Therefore, using part 2 of Lemma 1 , we get $\mathbb{P}(A)=1$.

Case 2: Suppose $1-\delta>\delta_{\text {perc }}$ (i.e. $\delta<1-\delta_{\text {perc }}$ ) in our coupled 2D grid. Consider an oriented bond percolation process (as described earlier) with $p=1-\delta$ that runs on the 2 D regular grid, where an edge is open if and only if the corresponding BSC is either copying or generating a 0 as the new bit (i.e. this BSC takes a $0_{c}$ to a $0_{\mathrm{c}}$, which happens with probability $1-\delta$ as shown in (20)). Let $B_{k}$ for $k \in \mathbb{N} \backslash\{0\}$ be the event that the BSC from $Y_{k-1,0}$ to $Y_{k, 0}$ generates a new bit which equals 0 . Then, $\mathbb{P}\left(B_{k}\right)=\delta$ and $\left\{B_{k}: k \in \mathbb{N} \backslash\{0\}\right\}$ are mutually independent. So, the second Borel-Cantelli lemma tells us that infinitely many of the events $\left\{B_{k}: k \in \mathbb{N} \backslash\{0\}\right\}$ occur almost surely. Furthermore, $B_{k} \subseteq\left\{Y_{k, 0}=0_{c}\right\}$ for every $k \in \mathbb{N} \backslash\{0\}$.

We next define the following sequence of random variables for all $i \in \mathbb{N} \backslash\{0\}$ :

$$
\begin{aligned}
& L_{i} \triangleq \min \left\{k \geq T_{i-1}+1: B_{k} \text { occurs }\right\} \\
& T_{i} \triangleq 1+\max \left\{k \geq L_{i}: \exists j \in[k+1], Y_{k, j} \text { is connected to } Y_{L_{i}, 0} \text { by an open path }\right\}
\end{aligned}
$$

where we set $T_{0} \triangleq 0$. Note that when $T_{i-1}=\infty$, we let $L_{i}=\infty$ almost surely. Furthermore, when $T_{i-1}<\infty$, $L_{i}<\infty$ almost surely, because infinitely many of the events $\left\{B_{k}: k \in \mathbb{N}\{0\}\right\}$ occur almost surely. We also note that when $L_{i}<\infty$, the set:

$$
\left\{k \geq L_{i}: \exists j \in[k+1], Y_{k, j} \text { is connected to } Y_{L_{i}, 0} \text { by an open path }\right\}
$$

is non-empty since $Y_{L_{i}, 0}$ is always connected to itself, and $T_{i}-L_{i}-1$ denotes the length of the longest open path connected to $Y_{L_{i}, 0}$ (which could be infinity). Lastly, when $L_{i}=\infty$, we let $T_{i}=\infty$ almost surely.

Let $\mathcal{F}_{k}$ for every $k \in \mathbb{N}$ be the $\sigma$-algebra generated by the random variables $\left(Y_{0}, \ldots, Y_{k}\right)$ and all the BSCs before level $k$ (where we include all events determining whether these BSCs are copies, and all events determining the independent bits they produce). Then, $\left\{\mathcal{F}_{k}: k \in \mathbb{N}\right\}$ is a filtration. It is straightforward to verify that $L_{i}$ and $T_{i}$ are stopping times with respect to $\left\{\mathcal{F}_{k}: k \in \mathbb{N}\right\}$ for all $i \in \mathbb{N} \backslash\{0\}$. We can show this inductively. $T_{0}=0$ is trivially a stopping time, and if $T_{i-1}$ is a stopping time, then $L_{i}$ is clearly a stopping time. So, it suffices to prove that $T_{i}$ is a stopping time given $L_{i}$ is a stopping time. For any finite $m \in \mathbb{N} \backslash\{0\},\left\{T_{i}=m\right\}$ is the event that $L_{i} \leq m-1$ and the length of the longest open path connected to $Y_{L_{i}, 0}$ is $m-1-L_{i}$. This event is contained in $\mathcal{F}_{m}$ because the event $\left\{L_{i} \leq m-1\right\}$ is contained in $\mathcal{F}_{m-1} \subseteq \mathcal{F}_{m}$ (since $L_{i}$ is a stopping time), and the length of the longest open path can be determined from $\mathcal{F}_{m}$ (rather than $\mathcal{F}_{m-1}$ ). Hence, $T_{i}$ is indeed a stopping time when $L_{i}$ is a stopping time.

Now observe that:

$$
\begin{aligned}
\mathbb{P}\left(\exists k \in \mathbb{N}\{0\}, T_{k}=\infty\right) & =\mathbb{P}\left(T_{1}=\infty\right)+\sum_{m=2}^{\infty} \mathbb{P}\left(\exists k \in \mathbb{N}\{0,1\}, T_{k}=\infty \mid T_{1}=m\right) \mathbb{P}\left(T_{1}=m\right) \\
& =\mathbb{P}\left(T_{1}=\infty\right)+\sum_{m=2}^{\infty} \mathbb{P}\left(\exists k \in \mathbb{N}\{0\}, T_{k}+m=\infty\right) \mathbb{P}\left(T_{1}=m\right)
\end{aligned}
$$

$$
\begin{equation*}
=\mathbb{P}\left(T_{1}=\infty\right)+\left(1-\mathbb{P}\left(T_{1}=\infty\right)\right) \mathbb{P}\left(\exists k \in \mathbb{N}\{0\}, T_{k}=\infty\right) \tag{23}
\end{equation*}
$$

where the first equality uses the law of total probability, the third equality follows from straightforward calculations, and the second equality follows from the fact that for all $m \in \mathbb{N} \backslash\{0,1\}$ :

$$
\mathbb{P}\left(\exists k \in \mathbb{N} \backslash\{0,1\}, T_{k}=\infty \mid T_{1}=m\right)=\mathbb{P}\left(\exists k \in \mathbb{N} \backslash\{0\}, T_{k}+m=\infty\right)
$$

This relation holds because the random variables $\left\{\left(L_{i}, T_{i}\right): i \in \mathbb{N} \backslash\{0,1\}\right\}$ given $T_{1}=m$ have the same distribution as the random variables $\left\{\left(L_{i-1}+m, T_{i-1}+m\right): i \in \mathbb{N} \backslash\{0,1\}\right\}$. In particular, the conditional distribution of $L_{i}$ given $T_{1}=m$ corresponds to the distribution of $L_{i-1}+m$, and the conditional distribution of $T_{i}$ given $T_{1}=m$ corresponds to the distribution of $T_{i-1}+m$. These distributional equivalences implicitly use the fact that $\left\{T_{i}: i \in \mathbb{N}\right\}$ are stopping times. Indeed, the conditioning on $\left\{T_{1}=m\right\}$ in these equivalences can be removed because the event $\left\{T_{1}=m\right\}$ is in $\mathcal{F}_{m}$ since $T_{1}$ is a stopping time, and $\left\{T_{1}=m\right\}$ is therefore independent of the events $\left\{B_{k}: k>m\right\}$ and the events that determine when the BSCs below level $m$ are open.

Next, rearranging (23), we get:

$$
\mathbb{P}\left(\exists k \in \mathbb{N} \backslash\{0\}, T_{k}=\infty\right) \mathbb{P}\left(T_{1}=\infty\right)=\mathbb{P}\left(T_{1}=\infty\right)
$$

Since $\mathbb{P}\left(T_{1}=\infty\right)=\mathbb{P}\left(\Omega_{\infty}\right)>0$ by part 1 of Lemma 1, we have:

$$
\begin{equation*}
\mathbb{P}\left(\exists k \in \mathbb{N} \backslash\{0\}, T_{k}=\infty\right)=1 \tag{24}
\end{equation*}
$$

For every $k \in \mathbb{N} \backslash\{0\}$, define the events:

$$
\begin{aligned}
\Omega_{k}^{\text {left }} \triangleq\left\{\text { there exists an infinite open path starting at the vertex } Y_{k, 0}\right\} \\
\Omega_{k}^{\text {right }} \triangleq\left\{\text { there exists an infinite open path starting at the vertex } Y_{k, k}\right\}
\end{aligned}
$$

If the event $\left\{\exists k \in \mathbb{N} \backslash\{0\}, T_{k}=\infty\right\}$ occurs, we can choose the smallest $m \in \mathbb{N} \backslash\{0\}$ such that $T_{m}=\infty$, and for this $m$, there is an infinite open path starting at $Y_{L_{m}, 0}=0_{c}$ (where $Y_{L_{m}, 0}=0_{c}$ because $B_{L_{m}}$ occurs). Hence, using (24), we have:

$$
\mathbb{P}\left(\exists k \in \mathbb{N},\left\{Y_{k, 0}=0_{c}\right\} \cap \Omega_{k}^{\text {left }}\right)=1
$$

Likewise, we can also prove that:

$$
\mathbb{P}\left(\exists k \in \mathbb{N},\left\{Y_{k, k}=0_{\mathrm{c}}\right\} \cap \Omega_{k}^{\text {right }}\right)=1
$$

which implies that:

$$
\begin{equation*}
\mathbb{P}\left(\exists k \in \mathbb{N}, \exists m \in \mathbb{N},\left\{Y_{k, 0}=Y_{m, m}=0_{c}\right\} \cap \Omega_{k}^{\text {left }} \cap \Omega_{m}^{\text {right }}\right)=1 \tag{25}
\end{equation*}
$$

To finish the proof, consider $k, m \in \mathbb{N}$ such that $Y_{k, 0}=Y_{m, m}=0_{c}$, and suppose $\Omega_{k}^{\text {left }}$ and $\Omega_{m}^{\text {right }}$ both happen. For every $n>\max \{k, m\}$, define the quantities:

$$
\begin{gathered}
R_{n}^{\text {left }} \triangleq \sup \left\{j \in[n+1]: \text { there is an open path from } Y_{k, 0} \text { to } Y_{n, j}\right\} \\
L_{n}^{\text {right }} \triangleq \inf \left\{j \in[n+1]: \text { there is an open path from } Y_{m, m} \text { to } Y_{n, j}\right\}
\end{gathered}
$$

which are the rightmost and leftmost vertices at level $n$ that are connected to $Y_{k, 0}$ and $Y_{m, m}$, respectively, by open paths. Using (22) from part 1 of Lemma 1, we know that almost surely:

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{R_{n}^{\text {left }}}{n} & =\lim _{n \rightarrow \infty} \frac{R_{n}^{\text {left }}}{n-k}=\frac{1+\alpha(1-\delta)}{2} \\
\lim _{n \rightarrow \infty} \frac{L_{n}^{\text {right }}}{n} & =\lim _{n \rightarrow \infty} \frac{L_{n}^{\text {right }}-m}{n-m}=\frac{1-\alpha(1-\delta)}{2}
\end{aligned}
$$

This implies that almost surely:

$$
\lim _{n \rightarrow \infty} \frac{R_{n}^{\text {left }}-L_{n}^{\text {right }}}{n}=\alpha(1-\delta)>0
$$

which means that for some sufficiently large level $n>\max \{k, m\}$, the rightmost open path from $Y_{k, 0}$ meets the leftmost open path from $Y_{m, m}$ :

$$
\left|R_{n}^{\text {left }}-L_{n}^{\text {right }}\right| \leq 1
$$

By construction, all the vertices in these two open paths are equal to $0_{c}$. Furthermore, since (20) and (21) demonstrate that AND gates and BSCs output $0_{c}$ 's or $1_{c}$ 's when their inputs are $0_{c}$ 's or $1_{c}$ 's, it is straightforward to inductively
establish that all vertices at level $n$ that are either to left of $R_{n}^{\text {left }}$ or to the right of $L_{n}^{\text {right }}$ take values in $\left\{0_{c}, 1_{c}\right\}$. This shows that every vertex at level $n$ must be equal to $0_{c}$ or $1_{c}$ because the two aforementioned open paths meet. Hence, there exists a level $n \in \mathbb{N}$ with no $1_{\mathrm{u}}$ 's, i.e. the event $A$ occurs. Therefore, we get $\mathbb{P}(A)=1$ using (25).

Combining the two cases completes the proof as $\mathbb{P}(A)=1$ for any $\delta \in\left(0, \frac{1}{2}\right)$.
We remark that this proof can be perceived as using the technique presented in [45, Theorem 5.2]. Indeed, let $T \triangleq \inf \left\{k \in \mathbb{N}: X_{k}^{+}=X_{k}^{-}\right\}$be a stopping time (with respect to the filtration $\left\{\mathcal{F}_{k}: k \in \mathbb{N}\right\}$ defined earlier) denoting the first time that the marginal Markov chains $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ meet. (Note that $\{T=\infty\}$ corresponds to the event that these chains never meet.) Since the events $\left\{X_{k}^{+}=X_{k}^{-}\right\}$for $k \in \mathbb{N}$ form a non-decreasing sequence of sets, $\{T>k\}=\left\{X_{k}^{+} \neq X_{k}^{-}\right\}$. We can use this relation to obtain the following bound on the TV distance between $P_{X_{k}}^{+}$and $P_{X_{k}}^{-}$:

$$
\begin{equation*}
\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}} \leq \mathbb{P}\left(X_{k}^{+} \neq X_{k}^{-}\right)=\mathbb{P}(T>k)=1-\mathbb{P}(T \leq k) \tag{26}
\end{equation*}
$$

where letting $k \rightarrow \infty$ and using the continuity of $\mathbb{P}$ produces:

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}} \leq 1-\mathbb{P}(\exists k \in \mathbb{N}, T \leq k)=1-\mathbb{P}(T<\infty) \tag{27}
\end{equation*}
$$

These bounds correspond to the ones shown in [45, Theorem 5.2]. Since the event $A=\{\exists k \in \mathbb{N}, T \leq k\}=\{T<$ $\infty\}$, our proof that $A$ happens almost surely also demonstrates that the two marginal Markov chains meet after a finite amount of time almost surely.

## IV. Coding Theoretic Analysis of 2D Regular Grid with XOR Processing Functions

We now turn to proving Theorem 2 . We will use some rudimentary coding theory ideas in this section, and refer readers to [48] for an introduction to the subject. We let $\mathbb{F}_{2}=\{0,1\}$ denote the Galois field of order 2 (i.e. integers with addition and multiplication modulo 2), $\mathbb{F}_{2}^{n}$ with $n \in \mathbb{N} \backslash\{0,1\}$ denote the vector space over $\mathbb{F}_{2}$ of column vectors with $n$ entries from $\mathbb{F}_{2}$, and $\mathbb{F}_{2}^{m \times n}$ with $m, n \in \mathbb{N} \backslash\{0,1\}$ denote the space of $m \times n$ matrices with entries in $\mathbb{F}_{2}$. (All matrix and vector operations in this section will be performed modulo 2.) Now fix some matrix $H \in \mathbb{F}_{2}^{m \times n}$ that has the following block structure:

$$
H=\left[\begin{array}{ll}
1 & B_{1}  \tag{28}\\
\mathbf{0} & B_{2}
\end{array}\right]
$$

where $\mathbf{0} \triangleq[0 \cdots 0]^{\mathrm{T}} \in \mathbb{F}_{2}^{m-1}$ denotes the zero vector (whose dimension will be understood from context in the sequel), $B_{1} \in \mathbb{F}_{2}^{1 \times(n-1)}$, and $B_{2} \in \mathbb{F}_{2}^{(m-1) \times(n-1)}$. Consider the following two problems:

1) Coding Problem: Let $\mathcal{C} \triangleq\left\{x \in \mathbb{F}_{2}^{n}: H x=0\right\}$ be the linear code defined by the parity check matrix $H$. Let $X=\left[\begin{array}{ll}X_{1} & X_{2}^{\mathrm{T}}\end{array}\right]^{\mathrm{T}}$ with $X_{1} \in \mathbb{F}_{2}$ and $X_{2} \in \mathbb{F}_{2}^{n-1}$ be a codeword drawn uniformly from $\mathcal{C}$. Assume that there exists a codeword $x=\left[1 x_{2}^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathcal{C}$ (i.e. $B_{1} x_{2}=1$ and $B_{2} x_{2}=\mathbf{0}$ ). Then, since $\mathcal{C} \ni x^{\prime} \mapsto x^{\prime}+x \in \mathcal{C}$ is a bijective map that flips the first bit of its input, $X_{1}$ is a $\operatorname{Bernoulli}\left(\frac{1}{2}\right)$ random variable. We observe the codeword $X$ through an additive noise channel model and see $Y_{1} \in \mathbb{F}_{2}$ and $Y_{2} \in \mathbb{F}_{2}^{n-1}$ :

$$
\left[\begin{array}{l}
Y_{1}  \tag{29}\\
Y_{2}
\end{array}\right]=X+\left[\begin{array}{l}
Z_{1} \\
Z_{2}
\end{array}\right]=\left[\begin{array}{l}
X_{1}+Z_{1} \\
X_{2}+Z_{2}
\end{array}\right]
$$

where $Z_{1} \in \mathbb{F}_{2}$ is a Bernoulli $\left(\frac{1}{2}\right)$ random variable, $Z_{2} \in \mathbb{F}_{2}^{n-1}$ is a vector of i.i.d. Bernoulli $(\delta)$ random variables that are independent of $Z_{1}$, and both $Z_{1}, Z_{2}$ are independent of $X$. Our problem is to decode $X_{1}$ with minimum probability of error after observing $Y_{1}, Y_{2}$. This can be achieved by using the ML decoder for $X_{1}$ based on $Y_{1}, Y_{2}$.
2) Inference Problem: Let $X^{\prime} \in \mathbb{F}_{2}$ be a Bernoulli $\left(\frac{1}{2}\right)$ random variable, and $Z \in \mathbb{F}_{2}^{n-1}$ be a vector of i.i.d. Bernoulli $(\delta)$ random variables that are independent of $X^{\prime}$. Suppose we see the observations $S_{1}^{\prime} \in \mathbb{F}_{2}$ and $S_{2}^{\prime} \in \mathbb{F}_{2}^{m-1}$ through the model:

$$
\left[\begin{array}{l}
S_{1}^{\prime}  \tag{30}\\
S_{2}^{\prime}
\end{array}\right]=H\left[\begin{array}{c}
X^{\prime} \\
Z
\end{array}\right]=\left[\begin{array}{c}
X^{\prime}+B_{1} Z \\
B_{2} Z
\end{array}\right]
$$

Our problem is to decode $X^{\prime}$ with minimum probability of error after observing $S_{1}^{\prime}, S_{2}^{\prime}$. This can be achieved by using the ML decoder for $X^{\prime}$ based on $S_{1}^{\prime}, S_{2}^{\prime}$.
As we will soon see, the inference problem above corresponds to our setting of reconstruction in the 2D regular grid with XOR processing functions. The next lemma illustrates that this inference problem is in fact "equivalent"
to the aforementioned coding problem, and this connection will turn out to be useful since the coding problem admits simpler analysis.

Lemma 2 (Equivalence of Problems). For the coding problem in (29) and the inference problem in (30), the following statements hold:

1) The minimum probabilities of error for the coding and inference problems are equal.
2) Suppose the random variables in the coding and inference problems are coupled so that $X_{1}=X^{\prime}$ and $Z_{2}=Z$ almost surely (i.e. these variables are shared by the two problems), $X_{2}$ is generated from a conditional distribution $P_{X_{2} \mid X_{1}}$ such that $X$ is uniform on $\mathcal{C}, Z_{1}$ is generated independently, $\left(Y_{1}, Y_{2}\right)$ is defined by (29), and $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ is defined by (30). Then, $S_{1}^{\prime}=B_{1} Y_{2}$ and $S_{2}^{\prime}=B_{2} Y_{2}$ almost surely.
3) Under the aforementioned coupling, $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ is a sufficient statistic of $\left(Y_{1}, Y_{2}\right)$ for performing inference about $X_{1}$ (in the coding problem).

## Proof.

Part 1: We first show that the minimum probabilities of error for the two problems are equal. The inference problem has prior $X^{\prime} \sim \operatorname{Bernoulli}\left(\frac{1}{2}\right)$, and the following likelihoods for every $s_{1}^{\prime} \in \mathbb{F}_{2}$ and every $s_{2}^{\prime} \in \mathbb{F}_{2}^{m-1}$ :

$$
\begin{align*}
& P_{S_{1}^{\prime}, S_{2}^{\prime} \mid X^{\prime}}\left(s_{1}^{\prime}, s_{2}^{\prime} \mid 0\right)=\sum_{z \in \mathbb{F}_{2}^{n-1}} P_{Z}(z) \mathbb{1}\left\{B_{1} z=s_{1}^{\prime}, B_{2} z=s_{2}^{\prime}\right\},  \tag{31}\\
& P_{S_{1}^{\prime}, S_{2}^{\prime} \mid X^{\prime}}\left(s_{1}^{\prime}, s_{2}^{\prime} \mid 1\right)=\sum_{z \in \mathbb{F}_{2}^{n-1}} P_{Z}(z) \mathbb{1}\left\{B_{1} z=s_{1}^{\prime}+1, B_{2} z=s_{2}^{\prime}\right\} . \tag{32}
\end{align*}
$$

On the other hand, the coding problem has prior $X_{1} \sim \operatorname{Bernoulli}\left(\frac{1}{2}\right)$, and the following likelihoods for every $y_{1} \in \mathbb{F}_{2}$ and every $y_{2} \in \mathbb{F}_{2}^{n-1}$ :

$$
\begin{align*}
P_{Y_{1}, Y_{2} \mid X_{1}}\left(y_{1}, y_{2} \mid 0\right) & =P_{Y_{1} \mid X_{1}}\left(y_{1} \mid 0\right) P_{Y_{2} \mid X_{1}}\left(y_{2} \mid 0\right) \\
& =\frac{1}{2} \sum_{x_{2} \in \mathbb{F}_{2}^{n-1}} P_{Y_{2} \mid X_{2}}\left(y_{2} \mid x_{2}\right) P_{X_{2} \mid X_{1}}\left(x_{2} \mid 0\right) \\
& =\frac{1}{2} \sum_{x_{2} \in \mathbb{F}_{2}^{n-1}} P_{Z_{2}}\left(y_{2}-x_{2}\right) \mathbb{1}\left\{B_{1} x_{2}=0, B_{2} x_{2}=\mathbf{0}\right\} \frac{2}{|\mathcal{C}|} \\
& =\frac{1}{|\mathcal{C}|} \sum_{z_{2} \in \mathbb{F}_{2}^{n-1}} P_{Z_{2}}\left(z_{2}\right) \mathbb{1}\left\{B_{1} z_{2}=B_{1} y_{2}, B_{2} z_{2}=B_{2} y_{2}\right\}  \tag{33}\\
P_{Y_{1}, Y_{2} \mid X_{1}}\left(y_{1}, y_{2} \mid 1\right) & =\frac{1}{|\mathcal{C}|} \sum_{z_{2} \in \mathbb{F}_{2}^{n-1}} P_{Z_{2}}\left(z_{2}\right) \mathbb{1}\left\{B_{1} z_{2}=B_{1} y_{2}+1, B_{2} z_{2}=B_{2} y_{2}\right\} \tag{34}
\end{align*}
$$

where the third equality uses the fact that $X_{2}$ is uniform over a set of cardinality $|\mathcal{C}| / 2$ given any value of $X_{1}$, because $X_{1} \sim \operatorname{Bernoulli}\left(\frac{1}{2}\right)$ and $X$ is uniform on $\mathcal{C}$. For the coding problem, define $S_{1} \triangleq B_{1} Y_{2}$ and $S_{2} \triangleq B_{2} Y_{2}$. Due to the Fisher-Neyman factorization theorem [49, Theorem 3.6], (33) and (34) demonstrate that $\left(S_{1}, S_{2}\right)$ is a sufficient statistic of $\left(Y_{1}, Y_{2}\right)$ for performing inference about $X_{1}$.

Continuing in the context of the coding problem, define the set:

$$
\mathcal{C}^{\prime} \triangleq\left\{x \in \mathbb{F}_{2}^{n-1}: B_{1} x=0, B_{2} x=\mathbf{0}\right\}
$$

which is also a linear code, and for any fixed $s_{1} \in \mathbb{F}_{2}$ and $s_{2} \in \mathbb{F}_{2}^{m-1}$, define the set:

$$
\mathcal{S}\left(s_{1}, s_{2}\right) \triangleq\left\{\left(y_{1}, y_{2}\right) \in \mathbb{F}_{2} \times \mathbb{F}_{2}^{n-1}: B_{1} y_{2}=s_{1}, B_{2} y_{2}=s_{2}\right\}
$$

If there exists $y_{2}^{\prime} \in \mathbb{F}_{2}^{n-1}$ such that $B_{1} y_{2}^{\prime}=s_{1}$ and $B_{2} y_{2}^{\prime}=s_{2}$, then $\mathcal{S}\left(s_{1}, s_{2}\right)=\left\{\left(y_{1}, y_{2}+y_{2}^{\prime}\right) \in \mathbb{F}_{2} \times \mathbb{F}_{2}^{n-1}\right.$ : $\left.y_{2} \in \mathcal{C}^{\prime}\right\}$, which means that $\left|\mathcal{S}\left(s_{1}, s_{2}\right)\right|=2\left|\mathcal{C}^{\prime}\right|=|\mathcal{C}|$ (where the final equality holds because each vector in $\mathcal{C}^{\prime}$ corresponds to a codeword in $\mathcal{C}$ whose first letter is 0 , and we have assumed that there are an equal number of codewords in $\mathcal{C}$ with first letter 1 ). Hence, for every $s_{1} \in \mathbb{F}_{2}$ and every $s_{2} \in \mathbb{F}_{2}^{m-1}$, the likelihoods of $\left(S_{1}, S_{2}\right)$ given $X_{1}$ can be computed from (33) and (34):

$$
P_{S_{1}, S_{2} \mid X_{1}}\left(s_{1}, s_{2} \mid 0\right)=\sum_{y_{1} \in \mathbb{F}_{2}, y_{2} \in \mathbb{F}_{2}^{n-1}} P_{Y_{1}, Y_{2} \mid X_{1}}\left(y_{1}, y_{2} \mid 0\right) \mathbb{1}\left\{B_{1} y_{2}=s_{1}, B_{2} y_{2}=s_{2}\right\}
$$

$$
\begin{align*}
& =\frac{\left|\mathcal{S}\left(s_{1}, s_{2}\right)\right|}{|\mathcal{C}|} \sum_{z_{2} \in \mathbb{F}_{2}^{n-1}} P_{Z_{2}}\left(z_{2}\right) \mathbb{1}\left\{B_{1} z_{2}=s_{1}, B_{2} z_{2}=s_{2}\right\} \\
& =\sum_{z_{2} \in \mathbb{F}_{2}^{n-1}} P_{Z_{2}}\left(z_{2}\right) \mathbb{1}\left\{B_{1} z_{2}=s_{1}, B_{2} z_{2}=s_{2}\right\},  \tag{35}\\
P_{S_{1}, S_{2} \mid X_{1}}\left(s_{1}, s_{2} \mid 1\right) & =\sum_{z_{2} \in \mathbb{F}_{2}^{n-1}} P_{Z_{2}}\left(z_{2}\right) \mathbb{1}\left\{B_{1} z_{2}=s_{1}+1, B_{2} z_{2}=s_{2}\right\}, \tag{36}
\end{align*}
$$

where the second equality follows from (33) and the third equality clearly holds in the $\left|\mathcal{S}\left(s_{1}, s_{2}\right)\right|=0$ case as well. The likelihoods (35) and (36) are exactly the same as the likelihoods (31) and (32), respectively, that we computed earlier for the inference problem. Thus, the sufficient statistic $\left(S_{1}, S_{2}\right)$ of $\left(Y_{1}, Y_{2}\right)$ for $X_{1}$ in the coding problem is equivalent to the observation $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ in the inference problem in the sense that they are defined by the same probability model. As a result, the minimum probabilities of error in these formulations must be equal.

Part 2: We now assume that the random variables in the two problems are coupled as in the lemma statement. To prove that $S_{1}^{\prime}=S_{1}$ and $S_{2}^{\prime}=S_{2}$ almost surely, observe that:

$$
\begin{aligned}
{\left[\begin{array}{l}
S_{1} \\
S_{2}
\end{array}\right] } & =\left[\begin{array}{l}
B_{1} Y_{2} \\
B_{2} Y_{2}
\end{array}\right] \\
& =\left[\begin{array}{c}
B_{1} X_{2}+B_{1} Z_{2} \\
B_{2} X_{2}+B_{2} Z_{2}
\end{array}\right] \\
& =\left[\begin{array}{c}
X_{1}+B_{1} Z_{2} \\
B_{2} Z_{2}
\end{array}\right] \\
& =H\left[\begin{array}{c}
X_{1} \\
Z_{2}
\end{array}\right] \\
& =\left[\begin{array}{c}
S_{1}^{\prime} \\
S_{2}^{\prime}
\end{array}\right]
\end{aligned}
$$

where the second equality uses (29), the third equality holds because $B_{1} X_{2}=X_{1}$ and $B_{2} X_{2}=\mathbf{0}$ since $X \in \mathcal{C}$ is a codeword, and the last equality uses (30) and the fact that $X_{1}=X^{\prime}$ and $Z_{2}=Z$ almost surely. This proves part 2.

Part 3: Since $\left(S_{1}, S_{2}\right)$ is a sufficient statistic of $\left(Y_{1}, Y_{2}\right)$ for performing inference about $X_{1}$ in the coding problem, and $S_{1}^{\prime}=S_{1}$ and $S_{2}^{\prime}=S_{2}$ almost surely under the coupling in the lemma statement, $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ is also a sufficient statistic of $\left(Y_{1}, Y_{2}\right)$ for performing inference about $X_{1}$ under this coupling. This completes the proof.

Recall that we are given a 2D regular grid where all Boolean processing functions with two inputs are the XOR rule, and all Boolean processing functions with one input are the identity rule, i.e. $f_{2}\left(x_{1}, x_{2}\right)=x_{1} \oplus x_{2}$ and $f_{1}(x)=x$. We next prove Theorem 2 using Lemma 2.

Proof of Theorem 2. We first prove that the problem of decoding the root bit in the XOR 2D regular grid is captured by the inference problem defined in (30). Let $E_{k}$ denote the set of all directed edges in the 2D regular grid above level $k \in \mathbb{N}$. Furthermore, let us associate each edge $e \in E_{k}$ with an independent $\operatorname{Bernoulli}(\delta)$ random variable $Z_{e} \in \mathbb{F}_{2}$. Since a $\operatorname{BSC}(\delta)$ can be modeled as addition of an independent $\operatorname{Bernoulli}(\delta)$ bit (in $\left.\mathbb{F}_{2}\right)$, the random variables $\left\{Z_{e}: e \in E_{k}\right\}$ define the BSCs of the 2D regular grid up to level $k$. Moreover, each vertex at level $k \in \mathbb{N} \backslash\{0\}$ of the XOR 2 D regular grid is simply a sum (in $\mathbb{F}_{2}$ ) of its parent vertices and the random variables on the edges between it and its parents:

$$
\begin{aligned}
\forall j \in\{1, \ldots, k-1\}, & X_{k, j}=X_{k-1, j-1} \oplus X_{k-1, j} \oplus Z_{\left(X_{k-1, j-1}, X_{k, j}\right)} \oplus Z_{\left(X_{k-1, j}, X_{k, j}\right)} \\
& X_{k, 0}=X_{k-1,0} \oplus Z_{\left(X_{k-1,0}, X_{k, 0}\right)} \\
& X_{k, k}=X_{k-1, k-1} \oplus Z_{\left(X_{k-1, k-1}, X_{k, k}\right)}
\end{aligned}
$$

These recursive formulae for each vertex in terms of its parent vertices can be unwound so that each vertex is represented as a linear combination (in $\mathbb{F}_{2}$ ) of the root bit and all the edge random variables:

$$
\begin{equation*}
\forall k \in \mathbb{N} \backslash\{0\}, \forall j \in[k+1], \quad X_{k, j}=\left(\binom{k}{j}(\bmod 2)\right) X_{0,0}+\sum_{e \in E_{k}} b_{k, j, e} Z_{e} \tag{37}
\end{equation*}
$$

where the coefficient of $X_{0,0}$ can be computed by realizing that the coefficients of the vertices in the " 2 D regular grid above $X_{k, j}$ " (with $X_{k, j}$ as the root) are defined by the recursion of Pascal's triangle, and $\left\{b_{k, j, e} \in \mathbb{F}_{2}: k \in\right.$ $\left.\mathbb{N} \backslash\{0\}, j \in[k+1], e \in E_{k}\right\}$ are some fixed coefficients. We do not require detailed knowledge of the values of $\left\{b_{k, j, e} \in \mathbb{F}_{2}: k \in \mathbb{N} \backslash\{0\}, j \in[k+1], e \in E_{k}\right\}$, but they can also be evaluated via straightforward counting if desired.

In the remainder of this proof, we will fix $k$ to be a power of 2 : $k=2^{m}$ for $m \in \mathbb{N} \backslash\{0\}$. Then, we have:

$$
\binom{k}{j} \equiv\binom{2^{m}}{j} \equiv\left\{\begin{array}{lll}
1 & , & j \in\{0, k\}  \tag{38}\\
0 & , & j \in\{1, \ldots, k-1\}
\end{array} \quad(\bmod 2)\right.
$$

since by Lucas' theorem (see [50]), the parity of $\binom{k}{j}$ is 0 if and only if at least one of the digits of $j$ in base 2 is strictly greater than the corresponding digit of $k$ in base 2 , and the base 2 representation of $k=2^{m}$ is $10 \cdots 0$ (with $m 0$ 's). So, for each $k$, we can define a binary matrix $H_{k} \in \mathbb{F}_{2}^{(k+1) \times\left(\left|E_{k}\right|+1\right)}$ whose rows are indexed by the vertices at level $k$ and columns are indexed by 1 (first index corresponding to $X_{0,0}$ ) followed by the edges in $E_{k}$, and whose rows are made up of the coefficients in (37) (where the first entry of each row is given by (38)). Clearly, we can write (37) in matrix-vector form using $H_{k}$ for every $k$ :

$$
\left[\begin{array}{c}
X_{k, 0}  \tag{39}\\
X_{k, 1} \\
\vdots \\
X_{k, k-1} \\
X_{k, k}
\end{array}\right]=\underbrace{\left[\begin{array}{cccc}
1 & - & b_{k, 0, e} & - \\
0 & - & b_{k, 1, e} & - \\
\vdots & & \vdots & \\
0 & - & b_{k, k-1, e} & - \\
1 & - & b_{k, k, e} & -
\end{array}\right]}_{\triangleq H_{k}}\left[\begin{array}{c}
X_{0,0} \\
\mid \\
Z_{e} \\
\mid
\end{array}\right]
$$

where the vector on the right hand side of (39) has first element $X_{0,0}$ followed by the random variables $\left\{Z_{e}: e \in E_{k}\right\}$ (indexed consistently with $H_{k}$ ). Our XOR 2D regular grid reconstruction problem is to decode $X_{0,0}$ from the observations $\left(X_{k, 0}, \ldots, X_{k, k}\right)$ with minimum probability of error. Note that we can apply a row operation to $H_{k}$ that replaces the last row of $H_{k}$ with the sum of the first and last rows of $H_{k}$ to get the binary matrix $H_{k}^{\prime} \in \mathbb{F}_{2}^{(k+1) \times\left(\left|E_{k}\right|+1\right)}$, and correspondingly, we can replace $X_{k, k}$ with $X_{k, 0}+X_{k, k}$ in (39) to get the "equivalent" formulation:

$$
\left[\begin{array}{c}
X_{k, 0}  \tag{40}\\
X_{k, 1} \\
\vdots \\
X_{k, k-1} \\
X_{k, 0}+X_{k, k}
\end{array}\right]=H_{k}^{\prime}\left[\begin{array}{c}
X_{0,0} \\
\mid \\
Z_{e} \\
\mid
\end{array}\right]
$$

for every $k$. Indeed, since we only perform invertible operations to obtain (40) from (39), the minimum probability of error for ML decoding $X_{0,0}$ from the observations ( $X_{k, 0}, \ldots, X_{k, k}$ ) under the model (39) is equal to the minimum probability of error for ML decoding $X_{0,0}$ from the observations ( $X_{k, 0}, \ldots, X_{k, k-1}, X_{k, 0}+X_{k, k}$ ) under the model (40). Furthermore, since $H_{k}^{\prime}$ is of the form (28), the equivalent XOR 2D regular grid reconstruction problem in (40) is exactly of the form of the inference problem in (30).

We next transform the XOR 2D regular grid reconstruction problem in (39), or equivalently, (40), into a coding problem. By Lemma 2, the inference problem in (40) is "equivalent" to a coupled coding problem analogous to (29). To describe this coupled coding problem, consider the linear code defined by the parity check matrix $H_{k}^{\prime}$ :

$$
\mathcal{C}_{k} \triangleq\left\{w \in \mathbb{F}_{2}^{\left|E_{k}\right|+1}: H_{k}^{\prime} w=\mathbf{0}\right\}=\left\{w \in \mathbb{F}_{2}^{\left|E_{k}\right|+1}: H_{k} w=\mathbf{0}\right\}
$$

where the second equality shows that the parity check matrix $H_{k}$ also generates $\mathcal{C}_{k}$ (because row operations do not change the nullspace of a matrix). As required by the coding problem, this linear code contains a codeword of the form $\left[1 w_{2}^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathcal{C}_{k}$ for some $w_{2} \in \mathbb{F}_{2}^{\left|E_{k}\right|}$. To prove this, notice that such a codeword exists if and only if the first column $[10 \cdots 0]^{\mathrm{T}}$ of $H_{k}^{\prime}$ is in the span of the remaining columns of $H_{k}^{\prime}$. Assume for the sake of contradiction that such a codeword does not exist. Then, we can decode $X_{0,0}$ in the setting of (40) with zero probability of error, because the observation vector on the left hand side of (40) is in the span of the second to last columns of $H_{k}^{\prime}$ if and only if $X_{0,0}=0 .{ }^{9}$ This leads to a contradiction since it is clear that we cannot decode the root bit with

[^6]zero probability of error in the XOR 2D regular grid. Hence, a codeword of the form $\left[\begin{array}{ll}1 & w_{2}^{\mathrm{T}}\end{array}\right]^{\mathrm{T}} \in \mathcal{C}_{k}$ for some $w_{2} \in \mathbb{F}_{2}^{\left|E_{k}\right|}$ always exists. Next, we let $W_{k}=\left[X_{0,0}-W_{k, e}-\right]^{\mathrm{T}} \in \mathcal{C}_{k}$ be a codeword that is drawn uniformly from $\mathcal{C}_{k}$, where the first element of $W_{k}$ is $X_{0,0}$ and the remaining elements of $W_{k}$ are $\left\{W_{k, e}: e \in E_{k}\right\}$. In the coupled coding problem, we observe $W_{k}$ through the additive noise channel model:

$$
Y_{k} \triangleq\left[\begin{array}{c}
Y_{0,0}^{k}  \tag{41}\\
\mid \\
Y_{k, e} \\
\mid
\end{array}\right]=W_{k}+\left[\begin{array}{c}
Z_{0,0}^{k} \\
\mid \\
Z_{e} \\
\mid
\end{array}\right]
$$

where $\left\{Z_{e}: e \in E_{k}\right\}$ are the BSC random variables that are independent of $W_{k}, Z_{0,0}^{k}$ is a completely independent Bernoulli $\left(\frac{1}{2}\right)$ random variable, $Y_{0,0}^{k}=X_{0,0} \oplus Z_{0,0}^{k}$, and $Y_{k, e}=W_{k, e} \oplus Z_{e}$ for $e \in E_{k}$. Our goal is to decode the first bit of the codeword, $X_{0,0}$, with minimum probability of error from the observation $Y_{k}$. Since we have coupled the coding problem (41) and the inference problem (40) according to the coupling in part 2 of Lemma 2, part 3 of Lemma 2 shows that $\left(X_{k, 0}, \ldots, X_{k, k-1}, X_{k, 0}+X_{k, k}\right)$, or equivalently:

$$
\left[\begin{array}{c}
X_{k, 0}  \tag{42}\\
\vdots \\
X_{k, k}
\end{array}\right]=\left[\begin{array}{c}
\sum_{e \in E_{k}} b_{k, 0, e} Y_{k, e} \\
\vdots \\
\sum_{e \in E_{k}} b_{k, k, e} Y_{k, e}
\end{array}\right]
$$

is a sufficient statistic of $Y_{k}$ for performing inference about $X_{0,0}$ in the coding problem (41). Hence, the ML decoder for $X_{0,0}$ based on the sufficient statistic ( $X_{k, 0}, \ldots, X_{k, k-1}, X_{k, 0}+X_{k, k}$ ) (without loss of generality), which achieves the minimum probability of error in the coding problem (41), makes an error if and only if the ML decision rule for $X_{0,0}$ based on $\left(X_{k, 0}, \ldots, X_{k, k-1}, X_{k, 0}+X_{k, k}\right)$, which achieves the minimum probability of error in the inference problem (40), makes an error. Therefore, as shown in part 1 of Lemma 2, the minimum probabilities of error in the XOR 2D regular grid reconstruction problem (39) and the coding problem (41) are equal, and it suffices to analyze the coding problem (41).

In the coding problem (41), we observe the codeword $W_{k}$ after passing it through memoryless BSCs. We now establish a "cleaner" model where $W_{k}$ is passed through memoryless BECs. Recall that each $\operatorname{BSC}(\delta)$ copies its input bit with probability $1-2 \delta$ and generates an independent Bernoulli $\left(\frac{1}{2}\right)$ output bit with probability $2 \delta$ (as shown in (18) in section III), i.e. for any $e \in E_{k}$, instead of setting $Z_{e} \sim \operatorname{Bernoulli}(\delta)$, we can generate $Z_{e}$ as follows:

$$
Z_{e}=\left\{\begin{array}{cl}
0 & , \quad \text { with probability } 1-2 \delta \\
\text { Bernoulli }\left(\frac{1}{2}\right) & , \quad \text { with probability } 2 \delta
\end{array}\right.
$$

where Bernoulli $\left(\frac{1}{2}\right)$ denotes an independent uniform bit. Suppose we know which BSCs among $\left\{Z_{e}: e \in E_{k}\right\}$ generate independent bits in (41). Then, we can perceive each BSC in $\left\{Z_{e}: e \in E_{k}\right\}$ as an independent binary erasure channel (BEC) with erasure probability $2 \delta$, denoted $\operatorname{BEC}(2 \delta)$, which erases its input with probability $2 \delta$ and produces the erasure symbol e if and only if the corresponding $\operatorname{BSC}(\delta)$ generates an independent bit, and copies its input with probability $1-2 \delta$ otherwise. (Note that the BSC defined by $Z_{0,0}^{k}$ corresponds to a BEC(1) which always erases its input.) Consider observing the codeword $W_{k}$ under this BEC model, where $X_{0,0}$ is erased almost surely, and the remaining bits of $W_{k}$ are erased independently with probability $2 \delta$, i.e. we observe $Y_{k}^{\prime}=\left[\mathrm{e}-Y_{k, e}^{\prime}-\right]^{\mathrm{T}} \in\{0,1, \mathrm{e}\}^{\left|E_{k}\right|+1}$, where the first entry corresponds to the erased value of $X_{0,0}$, and for every $e \in E_{k}, Y_{k, e}^{\prime}=W_{k, e}$ with probability $1-2 \delta$ and $Y_{k, e}^{\prime}=\mathrm{e}$ with probability $2 \delta$. Clearly, we can obtain $Y_{k}$ from $Y_{k}^{\prime}$ by replacing every instance of e in $Y_{k}^{\prime}$ with an independent Bernoulli $\left(\frac{1}{2}\right)$ bit. Since the BECs reveal additional information about which BSCs generate independent bits, the minimum probability of error in ML decoding $X_{0,0}$ based on $Y_{k}^{\prime}$ under the BEC model lower bounds the minimum probability of error in ML decoding $X_{0,0}$ based on $Y_{k}$ under the BSC model (41). ${ }^{10}$ In the rest of the proof, we establish conditions under which the minimum probability of error for the BEC model is $\frac{1}{2}$, and then show as a consequence that the minimum probability of error in the XOR 2D regular grid reconstruction problem in (39) tends to $\frac{1}{2}$ as $k \rightarrow \infty$.

[^7]Let $I_{k} \subseteq E_{k}$ denote the set of indices where the corresponding elements of $W_{k}$ are not erased in the BEC model:

$$
I_{k} \triangleq\left\{e \in E_{k}: Y_{k, e}^{\prime}=W_{k, e}\right\} .
$$

The ensuing lemma is a standard exercise in coding theory which shows that the ML decoder for $X_{0,0}$ only fails under the BEC model when a special codeword exists in $\mathcal{C}_{k}$; see the discussion in [48, Section 3.2].

Lemma 3 (Bit-wise ML Decoding [48, Section 3.2]). Suppose we condition on some realization of $Y_{k}^{\prime}$ (in the BEC model), which determines a corresponding realization of the set of indices $I_{k}$. Then, the ML decoder for $X_{0,0}$ based on $Y_{k}^{\prime}$ (with codomain $\mathbb{F}_{2}$ ) makes an error with probability $\frac{1}{2}$ if and only if there exists a codeword $w \in \mathcal{C}_{k}$ with first element $w_{1}=1$ and $w_{e}=0$ for all $e \in I_{k}$.

We next illustrate that such a special codeword exists whenever two particular erasures occur. Let $e_{1} \in E_{k}$ and $e_{2} \in E_{k}$ denote the edges $\left(X_{k-1,0}, X_{k, 0}\right)$ and $\left(X_{k-1, k-1}, X_{k, k}\right)$ in the 2D regular grid, respectively. Consider the vector $\omega^{k} \in \mathbb{F}_{2}^{\left|E_{k}\right|+1}$ such that $\omega_{1}^{k}=1$ (i.e. the first bit is 1 ), $\omega_{e_{1}}^{k}=\omega_{e_{2}}^{k}=1$, and all other elements of $\omega^{k}$ are 0 . Then, $\omega^{k} \in \mathcal{C}_{k}$ because:

$$
\begin{aligned}
H_{k} \omega^{k} & =\left[\begin{array}{cccc}
1 & - & b_{k, 0, e} & - \\
0 & - & b_{k, 1, e} & - \\
\vdots & & \vdots & \\
0 & - & b_{k, k-1, e} & - \\
1 & - & b_{k, k, e} & -
\end{array}\right] \omega^{k} \\
& =\left[\begin{array}{c}
1 \oplus b_{k, 0, e_{1}} \oplus b_{k, 0, e_{2}} \\
b_{k, 1, e_{1}} \oplus b_{k, 1, e_{2}} \\
\vdots \\
b_{k, k-1, e_{1}} \oplus b_{k, k-1, e_{2}} \\
1 \oplus b_{k, k, e_{1}} \oplus b_{k, k, e_{2}}
\end{array}\right] \\
& =\mathbf{0}
\end{aligned}
$$

where we use the facts that $b_{k, 0, e_{1}}=1, b_{k, 0, e_{2}}=0, b_{k, k, e_{1}}=0, b_{k, k, e_{2}}=1$, and for any $j \in\{1, \ldots, k-1\}$, $b_{k, j, e_{1}}=0$ and $b_{k, j, e_{2}}=0$. (Note that the value of $b_{k, j, e_{i}}$ for $i \in\{0,1\}$ and $j \in[k+1]$ is determined by checking the dependence of vertex $X_{k, j}$ on the variable $Z_{e_{i}}$ in (37), which is straightforward because $e_{i}$ is an edge between the last two layers at the side of the 2D regular grid up to level $k$ ). Since $\omega^{k}$ has two 1 's at the indices $e_{1}$ and $e_{2}$ (besides the first bit), if the BECs corresponding to the indices $e_{1}$ and $e_{2}$ erase their inputs, i.e. $e_{1}, e_{2} \notin I_{k}$, then $\omega^{k} \in \mathcal{C}_{k}$ satisfies the conditions of Lemma 3 and the ML decoder for $X_{0,0}$ based on $Y_{k}^{\prime}$ under the BEC model makes an error with probability $\frac{1}{2}$. Hence, we define the event:

$$
\begin{aligned}
B_{k} & \triangleq\left\{Y_{k, e_{1}}^{\prime}=Y_{k, e_{2}}^{\prime}=\mathrm{e}\right\} \\
& =\left\{\text { BECs corresponding to edges } e_{1} \in E_{k} \text { and } e_{2} \in E_{k} \text { erase their inputs }\right\} \\
& =\left\{\text { BSCs corresponding to edges } e_{1} \in E_{k} \text { and } e_{2} \in E_{k} \text { generate independent bits }\right\} .
\end{aligned}
$$

As the ML decoder for $X_{0,0}$ based on $Y_{k}^{\prime}$ under the BEC model makes an error with probability $\frac{1}{2}$ conditioned on $B_{k}$, we must have:

$$
P_{Y_{k}^{\prime} \mid X_{0,0}}\left(y^{\prime} \mid 0\right)=P_{Y_{k}^{\prime} \mid X_{0,0}}\left(y^{\prime} \mid 1\right)
$$

for all realizations $y^{\prime} \in\{0,1, \mathrm{e}\}^{\left|E_{k}\right|+1}$ of $Y_{k}^{\prime}$ such that $B_{k}$ occurs, i.e. $y_{1}^{\prime}=y_{e_{1}}^{\prime}=y_{e_{2}}^{\prime}=\mathrm{e}$. This implies that $Y_{k}^{\prime}$ is conditionally independent of $X_{0,0}$ given $B_{k}$ (where we also use the fact that $X_{0,0}$ is independent of $B_{k}$ ). Furthermore, it is straightforward to verify that $Y_{k}$ is also conditionally independent of $X_{0,0}$ given $B_{k}$, because $Y_{k}$ can be obtained from $Y_{k}^{\prime}$ by replacing e's with completely independent Bernoulli $\left(\frac{1}{2}\right)$ bits. Thus, since (42) shows that $X_{k}$ is a deterministic function of $Y_{k}, X_{k}$ is conditionally independent of $X_{0,0}$ given $B_{k}$.

To finish the proof, notice that $\mathbb{P}\left(B_{k}\right)=(2 \delta)^{2}$ for every $k$, and the events $\left\{B_{k}: k=2^{m}, m \in \mathbb{N} \backslash\{0\}\right\}$ are mutually independent because the BSCs in the 2D regular grid are all independent. So, infinitely many of the events $\left\{B_{k}: k=2^{m}, m \in \mathbb{N} \backslash\{0\}\right\}$ occur almost surely by the second Borel-Cantelli lemma. Let us define:

$$
\forall n \in \mathbb{N} \backslash\{0\}, \quad A_{n} \triangleq \bigcup_{m=1}^{n} B_{2^{m}}
$$



Fig. 2. A high-level schematic of the proposed martingale-based approach for establishing the impossibility of broadcasting on 2D regular grids with NAND processing functions.
where the continuity of the underlying probability measure $\mathbb{P}$ yields $\lim _{n \rightarrow \infty} \mathbb{P}\left(A_{n}\right)=1$. Then, since $X_{k}$ is conditionally independent of $X_{0,0}$ given $B_{k}$, and $X_{r}$ is conditionally independent of $X_{0,0}$ and $B_{k}$ given $X_{k}$ for any $r>k$, we have that $X_{2^{m}}$ is conditionally independent of $X_{0,0}$ given $A_{m}$ for every $m \in \mathbb{N} \backslash\{0\}$. Hence, we obtain:

$$
\forall m \in \mathbb{N} \backslash\{0\}, \quad \mathbb{P}\left(h_{\mathrm{ML}}^{2^{m}}\left(X_{2^{m}}\right) \neq X_{0,0} \mid A_{m}\right)=\frac{1}{2}
$$

where $h_{\mathrm{ML}}^{k}: \mathbb{F}_{2}^{k+1} \rightarrow \mathbb{F}_{2}$ denotes the ML decoder for $X_{0,0}$ based on $X_{k}$ for the XOR 2D regular grid reconstruction problem in (39). Finally, observe that:

$$
\begin{aligned}
\lim _{m \rightarrow \infty} \mathbb{P}\left(h_{\mathrm{ML}}^{2^{m}}\left(X_{2^{m}}\right) \neq X_{0,0}\right)= & \lim _{m \rightarrow \infty} \mathbb{P}\left(h_{\mathrm{ML}}^{2^{m}}\left(X_{2^{m}}\right) \neq X_{0,0} \mid A_{m}\right) \mathbb{P}\left(A_{m}\right) \\
& +\mathbb{P}\left(h_{\mathrm{ML}}^{2^{m}}\left(X_{2^{m}}\right) \neq X_{0,0} \mid A_{m}^{c}\right) \mathbb{P}\left(A_{m}^{c}\right) \\
= & \lim _{m \rightarrow \infty} \mathbb{P}\left(h_{\mathrm{ML}}^{2^{m}}\left(X_{2^{m}}\right) \neq X_{0,0} \mid A_{m}\right) \\
= & \frac{1}{2}
\end{aligned}
$$

This completes the proof since the above condition establishes (5).

## V. Martingale Approach for 2D Regular Grid with NAND Processing Functions

Finally, we consider the 2D regular grid where all Boolean processing functions with two inputs are the NAND rule, and all Boolean processing functions with one input are the identity rule, i.e. $f_{2}\left(x_{1}, x_{2}\right)=\neg\left(x_{1} \wedge x_{2}\right)$ and $f_{1}(x)=x$, where $\neg$ denotes the logical NOT operation. In this section, we will illustrate a promising program for proving Conjecture 1 (which is generalizable to other Boolean processing functions). In particular, we will establish Theorem 3, derive results required to generate the accompanying numerical evidence in Table I, and in the process, present several peripheral results for completeness.

To guide the readers, we briefly outline the ensuing subsections. In subsection V-A, we will describe the Markovian coupling setup of subsection II-B, and then show that the existence of certain structured supermartingales implies Conjecture 1. In subsection V-B, inspired by [39], we will introduce counting forms and elucidate how they define the structured superharmonic potential functions that produce the aforementioned supermartingales. Then, to efficiently test the desired structural properties of counting forms, we will develop cyclic evaluation of counting forms and derive corresponding graph theoretic characterizations in subsection V-C. With these required pieces in place, we will prove Theorem 3 in subsection V-D. Lastly, we will transform the graph theoretic tests for the desired superharmonic potential functions in subsection V-C into simple LPs in subsection V-E. These LPs are then solved to generate Table I. This chain of ideas is illustrated in Figure 2 so that readers may refer back to it as they proceed through this section.

## A. Existence of Supermartingale

As mentioned in subsection II-B, similar to our proof of Theorem 1 in section III, we begin by constructing a Markovian coupling. Let $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ denote versions of the Markov chain $\left\{X_{k}: k \in\right.$ $\mathbb{N}\}$ initialized at $X_{0}^{+}=1$ and $X_{0}^{-}=0$, respectively. Moreover, define the coupled 2D grid variables $\left\{Y_{k, j}=\right.$ $\left.\left(X_{k, j}^{-}, X_{k, j}^{+}\right): k \in \mathbb{N}, j \in[k+1]\right\}$, which yield the Markovian coupling $\left\{Y_{k}=\left(Y_{k, 0}, \ldots, Y_{k, k}\right): k \in \mathbb{N}\right\}$. Recall from (18) that each edge $\operatorname{BSC}(\delta)$ either copies its input bit with probability $1-2 \delta$, or generates an independent Bernoulli $\left(\frac{1}{2}\right)$ output bit with probability $2 \delta$. As before, the Markovian coupling $\left\{Y_{k}: k \in \mathbb{N}\right\}$ "runs" its "marginal" Markov chains $\left\{X_{k}^{+}: k \in \mathbb{N}\right\}$ and $\left\{X_{k}^{-}: k \in \mathbb{N}\right\}$ on a common underlying 2 D regular grid so that along any edge BSC, either both inputs are copied with probability $1-2 \delta$, or a shared independent Bernoulli $\left(\frac{1}{2}\right)$ output bit is produced with probability $2 \delta$. We will analyze the Markov chain $\left\{Y_{k}: k \in \mathbb{N}\right\}$ in the remainder of this section.

To simplify our analysis, we will again keep track of $\left\{Y_{k}: k \in \mathbb{N}\right\}$ in a single coupled 2D regular grid. The edge configuration of the underlying graph of the coupled 2D regular grid is exactly as described in subsection I-C. The vertices of the coupled 2D regular grid are indexed by the coupled 2D regular grid variables $\left\{Y_{k, j} \in \mathcal{Y}: k \in\right.$ $\mathbb{N}, j \in[k+1]\}$, where we simplify the alphabet set of these variables from $\{(0,0),(0,1),(1,0),(1,1)\}$ to:

$$
\begin{equation*}
\mathcal{Y} \triangleq\{0,1, u\} \tag{43}
\end{equation*}
$$

such that for any $k \in \mathbb{N}$ and any $j \in[k+1], Y_{k, j}=0$ if and only if $X_{k, j}^{-}=X_{k, j}^{+}=0, Y_{k, j}=1$ if and only if $X_{k, j}^{-}=X_{k, j}^{+}=1$, and $Y_{k, j}=u$ if and only if $X_{k, j}^{-} \neq X_{k, j}^{+}{ }^{11}$ So, $Y_{k, j}=u$ represents that the variables $X_{k, j}^{-}$ and $X_{k, j}^{+}$are uncoupled; in particular, we do not distinguish between the two possible values the pair $\left(X_{k, j}^{-}, X_{k, j}^{+}\right)$ might have. Furthermore, much like (20), each shared edge $\operatorname{BSC}(\delta)$ of the coupled 2D regular grid is described by a row stochastic matrix, or channel, on the alphabet $\mathcal{Y}$ :

$$
W=\begin{gather*}
 \tag{44}\\
0 \\
u \\
1
\end{gather*}\left[\begin{array}{ccc}
0 & u & 1 \\
1-\delta & 0 & \delta \\
\delta & 1-2 \delta & \delta \\
\delta & 0 & 1-\delta
\end{array}\right]
$$

and $W$ appropriately captures the aforementioned Markovian coupling. Lastly, the NAND rule can be equivalently described on the alphabet $\mathcal{Y}$ as: ${ }^{12}$

| $y_{1}$ | $y_{2}$ | $\neg\left(y_{1} \wedge y_{2}\right)$ |
| :---: | :---: | :---: |
| 0 | $\star$ | 1 |
| $u$ | $u$ | $u$ |
| $u$ | 1 | $u$ |
| 1 | 1 | 0 |

where $\star$ denotes any letter in $\mathcal{Y}$ as before, and the symmetry of the NAND rule covers all other possible input combinations. This coupled 2D regular grid model completely characterizes the Markov chain $\left\{Y_{k}: k \in \mathbb{N}\right\}$, which starts at $Y_{0}=u$ almost surely.

For every $k \in \mathbb{N}$, define the random variable:

$$
\begin{equation*}
N_{k} \triangleq \sum_{j=0}^{k} \mathbb{1}\left\{Y_{k, j}=u\right\} \tag{46}
\end{equation*}
$$

which gives the number of $u$ 's at level $k$ in the coupled 2 D regular grid, where $\mathbb{1}\{\cdot\}$ denotes the indicator function which equals 1 if its input proposition is true, and 0 otherwise. The next lemma illustrates that the almost sure convergence of $\left\{N_{k}: k \in \mathbb{N}\right\}$ to zero implies Conjecture 1.

Lemma 4 (Vanishing $u$ 's Condition). For any noise level $\delta \in\left(0, \frac{1}{2}\right)$, if we have $\lim _{k \rightarrow \infty} N_{k}=0$ almost surely, then Conjecture 1 is true, i.e. $\lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}}=0$.

Proof. Define the event:

$$
A \triangleq\left\{\exists k \in \mathbb{N}, \forall j \in[k+1], \quad Y_{k, j} \in\{0,1\}\right\}
$$

[^8]$$
=\{\exists k \in \mathbb{N}, \text { there are no } u \text { 's in level } k \text { of the coupled } 2 \mathrm{D} \text { regular grid }\}
$$
and observe that, as shown at the outset of the proof of Theorem 1 in section III, we have:
$$
\lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}} \leq 1-\lim _{k \rightarrow \infty} \mathbb{P}\left(X_{k}^{+}=X_{k}^{-}\right)=1-\mathbb{P}(A)
$$
where we use (44), (45), and the continuity of probability measures. Therefore, similar to section III, it suffices to prove that if $\mathbb{P}\left(\lim _{k \rightarrow \infty} N_{k}=0\right)=1$, then $\mathbb{P}(A)=1$. To establish this implication, we next show that $\left\{\lim _{k \rightarrow \infty} N_{k}=0\right\}=A$.

Indeed, suppose $\lim _{k \rightarrow \infty} N_{k}=0$. Then, there exists $K \in \mathbb{N}$ such that for all $k \in \mathbb{N} \backslash[K+1], N_{k} \leq \frac{1}{2}$. However, since $N_{m} \in \mathbb{N}$ for all $m \in \mathbb{N}$, we must have $N_{k}=0$ for all $k \in \mathbb{N} \backslash[K+1]$. This implies that $A$ occurs. Conversely, suppose $A$ occurs, i.e. there exists a level $k \in \mathbb{N}$ with no $u$ 's. Then, $N_{k}=0$, and due to (44) and (45), $N_{m}=0$ for all $m \geq k$. Hence, $\lim _{k \rightarrow \infty} N_{k}=0$. This completes the proof.

Due to Lemma 4, we now focus our attention on establishing the almost sure convergence of $\left\{N_{k}: k \in \mathbb{N}\right\}$ to zero. A general approach to establishing such almost sure convergence of random variables is via Doob's martingale convergence theorem, cf. [51, Chapter V, Theorem 4.1]. However, since we do not have a martingale in our problem, we need to construct one from our Markov chain $\left\{Y_{k}: k \in \mathbb{N}\right\}$. It is well-known in probability theory that ergodicity and related properties of discrete-time time-homogeneous Markov chains with countable state spaces can be analyzed using tools from the intimately related fields of martingale theory, Lyapunov theory, and potential theory (see e.g. [52, Chapter 5]). For example, Foster's theorem, cf. [52, Theorem 1.1, Chapter 5], characterizes the positive recurrence of Markov chains via the existence of Lyapunov functions with certain properties. It is also closely related to martingale convergence based criteria for recurrence of Markov chains. Indeed, such martingale arguments typically construct a martingale from the Markov chain under consideration. The canonical approach of doing this is to apply a harmonic function, which is an eigen-function of the Markov chain's conditional expectation operator with eigenvalue 1 , to the random variables defining the Markov chain-this is a specialization of the so called Dynkin martingale (or Lévy's martingale). Likewise, a supermartingale can be constructed by applying a superharmonic function to the Markov chain. The study of (super)harmonic functions in classical analysis is known as potential theory, and variants of harmonic functions turn out to be precisely the desired Lyapunov functions of Foster's theorem.

The ensuing conjecture presents a particularly useful family of superharmonic functions with special properties that can be used to construct a supermartingale from $\left\{Y_{k}: k \in \mathbb{N}\right\}$.

Conjecture 2 (Existence of Supermartingale). Let $\mathcal{Y}^{*}=\cup_{k \in \mathbb{N} \backslash\{0\}} \mathcal{Y}^{k}$ denote the set of all non-empty finite-length strings with letters in $\mathcal{Y}$, and as in the proof of Theorem 1 in section III, let $\left\{\mathcal{F}_{k}: k \in \mathbb{N}\right\}$ denote a filtration such that each $\mathcal{F}_{k}$ is the $\sigma$-algebra generated by the random variables $\left(Y_{0}, \ldots, Y_{k}\right)$ and all the BSCs before level $k$ (where we include all events determining whether these BSCs are copies, and all events determining the independent bits they produce) for $k \in \mathbb{N}$. Then, there exists a parametrized family of Borel measurable superharmonic functions $\left\{f_{\delta}: \mathcal{Y}^{*} \rightarrow \mathbb{R} \left\lvert\, \delta \in\left(0, \frac{1}{2}\right)\right.\right\}$ with parameter $\delta$ such that the discrete-time stochastic process $\left\{f_{\delta}\left(Y_{k}\right): k \in \mathbb{N}\right\}$ satisfies the following properties for every $\delta \in\left(0, \frac{1}{2}\right)$ :

1) $\left\{f_{\delta}\left(Y_{k}\right): k \in \mathbb{N}\right\}$ is a supermartingale adapted to $\left\{\mathcal{F}_{k}: k \in \mathbb{N}\right\}$, which implies that:

$$
\begin{equation*}
\forall k \in \mathbb{N}, \mathbb{E}\left[f_{\delta}\left(Y_{k+1}\right) \mid \mathcal{F}_{k}\right]=\mathbb{E}\left[f_{\delta}\left(Y_{k+1}\right) \mid Y_{k}\right] \leq f_{\delta}\left(Y_{k}\right) \tag{47}
\end{equation*}
$$

since $\left\{Y_{k}: k \in \mathbb{N}\right\}$ forms a Markov chain.
2) There exists a constant $C=C(\delta)>0$ (which may depend on $\delta$ ) such that:

$$
\begin{equation*}
\forall k \in \mathbb{N}, \quad f_{\delta}\left(Y_{k}\right) \geq C N_{k} \text { almost surely } \tag{48}
\end{equation*}
$$

The aforementioned properties should be reminiscent of the conditions in Theorem 3. If existence of the supermartingale in Conjecture 2 can be rigorously established, then the impossibility of broadcasting on 2D regular grids with NAND processing functions follows as a consequence. The next proposition justifies this implication.

Proposition 4 (Supermartingale Sufficient Condition). If Conjecture 2 is true, then Conjecture 1 is true. Equivalently, if $\delta \in\left(0, \frac{1}{2}\right)$, and there exists a Borel measurable superharmonic function $f_{\delta}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ such that $\left\{f_{\delta}\left(Y_{k}\right): k \in \mathbb{N}\right\}$ is a supermartingale adapted to $\left\{\mathcal{F}_{k}: k \in \mathbb{N}\right\}$ and (48) holds for some constant $C=C(\delta)>0$, then broadcasting is impossible on the $2 D$ regular grid with NAND processing functions in the sense of (5).

Proof. Fix any noise level $\delta \in\left(0, \frac{1}{2}\right)$, and consider the supermartingale $\left\{f_{\delta}\left(Y_{k}\right): k \in \mathbb{N}\right\}$ (see property (47)). Letting $X^{-} \triangleq-\min \{X, 0\}$ denote the "negative part" of a random variable $X$, observe that:

$$
0=\sup _{k \in \mathbb{N}} \mathbb{E}\left[f_{\delta}\left(Y_{k}\right)^{-}\right]<+\infty
$$

because (48) implies that for all $k \in \mathbb{N}, f_{\delta}\left(Y_{k}\right) \geq 0$ almost surely since $N_{k} \geq 0$ almost surely. Hence, applying Doob's martingale convergence theorem, cf. [51, Chapter V, Theorem 4.1], we have that the limiting random variable:

$$
F_{\delta} \triangleq \lim _{k \rightarrow \infty} f_{\delta}\left(Y_{k}\right) \geq 0
$$

exists (almost surely) and is integrable, i.e. $\mathbb{E}\left[F_{\delta}\right]<+\infty$. Define the limiting random variable:

$$
N_{\infty} \triangleq \limsup _{k \rightarrow \infty} N_{k} \geq 0
$$

(almost surely). Then, taking limits in (48), we have that:

$$
F_{\delta} \geq C N_{\infty} \text { almost surely }
$$

which in turn produces:

$$
\mathbb{E}\left[F_{\delta}\right] \geq C \mathbb{E}\left[N_{\infty}\right]
$$

Thus, we get that $\mathbb{E}\left[N_{\infty}\right]<+\infty$. Now notice that it suffices to prove that $N_{\infty}=0$ almost surely, since we can then employ Lemma 4 to establish Conjecture 1, and hence, this proposition.

We next prove that $N_{\infty}=0$ almost surely. Observe that if $\mathbb{E}\left[N_{\infty}\right]=0$, then $N_{\infty}=0$ almost surely, because the expectation of a non-negative random variable is zero when the random variable itself is zero. Thus, we assume, for the sake of contradiction, that $\mathbb{E}\left[N_{\infty}\right]>0$. Using Markov's inequality and the definition of superior limits, we obtain that for all $t>0$ :

$$
\mathbb{P}\left(\exists K=K(t) \in \mathbb{N}, \forall k \in \mathbb{N} \backslash[K+1], N_{k} \leq 2 t\right) \geq \mathbb{P}\left(N_{\infty} \leq t\right) \geq 1-\frac{\mathbb{E}\left[N_{\infty}\right]}{t}
$$

Fix any (small) $\epsilon \in(0,1)$, and let $t=\mathbb{E}\left[N_{\infty}\right] / \epsilon$. Then, the previous inequality can be written as:

$$
\begin{equation*}
\mathbb{P}\left(\exists K=K(\epsilon) \in \mathbb{N}, \forall k \in \mathbb{N} \backslash[K+1], N_{k} \leq \frac{2 \mathbb{E}\left[N_{\infty}\right]}{\epsilon}\right) \geq 1-\epsilon \tag{49}
\end{equation*}
$$

Now, for every $k \in \mathbb{N} \backslash\{0\}$, define the stopping time (with respect to the filtration $\left\{\mathcal{F}_{k}: k \in \mathbb{N}\right\}$ ):

$$
T_{k} \triangleq \inf \left\{m \in \mathbb{N}: \exists m_{1}<\cdots<m_{k}=m, \forall i \in\{1, \ldots, k\}, N_{m_{i}} \leq \frac{2 \mathbb{E}\left[N_{\infty}\right]}{\epsilon}\right\}
$$

which denotes the level index of the $k$ th occasion when the number of $u$ 's is bounded by $2 \mathbb{E}\left[N_{\infty}\right] / \epsilon$. Note that $T_{k}=+\infty$ for outcomes in the sample space where there do not exist $k$ levels $m_{1}<\cdots<m_{k}$ such that $N_{m_{i}} \leq 2 \mathbb{E}\left[N_{\infty}\right] / \epsilon$ for all $i \in\{1, \ldots, k\}$. Moreover, for $i \in\{0,1\}$, define the corresponding sequences of events $\left\{A_{k, i}: k \in \mathbb{N} \backslash\{0\}\right\}$ as follows:

$$
A_{k, i} \triangleq\left\{T_{k}=+\infty\right\} \cup\left\{\begin{array}{l}
\text { every } \operatorname{BSC}(\delta) \text { on an outgoing edge starting at a vertex at } \\
\text { level } T_{k}+i<+\infty \text { that equals } u \text { outputs an independent bit }
\end{array}\right\}
$$

which contains all outcomes of the sample space where there are no $u$ 's at level $T_{k}+i$. Observe that for every $k \in \mathbb{N} \backslash\{0\}, A_{k, 0} \subseteq A_{k+1,0}$. Indeed, if $A_{k, 0}$ occurs, then either $T_{k}=+\infty$ or there are no $u$ 's at level $T_{k}+1$ due to (44) and (45). In the former case, we have $T_{k+1}=+\infty$, and in the latter case, there are no $u$ 's at level $T_{k+1}$ due to (44) and (45). So, $A_{k+1,0}$ must occur.

It is straightforward to verify that for every $k \in \mathbb{N} \backslash\{0\}$ :

$$
\begin{align*}
\mathbb{P}\left(A_{k+1,0} \mid A_{k, 0}^{c}\right) & \geq \mathbb{P}\left(A_{k+1,0} \cap\left\{T_{k+1}<+\infty\right\} \mid A_{k, 0}^{c}\right) \\
& =\mathbb{P}\left(A_{k+1,0} \mid\left\{T_{k+1}<+\infty\right\} \cap A_{k, 0}^{c}\right) \mathbb{P}\left(T_{k+1}<+\infty \mid A_{k, 0}^{c}\right) \\
& \geq(2 \delta)^{4 \mathbb{E}\left[N_{\infty}\right] / \epsilon} \mathbb{P}\left(T_{k+1}<+\infty \mid A_{k, 0}^{c}\right) \tag{50}
\end{align*}
$$

where to obtain (50), we argue that if $A_{k, 0}^{c}$ occurs (which means that $T_{k}$ is finite and the collection of outgoing BSCs from level $T_{k}$ output at least one $\left.u\right)$ and $T_{k+1}$ is finite, then $\mathbb{P}\left(A_{k+1,0} \mid\left\{T_{k+1}<+\infty\right\} \cap A_{k, 0}^{c}\right)$ is the probability that the BSCs on the pairs of outgoing edges of $u$ 's at level $T_{k+1}$ produce independent bits. Since there are at
most $2 \mathbb{E}\left[N_{\infty}\right] / \epsilon u$ 's at level $T_{k+1}$, and the outgoing BSCs from these $u$ 's are conditionally independent given $A_{k, 0}^{c}$ and $\left\{T_{k+1}<+\infty\right\}$, we obtain the bound in (50). Next, notice that for every $k \in \mathbb{N} \backslash\{0\}$, conditioned on $A_{k, 0}^{c}$ happening, we have the inclusion relation $A_{k, 1} \subseteq\left\{T_{k+1}<+\infty\right\}$. Indeed, if $A_{k, 0}^{c}$ occurs, then $T_{k}$ must be finite. Suppose $A_{k, 1}$ also occurs. Then, either there are no $u$ 's at level $T_{k}+1$, or there are $u$ 's at level $T_{k}+1$ whose outgoing BSCs generate independent bits. In both cases, there are no $u$ 's at level $T_{k}+2$ (due to (44) and (45)). Hence, $T_{k+1}$ must be finite. Using this conditional inclusion relation, continuing from (50), we have that for every $k \in \mathbb{N} \backslash\{0\}:{ }^{13}$

$$
\begin{align*}
\mathbb{P}\left(A_{k+1,0} \mid A_{k, 0}^{c}\right) & \geq(2 \delta)^{4 \mathbb{E}\left[N_{\infty}\right] / \epsilon} \mathbb{P}\left(A_{k, 1} \mid A_{k, 0}^{c}\right) \\
& \geq(2 \delta)^{4 \mathbb{E}\left[N_{\infty}\right] / \epsilon}(2 \delta)^{8 \mathbb{E}\left[N_{\infty}\right] / \epsilon} \\
& =(2 \delta)^{12 \mathbb{E}\left[N_{\infty}\right] / \epsilon} \tag{51}
\end{align*}
$$

where to obtain (51), we argue akin to (50) that if $A_{k, 0}^{c}$ occurs, then $\mathbb{P}\left(A_{k, 1} \mid A_{k, 0}^{c}\right)$ is the probability that the BSCs on the pairs of outgoing edges of $u$ 's at level $T_{k}+1$ produce independent bits. Since there are at most $4 \mathbb{E}\left[N_{\infty}\right] / \epsilon$ $u$ 's at level $T_{k}+1$ due to (44) and (45) (because there are at most $2 \mathbb{E}\left[N_{\infty}\right] / \epsilon u$ 's at level $T_{k}$ that can propagate to the next level via two outgoing edges each), and the outgoing BSCs from these $u$ 's are conditionally independent given $A_{k, 0}^{c}$, we obtain the bound in (51).

To conclude the proof, we will employ the "counterpart of the Borel-Cantelli lemma," cf. [53, Lemma 1]. To this end, notice using (51) that:

$$
\sum_{k=1}^{\infty} \mathbb{P}\left(A_{k+1,0} \mid A_{k, 0}^{c}\right) \geq \sum_{k=1}^{\infty}(2 \delta)^{12 \mathbb{E}\left[N_{\infty}\right] / \epsilon}=+\infty
$$

Then, since $\left\{A_{k, 0}: k \in \mathbb{N} \backslash\{0\}\right\}$ is a non-decreasing sequence of sets, the counterpart of the Borel-Cantelli lemma yields:

$$
\begin{equation*}
\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_{k, 0}\right)=1 \tag{52}
\end{equation*}
$$

(Note that if there exists some $k \in \mathbb{N} \backslash\{0\}$ such that $\mathbb{P}\left(A_{k, 0}^{c}\right)=0$ and $\mathbb{P}\left(A_{k+1,0} \mid A_{k, 0}^{c}\right)$ is not well-defined, then $\mathbb{P}\left(A_{k, 0}\right)=1$, and the above equality trivially holds.) On the other hand, (49) yields the following inequality:

$$
\begin{equation*}
\mathbb{P}\left(\forall k \in \mathbb{N} \backslash\{0\}, T_{k}<+\infty\right) \geq \mathbb{P}\left(\exists K=K(\epsilon) \in \mathbb{N}, \forall k \in \mathbb{N} \backslash[K+1], N_{k} \leq \frac{2 \mathbb{E}\left[N_{\infty}\right]}{\epsilon}\right) \geq 1-\epsilon \tag{53}
\end{equation*}
$$

Finally, using both (52) and (53), we get that:

$$
\mathbb{P}\left(N_{\infty}=0\right) \geq \mathbb{P}\left(\left(\bigcup_{k=1}^{\infty} A_{k, 0}\right) \cap\left(\bigcap_{k=1}^{\infty}\left\{T_{k}<+\infty\right\}\right)\right) \geq 1-\epsilon
$$

where the first inequality holds because if there exists a finite level $T_{k}$ such that all outgoing BSCs from this level do not output $u$ 's, then we must have $N_{T_{k}+1}=0$, and therefore, $N_{\infty}=0$ (due to (44) and (45)). Letting $\epsilon \rightarrow 0$ produces:

$$
\mathbb{P}\left(N_{\infty}=0\right)=1,
$$

which implies that $\mathbb{E}\left[N_{\infty}\right]=0$. This contradicts the assumption that $\mathbb{E}\left[N_{\infty}\right]>0$. Hence, $N_{\infty}=0$ almost surely, which completes the proof.

Although Proposition 4 shows that it suffices to prove Conjecture 2 in order to establish the impossibility of broadcasting on 2D regular grids with NAND processing functions, constructing the supermartingales in Conjecture 2 turns out to be nontrivial. Therefore, in the ensuing subsections, we propose an approach for constructing the structured supermartingales outlined in Conjecture 2, and bolster this proposal by demonstrating how to compute them efficiently.

[^9]
## B. Counting Forms and Potential Functions

In this subsection, we will explain an extended version of the potential function technique in [39] that offers an avenue to construct the superharmonic functions delineated in Conjecture 2. In particular, we will consider superharmonic potential functions composed of so called "counting forms." To this end, we begin with a simple motivating example. Consider the potential function $w: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ given by:

$$
\begin{equation*}
\forall y \in \mathcal{Y}^{*}, w(y)=\left[\text { number of }(u)^{\prime} \text { 's in } y\right]-\left[\text { number of }(01 u)^{\prime} \text { 's in } y\right] \tag{54}
\end{equation*}
$$

This function counts the number of $u$ 's that appear in an input string $y$ and subtracts the number of times the substring $(01 u)$ appears in $y$. So, for example, if the input string is $y=(0010 u u 01101 u 0)$, then $w(y)=3-1=2$. Since computing $w$ requires a count of sub-strings of length (at most) 3 , $w$ can only be meaningfully evaluated for strings of length at least 3 , and we say that $w$ has "rank" 3 . Potential functions obtained by such linear combinations of counts will be the subject of much of our discussion from hereon. The ensuing definition formally generalizes the aforementioned example.

Definition 2 (Basis Clauses and Counting Forms). For every finite string $v \in \mathcal{Y}^{*}$ with length $r \in \mathbb{N} \backslash\{0\}$, we define an associated basis clause $\{v\}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ via:

$$
\forall k \in \mathbb{N} \backslash\{0\}, \forall y=\left(y_{1} \cdots y_{k}\right) \in \mathcal{Y}^{k}, \quad\{v\}(y) \triangleq \begin{cases}\sum_{i=1}^{k-r+1} \mathbb{1}\left\{\left(y_{i} \cdots y_{i+r-1}\right)=v\right\}, & r \leq k \\ 0, & r>k\end{cases}
$$

which is a map that counts the number of sub-strings $v$ that can be found in the input string $y$. (Note that when explicitly writing out the letters of a string $v$, we will often use parentheses for clarity, e.g. $v=(01 u) \in \mathcal{Y}^{3}$. On the other hand, curly braces will be used to distinguish the string $v$ from its associated basis clause $\{v\}$, e.g. $\{01 u\}$ denotes the basis clause associated with the string $(01 u)$. Moreover, throughout this paper, we will identify a basis clause $\{v\}$ with the string $v$.) The length of such a basis clause $\{v\}$ is called its rank, which we denote as $\operatorname{rank}(\{v\})=r$. For any finite set of basis clauses $\left\{v_{1}\right\}, \ldots,\left\{v_{m}\right\}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ and coefficients $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R}$ (with $m \in \mathbb{N} \backslash\{0\}$ ), the formal sum $w=\alpha_{1}\left\{v_{1}\right\}+\cdots+\alpha_{m}\left\{v_{m}\right\}$ is said to be a counting form, and $w: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ is a potential function defined by:

$$
\forall y \in \mathcal{Y}^{*}, w(y) \triangleq \sum_{j=1}^{m} \alpha_{j}\left\{v_{j}\right\}(y)
$$

(Note that we omit the curly braces notation for general counting forms since there is no cause for confusion between strings and counting forms.) The rank of a counting form $w$ is defined as the maximal rank of its basis clauses:

$$
\operatorname{rank}(w) \triangleq \max _{\substack{i \in\{1, \ldots, m\}: \\ \alpha_{i} \neq 0}} \operatorname{rank}\left(\left\{v_{i}\right\}\right)
$$

and we say that $w$ has pure rank if $\operatorname{rank}(w)=\operatorname{rank}\left(\left\{v_{i}\right\}\right)$ for every $i \in\{1, \ldots, m\}$ such that $\alpha_{i} \neq 0$. Furthermore, we define a counting form $w$ to be $u$-only if all the strings $v_{1}, \ldots, v_{m}$ defining its basis clauses contain a $u$, i.e. $w$ is not $u$-only if:

$$
\exists i \in\{1, \ldots, m\}, \quad v_{i} \in\{0,1\}^{\operatorname{rank}\left(\left\{v_{i}\right\}\right)}
$$

As another concrete illustration of the concepts in Definition 2, consider the $u$-only counting form:

$$
\begin{equation*}
w_{*} \triangleq 2\{u\}+\{u 1\}+\{1 u\}+\{u 10\}+\{01 u\}-2\{0 u 0\} \tag{55}
\end{equation*}
$$

This counting form has $\operatorname{rank}\left(w_{*}\right)=3$, and basis clauses $\{u\},\{u 1\},\{1 u\},\{u 10\},\{01 u\},\{0 u 0\}$ with coefficients $2,1,1,1,1,-2$, respectively. Moreover, the potential of the string $y=(01 u u 1 u 001 u 1)$ is $w_{*}(y)=2(4)+(2)+$ $(3)+(0)+(2)-2(0)=15$.

As mentioned earlier, we will restrict our search for superharmonic functions satisfying conditions (47) and (48) in Conjecture 2 to the class of $u$-only potential functions defined above. This restriction to $u$-only counting forms is intuitively sound, because our goal is to show that (48) is satisfied and the number of $u$ 's per layer of the coupled NAND 2D regular grid vanishes almost surely as the depth tends to infinity-see Lemma 4. Let us fix a rank $r \in \mathbb{N} \backslash\{0\}$, and only consider $u$-only counting forms with at most this rank. Moreover, define the set of finite strings $\mathcal{Y}_{r}^{*} \subset \cup_{k \geq r} \mathcal{Y}^{k}$ such that for any $y \in \mathcal{Y}_{r}^{*}$, the first and last $r$ letters of $y$ do not contain $u$ 's, or equivalently, $y=\left(y_{1} \cdots y_{k}\right) \in \mathcal{Y}_{r}^{*}$ with $k \geq r$ if and only if $y_{1}, \ldots, y_{r}, y_{k-r+1}, \ldots, y_{k} \in\{0,1\}$. Since the BSCs in
the coupled NAND 2D regular grid are independent, a simple second Borel-Cantelli lemma argument can be used to show that with probability 1 , there exists a level $t \in \mathbb{N}$ such that $Y_{t, 0}, \ldots, Y_{t, r-1}, Y_{t, t-r+1}, \ldots, Y_{t, t} \in\{0,1\}$ in the coupled 2D regular grid. (This is because $Y_{t, 0}, \ldots, Y_{t, r-1}, Y_{t, t-r+1}, \ldots, Y_{t, t} \in\{0,1\}$ if the BSCs above these vertices generate independent bits.) Define the event:

$$
\begin{equation*}
A \triangleq\left\{\exists t \in \mathbb{N}, \forall l \geq t, Y_{l, 0}, \ldots, Y_{l, r-1}, Y_{l, l-r+1}, \ldots, Y_{l, l} \in\{0,1\}\right\} \tag{56}
\end{equation*}
$$

Then, due to (44) and (45), we have that $\mathbb{P}(A)=1$, i.e. $A$ occurs almost surely. For convenience in our ensuing exposition, we will "condition the event $A$ " and make the simplifying assumption that our $u$-only counting forms are potential functions with domain $\mathcal{Y}_{r}^{*}$ (instead of $\mathcal{Y}^{*}$ ). We will rigorize what we mean by "conditioning on $A$ " during the proof of Theorem 3 in subsection V-D.

Therefore, we seek a $u$-only counting form $w_{\delta}=\alpha_{1}(\delta)\left\{v_{1}\right\}+\cdots+\alpha_{m}(\delta)\left\{v_{m}\right\}$ with $m \in \mathbb{N} \backslash\{0\}$ basis clauses $\left\{v_{1}\right\}, \ldots,\left\{v_{m}\right\}$ which contain $u$ 's, and corresponding $\delta$-dependent coefficients $\alpha_{1}(\delta), \ldots, \alpha_{m}(\delta) \in \mathbb{R} \backslash\{0\}$, respectively, such that for all $\delta \in\left(0, \frac{1}{2}\right)$, the potential function $w_{\delta}: \mathcal{Y}_{r}^{*} \rightarrow \mathbb{R}$ satisfies:

1) The supermartingale conditions:

$$
\begin{array}{r}
\forall k \in \mathbb{N}, \mathbb{E}\left[\left|w_{\delta}\left(Y_{k}\right)\right|\right]<+\infty \\
\forall k \geq r-1, \forall y \in \mathcal{Y}^{k+1} \cap \mathcal{Y}_{r}^{*}, \quad \mathbb{E}\left[w_{\delta}\left(Y_{k+1}\right) \mid Y_{k}=y\right] \leq w_{\delta}(y) . \tag{58}
\end{array}
$$

2) There exists a constant $C=C(\delta)>0$ such that:

$$
\begin{equation*}
\forall y \in \mathcal{Y}_{r}^{*}, w_{\delta}(y) \geq C\{u\}(y) \tag{59}
\end{equation*}
$$

where $\{u\}$ is a basis clause with rank 1.
The condition (57) holds trivially, because for all $k \in \mathbb{N}$ :

$$
\begin{equation*}
\mathbb{E}\left[\left|w_{\delta}\left(Y_{k}\right)\right|\right] \leq \sum_{i=1}^{m}\left|\alpha_{i}(\delta)\right| \mathbb{E}\left[\left|\left\{v_{i}\right\}\left(Y_{k}\right)\right|\right] \leq(k+1) \sum_{i=1}^{m}\left|\alpha_{i}(\delta)\right|<+\infty \tag{60}
\end{equation*}
$$

using the triangle inequality and the fact that $0 \leq\left\{v_{i}\right\}(y) \leq k+1$ for all $i \in\{1, \ldots, m\}$ and all $y \in \mathcal{Y}^{k+1}$. Moreover, it is intuitively clear that akin to the supermartingale depicted in (47), conditions (57) and (58) essentially define a supermartingale $\left\{w_{\delta}\left(Y_{k}\right): k \geq r-1\right\}$ "conditioned on $A$." Likewise, (59) is essentially equivalently to (48) "conditioned on $A$." Hence, by extending the argument in Proposition 4, the impossibility of broadcasting on the NAND 2D regular grid can be established from conditions (57), (58), and (59) (as we will explain in subsection V-D).

The discussion heretofore reveals that to prove the impossibility of broadcasting on the NAND 2D regular grid, it suffices to prove the existence of a $u$-only counting form $w_{\delta}$ (with rank at most $r$ ) such that (58) and (59) are satisfied. Naturally, one approach towards proving the existence of such a $w_{\delta}$ is to construct an explicit example. In the remainder of this section, we will make some partial progress on this important problem of constructing an example.

Both the constraints (58) and (59) require us to verify inequalities between certain potentials of strings in $\mathcal{Y}_{r}^{*}$. To state these constraints directly in terms of counting forms, we will transform them using the next two definitions. The first of these is an equivalent linear operator on counting forms that captures the action of the conditional expectation operators in (58).

Definition 3 (Conditional Expectation Operator). For any input basis clause $\{v\}$ with rank $s \in \mathbb{N} \backslash\{0\}$, the conditional expectation operator $\mathcal{E}$ outputs the following counting form which has pure rank $s+1$ :

$$
\mathcal{E}(\{v\}) \triangleq \sum_{z \in \mathcal{Y}^{s+1}} \mathbb{P}\left(\left(Y_{s+1,1}, \ldots, Y_{s+1, s}\right)=v \mid Y_{s}=z\right)\{z\}
$$

where the coefficients $\left\{\mathbb{P}\left(\left(Y_{s+1,1}, \ldots, Y_{s+1, s}\right)=v \mid Y_{s}=z\right): z \in \mathcal{Y}^{s+1}\right\}$ in the above formal sum of basis clauses are given by the Markov transition probabilities of the coupled NAND 2D regular grid (see (44) and (45)). Furthermore, for any input counting form $w=\alpha_{1}\left\{v_{1}\right\}+\cdots+\alpha_{m}\left\{v_{m}\right\}$ with basis clauses $\left\{v_{1}\right\}, \ldots,\left\{v_{m}\right\}$ and coefficients $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R} \backslash\{0\}$ (with $m \in \mathbb{N} \backslash\{0\}$ ), $\mathcal{E}$ outputs the following counting form:

$$
\mathcal{E}(w) \triangleq \alpha_{1} \mathcal{E}\left(\left\{v_{1}\right\}\right)+\cdots+\alpha_{m} \mathcal{E}\left(\left\{v_{m}\right\}\right)
$$

which has $\operatorname{rank}(\mathcal{E}(w))=\operatorname{rank}(w)+1$.

We note that the set of all counting forms is a vector space over $\mathbb{R}$, and the sets of all $u$-only counting forms and all counting forms with rank at most $r$ are linear subspaces of this larger vector space. Specifically, the intersection of these two linear subspaces, namely, the set of all $u$-only counting forms with rank at most $r$, is also a linear subspace. In particular, for any real scalar $\gamma \in \mathbb{R}$ and any pair of counting forms $w_{1}=\alpha_{1}\left\{v_{1}\right\}+\cdots+\alpha_{m}\left\{v_{m}\right\}$ and $w_{2}=\beta_{1}\left\{v_{1}\right\}+\cdots+\beta_{m}\left\{v_{m}\right\}$, which have a common set of basis clauses $\left\{v_{1}\right\}, \ldots,\left\{v_{m}\right\}$ without loss of generality, and coefficients $\alpha_{1}, \ldots, \alpha_{m}$ and $\beta_{1}, \ldots, \beta_{m}$, respectively, we define vector addition by $w_{1}+w_{2}=$ $\left(\alpha_{1}+\beta_{1}\right)\left\{v_{1}\right\}+\cdots+\left(\alpha_{m}+\beta_{m}\right)\left\{v_{m}\right\}$ and scalar multiplication by $\gamma w_{1}=\gamma \alpha_{1}\left\{v_{1}\right\}+\cdots+\gamma \alpha_{m}\left\{v_{m}\right\}$. It is straightforward to verify that $\operatorname{rank}\left(w_{1}+w_{2}\right) \leq \max \left\{\operatorname{rank}\left(w_{1}\right), \operatorname{rank}\left(w_{2}\right)\right\}$ and $\operatorname{rank}\left(\gamma w_{1}\right)=\operatorname{rank}\left(w_{1}\right)$ (for $\gamma \neq 0$ ). Moreover, we will use the notation $w_{1}-w_{2}$ to mean $w_{1}-w_{2}=w_{1}+(-1) w_{2}$ in the sequel.

In Definition 3, if the basis clause $\{v\}$ contains $u$ 's, then the coefficient $\mathbb{P}\left(\left(Y_{s+1,1}, \ldots, Y_{s+1, s}\right)=v \mid Y_{s}=z\right)=0$ when $z \in\{0,1\}^{s+1}$ (due to (44) and (45)). Thus, the conditional expectation operator $\mathcal{E}$ maps the vector space of $u$-only counting forms with rank at most $r$ to the vector space of $u$-only counting forms with rank at most $r+1$. The ensuing proposition illustrates that the action of the conditional expectations in (58) is equivalent to the action of $\mathcal{E}$ in Definition 3.

Proposition 5 (Equivalence of $\mathcal{E}$ ). Consider any u-only counting form $w_{\delta}$ with rank at most $r$. Then, for all $k \geq r-1$ and for every $y \in \mathcal{Y}^{k+1} \cap \mathcal{Y}_{r}^{*}$, we have:

$$
\left(\mathcal{E}\left(w_{\delta}\right)\right)(y)=\mathbb{E}\left[w_{\delta}\left(Y_{k+1}\right) \mid Y_{k}=y\right]
$$

Proof. First, consider any basis clause $\{v\}$ that contains $u$ 's and has $\operatorname{rank}(\{v\})=s \leq r$, and fix any $k \geq r-1$ and any $y=\left(y_{0} \cdots y_{k}\right) \in \mathcal{Y}^{k+1} \cap \mathcal{Y}_{r}^{*}$. Then, observe that:

$$
\begin{aligned}
\mathbb{E}\left[\{v\}\left(Y_{k+1}\right) \mid Y_{k}=y\right] & =\sum_{i=0}^{k-s+2} \mathbb{P}\left(\left(Y_{k+1, i}, \ldots, Y_{k+1, i+s-1}\right)=v \mid Y_{k}=y\right) \\
& =\sum_{i=1}^{k-s+1} \mathbb{P}\left(\left(Y_{k+1, i}, \ldots, Y_{k+1, i+s-1}\right)=v \mid Y_{k, i-1}=y_{i-1}, \ldots, Y_{k, i+s-1}=y_{i+s-1}\right) \\
& =\sum_{i=1}^{k-s+1} \mathbb{P}\left(\left(Y_{s+1,1}, \ldots, Y_{s+1, s}\right)=v \mid Y_{s}=\left(y_{i-1} \cdots y_{i+s-1}\right)\right) \\
& =\sum_{i=1}^{k-s+1} \sum_{z \in \mathcal{Y}^{s+1}} \mathbb{P}\left(\left(Y_{s+1,1}, \ldots, Y_{s+1, s}\right)=v \mid Y_{s}=z\right) \mathbb{1}\left\{\left(y_{i-1} \cdots y_{i+s-1}\right)=z\right\} \\
& =\sum_{z \in \mathcal{Y}^{s+1}} \mathbb{P}\left(\left(Y_{s+1,1}, \ldots, Y_{s+1, s}\right)=v \mid Y_{s}=z\right)\{z\}(y) \\
& =(\mathcal{E}(\{v\}))(y)
\end{aligned}
$$

where the first equality follows from Definition 2, the second equality holds because the coupled 2D grid variables $Y_{k+1, i}, \ldots, Y_{k+1, i+s-1}$ only depend on the variables $Y_{k, i-1}, \ldots, Y_{k, i+s-1}$ in the previous layer, and because $\mathbb{P}\left(\left(Y_{k+1,0}, \ldots, Y_{k+1, s-1}\right)=v \mid Y_{k, 0}=y_{0}, \ldots, Y_{k, s-1}=y_{s-1}\right)=0$ and $\mathbb{P}\left(\left(Y_{k+1, k-s+2}, \ldots, Y_{k+1, k+1}\right)=\right.$ $\left.v \mid Y_{k, k-s+1}=y_{k-s+1}, \ldots, Y_{k, k}=y_{k}\right)=0$ since $v$ contains $u$ 's and $y \in \mathcal{Y}_{r}^{*}$, the third equality holds because (44) and (45) determine the Markov transition kernels between all pairs of consecutive layers in the coupled NAND 2D regular grid, the fifth equality follows from Definition 2 (where we treat the string $z$ as a basis clause $\{z\}$ ), and the final equality follows from Definitions 2 and 3.

This implies that for all $k \geq r-1$ and for every $y=\left(y_{0} \cdots y_{k}\right) \in \mathcal{Y}^{k+1} \cap \mathcal{Y}_{r}^{*}$, we also have:

$$
\begin{aligned}
\mathbb{E}\left[w_{\delta}\left(Y_{k+1}\right) \mid Y_{k}=y\right] & =\sum_{i=1}^{m} \alpha_{i}(\delta) \mathbb{E}\left[\left\{v_{i}\right\}\left(Y_{k+1}\right) \mid Y_{k}=y\right] \\
& =\sum_{i=1}^{m} \alpha_{i}(\delta)\left(\mathcal{E}\left(\left\{v_{i}\right\}\right)\right)(y) \\
& =\left(\mathcal{E}\left(w_{\delta}\right)\right)(y)
\end{aligned}
$$

where we use Definitions 2 and 3, and we let $w_{\delta}=\alpha_{1}(\delta)\left\{v_{1}\right\}+\cdots+\alpha_{m}(\delta)\left\{v_{m}\right\}$ (without loss of generality) with $m \in \mathbb{N} \backslash\{0\}$ basis clauses $\left\{v_{1}\right\}, \ldots,\left\{v_{m}\right\}$ which contain $u$ 's, and corresponding $\delta$-dependent coefficients $\alpha_{1}(\delta), \ldots, \alpha_{m}(\delta) \in \mathbb{R}$, respectively. This completes the proof.

The second definition we require to transform the constraints (58) and (59) into inequalities in terms of counting forms is that of a partial order over counting forms.

Definition 4 (Equivalence Classes and Partial Order). Let $\mathbf{0}$ denote the zero counting form with no basis clauses and $\operatorname{rank}(\mathbf{0})=0$ (with abuse of notation), which outputs 0 on every input string. Define the equivalence relation $\simeq$ over $u$-only counting forms as follows. For any pair of $u$-only counting forms $w_{1}$ and $w_{2}$ with $\max \left\{\operatorname{rank}\left(w_{1}\right), \operatorname{rank}\left(w_{2}\right)\right\}$ $\leq r, w_{1} \simeq w_{2}$ if we have:

$$
\forall y \in \mathcal{Y}_{r}^{*}, \quad w_{1}(y)=w_{2}(y)
$$

Due to this equivalence relation, the aforementioned vector space of $u$-only counting forms with rank at most $r$ is actually a vector space of equivalence classes of such counting forms, where any specific $u$-only counting form $w$ with rank at most $r$ corresponds to the equivalence class $\{w+z: z$ is $u$-only, $\operatorname{rank}(z) \leq r, z \simeq \mathbf{0}\}$. Furthermore, we define the partial order $\succeq$ over $u$-only counting forms as follows. For any pair of $u$-only counting forms $w_{1}$ and $w_{2}$ with $\max \left\{\operatorname{rank}\left(w_{1}\right), \operatorname{rank}\left(w_{2}\right)\right\} \leq r, w_{1} \succeq w_{2}$ if we have:

$$
\forall y \in \mathcal{Y}_{r}^{*}, \quad w_{1}(y) \geq w_{2}(y)
$$

This makes the set of equivalence classes of $u$-only counting forms with rank at most $r$ a partially ordered vector space. Lastly, we say that a $u$-only counting form $w$ with rank at most $r$ is non-negative if $w \succeq \mathbf{0}$, and we will often write $w_{1} \succeq w_{2}$ equivalently as $w_{1}-w_{2} \succeq \mathbf{0}$.

We remark that Definition 4 illustrates that the rank of a $u$-only counting form is a property of the counting form itself, and not the corresponding equivalence class. For example, when $r \geq 2$, the basis clause $\{u\}$ has unit rank, and is equivalent to the following $u$-only counting forms with pure rank 2 :

$$
\begin{align*}
& \{u\} \simeq\{0 u\}+\{1 u\}+\{u u\},  \tag{61}\\
& \{u\} \simeq\{u 0\}+\{u 1\}+\{u u\} \tag{62}
\end{align*}
$$

In fact, given a $u$-only counting form $w_{1}$ with rank $s \leq r$, these equations exemplify a more general and rather useful trick for constructing an equivalent $u$-only counting form $w_{2}$ with pure rank $t$ such that $w_{1} \simeq w_{2}$ and $s \leq t \leq r$. The next lemma formally presents this trick.

Lemma 5 (Purification of Counting Forms). Consider any $u$-only counting form $w_{1}=\alpha_{1}\left\{v_{1}\right\}+\cdots+\alpha_{m}\left\{v_{m}\right\}$ with basis clauses $\left\{v_{1}\right\}, \ldots,\left\{v_{m}\right\}$ and coefficients $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R} \backslash\{0\}$ for some $m \in \mathbb{N} \backslash\{0\}$, and rank $s \leq r$. For any integer $t \in[s, r]$, construct the $u$-only counting forms:

$$
\begin{align*}
& w_{2}=\sum_{i=1}^{m} \alpha_{i} \sum_{z \in \mathcal{Y}^{t-\operatorname{rank}\left(\left\{v_{i}\right\}\right)}}\left\{v_{i}, z\right\},  \tag{63}\\
& w_{3}=\sum_{i=1}^{m} \alpha_{i} \sum_{z \in \mathcal{Y}^{t-\operatorname{rank}\left(\left\{v_{i}\right\}\right)}}\left\{z, v_{i}\right\}, \tag{64}
\end{align*}
$$

where $\left\{v_{i}, z\right\},\left\{z, v_{i}\right\} \in \mathcal{Y}^{t}$ denote the basis clauses obtained by concatenating the strings $v_{i} \in \mathcal{Y}^{\operatorname{rank}\left(\left\{v_{i}\right\}\right)}$ and $z \in \mathcal{Y}^{t-\operatorname{rank}\left(\left\{v_{i}\right\}\right)}$, and the inner summations equal $\left\{v_{i}\right\}$ when $\operatorname{rank}\left(\left\{v_{i}\right\}\right)=t$. Then, $w_{1} \simeq w_{2} \simeq w_{3}$, and $w_{2}, w_{3}$ have pure rank $t$.

Proof. It is obvious that $w_{2}, w_{3}$ have pure rank $t$. Moreover, the proof of $w_{1} \simeq w_{3}$ is the same as the proof of $w_{1} \simeq w_{2}$ by symmetry. So, we only need to establish that $w_{1} \simeq w_{2}$. To prove this, it suffices to show that for any basis clause $\left\{v_{i}\right\}$ of $w_{1}$ with $\operatorname{rank}\left(\left\{v_{i}\right\}\right)<t$ :

$$
\begin{equation*}
\left\{v_{i}\right\} \simeq \sum_{z \in \mathcal{Y}^{t-\operatorname{rank}\left(\left\{v_{i}\right\}\right)}}\left\{v_{i}, z\right\} \tag{65}
\end{equation*}
$$

To this end, consider any basis clause $\left\{v_{i}\right\}$ with $s^{\prime}=\operatorname{rank}\left(\left\{v_{i}\right\}\right)<t$. Then, for any $k \geq r$ and any string $y=\left(y_{1} \cdots y_{k}\right) \in \mathcal{Y}^{k} \cap \mathcal{Y}_{r}^{*}$, we have:

$$
\begin{aligned}
\sum_{z \in \mathcal{Y}^{t-s^{\prime}}}\left\{v_{i}, z\right\}(y) & =\sum_{z \in \mathcal{Y}^{t-s^{\prime}}} \sum_{j=1}^{k-t+1} \mathbb{1}\left\{\left(y_{j} \cdots y_{j+t-1}\right)=\left(v_{i}, z\right)\right\} \\
& =\sum_{j=1}^{k-t+1} \mathbb{1}\left\{\left(y_{j} \cdots y_{j+s^{\prime}-1}\right)=v_{i}\right\} \underbrace{\sum_{z \in \mathcal{Y}^{t-s^{\prime}}} \mathbb{1}\left\{\left(y_{j+s^{\prime}} \cdots y_{j+t-1}\right)=z\right\}}_{=1} \\
& =\sum_{j=1}^{k-s^{\prime}+1} \mathbb{1}\left\{\left(y_{j} \cdots y_{j+s^{\prime}-1}\right)=v_{i}\right\}-\underbrace{\sum_{j=k-t+2}^{k-s^{\prime}+1} \mathbb{1}\left\{\left(y_{j} \cdots y_{j+s^{\prime}-1}\right)=v_{i}\right\}}_{=0} \\
& =\left\{v_{i}\right\}(y)
\end{aligned}
$$

where $\left(v_{i}, z\right)$ denotes the concatenation of the strings $v_{i}$ and $z$, the first and final equalities follow from Definition 2, the second equality follows from swapping the order of summations, and the second summation in the third equality equals zero, because it only depends on the sub-string $\left(y_{k-t+2} \cdots y_{k}\right)$ with length $k-(k-t+2)+1=t-1 \leq r$, and $\left(y_{k-t+2} \cdots y_{k}\right)$ contains no $u$ 's since $y \in \mathcal{Y}_{r}^{*}$ while $v_{i}$ contains $u$ 's since $w_{1}$ is $u$-only. This establishes (65), and therefore, completes the proof.

Returning to our main discussion, with Definitions 3 and 4 in place, we can recast the constraints (58) and (59) as follows. We seek to construct a $u$-only counting form $w_{\delta}$ with rank at most $r$ such that for all $\delta \in\left(0, \frac{1}{2}\right)$ :

$$
\begin{align*}
w_{\delta}-\mathcal{E}\left(w_{\delta}\right) & \succeq \mathbf{0}  \tag{66}\\
\exists C=C(\delta)>0, w_{\delta}-C\{u\} & \succeq \mathbf{0} \tag{67}
\end{align*}
$$

where (66) is equivalent to (58) using Proposition 5, and (67) is equivalent to (59). In order to verify (66) and (67) using a computer program, we need to choose an appropriate value of $r$, and develop an efficient algorithm to test the partial order $\succeq$. We conclude this subsection by doing the former, and leave the development of the latter for ensuing subsections.

Observe using Definition 3 that the conditional expectation operator $\mathcal{E}$ evidently depends on $\delta$ (because the coefficients of the counting form $\mathcal{E}(\{v\})$ for a basis clause $\{v\}$ depend on the transition probabilities of the coupled NAND 2D regular grid). Thus, it is instructive to consider the noiseless setting where $\delta=0$. In this case, (66) is clearly satisfied if we find a $u$-only counting form $w$ that is a fixed point of $\mathcal{E}$, i.e. $w \simeq \mathcal{E}(w)$. Inspired by the "weight function" used in [39, Equation (2.3)], we consider the $u$-only harmonic counting form $w_{*}$ defined in (55) (which is an adjusted and symmetrized version of [39, Equation (2.3)]). The next proposition establishes that $w_{*}$ is a fixed point of $\mathcal{E}$ when $\delta=0$.

Proposition 6 (Harmonic Counting Form). If $\delta=0$ and $r \geq 3$, then the harmonic counting form $w_{*}$ in (55) satisfies:

$$
w_{*} \simeq \mathcal{E}\left(w_{*}\right)
$$

Proof. Using (55) and Definition 3, we have:

$$
\begin{aligned}
\mathcal{E}\left(w_{*}\right)= & 2 \mathcal{E}(\{u\})+\mathcal{E}(\{u 1\})+\mathcal{E}(\{1 u\})+\mathcal{E}(\{u 10\})+\mathcal{E}(\{01 u\})-2 \mathcal{E}(\{0 u 0\}) \\
= & 2(\{u u\}+\{1 u\}+\{u 1\})+(\{u 10\}+\{u u 0\}+\{1 u 0\}) \\
& +(\{01 u\}+\{0 u u\}+\{0 u 1\})+\mathbf{0}+\mathbf{0}-2(\mathbf{0}) \\
\simeq & (\underbrace{\{u u\}+\{1 u\}+\{0 u\}}_{\simeq\{u\}})-\{0 u\}+(\underbrace{\{u u\}+\{u 1\}+\{u 0\}}_{\simeq\{u\}})-\{u 0\}+\{1 u\}+\{u 1\} \\
& +\{u 10\}+(\underbrace{\{u u 0\}+\{1 u 0\}+\{0 u 0\}}_{\simeq\{u 0\}})+\{01 u\}+(\underbrace{\{0 u u\}+\{0 u 1\}+\{0 u 0\}}_{\simeq\{0 u\}}) \\
& -2\{0 u 0\}
\end{aligned}
$$

$$
\begin{aligned}
& \simeq 2\{u\}+\{u 1\}+\{1 u\}+\{u 10\}+\{01 u\}-2\{0 u 0\} \\
& =w_{*}
\end{aligned}
$$

where to obtain the second equality, for each basis clause $\left\{v_{1}\right\}$ of $w_{*}$, we find all basis clauses $\left\{v_{2}\right\}$ with $\operatorname{rank}\left(\left\{v_{2}\right\}\right)=\operatorname{rank}\left(\left\{v_{1}\right\}\right)+1$ such that consecutive noiseless NAND gates map $v_{2}$ to $v_{1}$ (e.g. (u10), (uu0), and $(1 u 0)$ form the set of strings that are mapped to $(u 1)$ via (45)), and when there are no such $\left\{v_{2}\right\}$ 's, $\mathcal{E}$ outputs the zero counting form $\mathbf{0}$, and to obtain the third equivalence, we utilize Lemma 5. This completes the proof.

Propelled by the elegant fixed point property of $w_{*}$ in Proposition 6, since $\operatorname{rank}\left(w_{*}\right)=3$, we will try to construct $u$-only counting forms $w_{\delta}$ with rank 3 such that (66) and (67) hold for all $\delta \in\left(0, \frac{1}{2}\right)$. Specifically, when performing computer simulations, e.g. for Table I, we will choose $r=4$ to ensure all our calculations are correct, because $\mathcal{E}\left(w_{\delta}\right)$ has rank at most 4 when $\operatorname{rank}\left(w_{\delta}\right)=3$. However, we continue our development for general fixed $r$ for now.

## C. Cyclic Evaluation and Graph Theoretic Characterizations

In order to efficiently test $\simeq$ and $\succeq$ computationally, we introduce the notion of cyclic evaluation of counting forms in this subsection.

Definition 5 (Cyclic Evaluation). In addition to the standard (acyclic) evaluation over strings presented in Definition 2 , a basis clause $\{v\}$ with rank $s \in \mathbb{N} \backslash\{0\}$ can be also evaluated cyclically over strings, i.e. it can operate on input strings in a periodic fashion:

$$
\forall k \in \mathbb{N} \backslash\{0\}, \forall y=\left(y_{0} \cdots y_{k-1}\right) \in \mathcal{Y}^{k}, \quad\{v\}[y] \triangleq \begin{cases}\sum_{i=0}^{k-1} \mathbb{1}\left\{\left(y_{(i)_{k}} \cdots y_{(i+s-1)_{k}}\right)=v\right\}, & s \leq k \\ 0, & s>k\end{cases}
$$

where $(i)_{k} \equiv i(\bmod k)$ for every $i \in \mathbb{N}$. In particular, we utilize the square bracket notation [•] to represent such cyclic evaluation, as opposed to the usual parentheses $(\cdot)$ used to represent acyclic evaluation. Furthermore, a counting form $w=\alpha_{1}\left\{v_{1}\right\}+\cdots+\alpha_{m}\left\{v_{m}\right\}$ with rank $s \in \mathbb{N} \backslash\{0\}$, basis clauses $\left\{v_{1}\right\}, \ldots,\left\{v_{m}\right\}$, and coefficients $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R}$ (with $m \in \mathbb{N} \backslash\{0\}$ ) can also be evaluated cyclically over strings, and in this case, we say that $w$ is a cyclic potential function $w: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ which operates on input strings as follows:

$$
\forall y \in \mathcal{Y}^{*}, \quad w[y] \triangleq \sum_{j=1}^{m} \alpha_{j}\left\{v_{j}\right\}[y]
$$

For any counting form $w$ with rank at most $r$, we will assume that the domain of its corresponding cyclic potential function is $\cup_{k \geq r} \mathcal{Y}^{r}$. This is in contrast to standard (acyclic) potential functions, where the domain is $\mathcal{Y}_{r}^{*}$. We will see that allowing strings which contain $u$ 's in the first and last $r$ positions to remain in the domain will lead to lucid algebraic conditions in the sequel. The next definition presents analogs of $\simeq$ and $\succeq$ for cyclic evaluation of counting forms $w: \cup_{k \geq r} \mathcal{Y}^{r} \rightarrow \mathbb{R}$ with rank at most $r$. (Although we will primarily be concerned with the vector space of $u$-only counting forms with rank at most $r$, unlike Definition 4, it is convenient to state the ensuing definition for general counting forms with rank at most $r$.)

Definition 6 (Cyclic Equivalence Classes and Partial Order). As before, let 0 denote the zero counting form, which outputs 0 on every input string after cyclic evaluation. We define the cyclic equivalence relation $\simeq_{c}$ and the cyclic partial order $\succeq_{c}$ over counting forms as follows. For any pair of counting forms $w_{1}$ and $w_{2}$ with $\max \left\{\operatorname{rank}\left(w_{1}\right), \operatorname{rank}\left(w_{2}\right)\right\} \leq r$, we write $w_{1} \simeq_{\mathrm{c}} w_{2}$ if we have:

$$
\forall y \in \bigcup_{k \geq r} \mathcal{Y}^{k}, w_{1}[y]=w_{2}[y]
$$

and we write $w_{1} \succeq_{\mathrm{c}} w_{2}$ if we have:

$$
\forall y \in \bigcup_{k \geq r} \mathcal{Y}^{k}, \quad w_{1}[y] \geq w_{2}[y]
$$

Hence, the aforementioned vector space of counting forms with rank at most $r$ is also a partially ordered vector space of cyclic equivalence classes $\left\{\left\{w+z: \operatorname{rank}(z) \leq r, z \simeq_{c} \mathbf{0}\right\}: \operatorname{rank}(w) \leq r\right\}$. Finally, we say that a counting form $w$ with rank at most $r$ is cyclically non-negative if $w \succeq_{c} \mathbf{0}$, and we will often write $w_{1} \succeq_{c} w_{2}$ equivalently as $w_{1}-w_{2} \succeq_{c} \mathbf{0}$.

The proposition below illustrates that for the special case of $u$-only counting forms, the relations $\simeq$ and $\succeq$ can be deduced from the (stronger) relations $\simeq_{c}$ and $\succeq_{c}$, respectively.

Proposition 7 (Sufficient Conditions for $\simeq$ and $\succeq$ ). For any pair of $u$-only counting forms $w_{1}$ and $w_{2}$ with rank at most $r$, we have:

1) $w_{1} \simeq_{c} w_{2}$ implies $w_{1} \simeq w_{2}$,
2) $w_{1} \succeq_{c} w_{2}$ implies $w_{1} \succeq w_{2}$.

Proof. Using Definitions 4 and 6, it suffices to prove this proposition for $w_{2}=\mathbf{0}$. So, given a $u$-only counting form $w=\alpha_{1}\left\{v_{1}\right\}+\cdots+\alpha_{m}\left\{v_{m}\right\}$ with rank at most $r, m \in \mathbb{N} \backslash\{0\}$, basis clauses $\left\{v_{1}\right\}, \ldots,\left\{v_{m}\right\}$ that contain $u$ 's, and coefficients $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R}$, we will show that:

1) $w \simeq{ }_{c} 0$ implies $w \simeq 0$,
2) $w \succeq{ }_{c} 0$ implies $w \succeq 0$.

Part 1: Observe that for any basis clause $\{v\}$ that contains $u$ 's and has rank $s \leq r$, and any string $y=$ $\left(y_{0} \cdots y_{k-1}\right) \in \mathcal{Y}^{k} \cap \mathcal{Y}_{r}^{*}$ with $k \geq r$, we have:

$$
\begin{aligned}
\{v\}[y] & =\sum_{i=0}^{k-1} \mathbb{1}\left\{\left(y_{(i)_{k}} \cdots y_{(i+s-1)_{k}}\right)=v\right\} \\
& =\sum_{i=0}^{k-s} \mathbb{1}\left\{\left(y_{i} \cdots y_{i+s-1}\right)=v\right\}+\sum_{i=k-s+1}^{k-1} \underbrace{\mathbb{1}\left\{\left(y_{(i)_{k}} \cdots y_{(i+s-1)_{k}}\right)=v\right\}}_{=0} \\
& =\{v\}(y)
\end{aligned}
$$

where the first equality follows from Definition 5, and the third equality follows from Definition 2 and the fact that $v$ contains $u$ 's and $y \in \mathcal{Y}_{r}^{*}$. Hence, using Definitions 2 and 5, we obtain:

$$
\begin{equation*}
\forall y \in \mathcal{Y}_{r}^{*}, \quad w(y)=w[y] \tag{68}
\end{equation*}
$$

Since $w \simeq{ }_{c} \mathbf{0}$, we have $w[y]=0$ for all $y \in \cup_{k \geq r} \mathcal{Y}^{k}$ using Definition 6. This implies that $w(y)=0$ for all $y \in \mathcal{Y}_{r}^{*}$ via (68). This yields $w \simeq \mathbf{0}$ using Definition 4 .

Part 2: Since $w \succeq_{c} \mathbf{0}$, we have $w[y] \geq 0$ for all $y \in \cup_{k \geq r} \mathcal{Y}^{k}$ using Definition 6. This implies that $w(y) \geq 0$ for all $y \in \mathcal{Y}_{r}^{*}$ via (68). This yields $w \succeq \mathbf{0}$ using Definition 4 .

With Proposition 7 at our disposal, we can write down the ensuing sufficient conditions for (66) and (67); specifically, we seek to construct a $u$-only counting form $w_{\delta}$ such that for all $\delta \in\left(0, \frac{1}{2}\right)$, the following cyclic non-negativity constraints are satisfied:

$$
\begin{array}{r}
w_{\delta}-\mathcal{E}\left(w_{\delta}\right) \succeq_{\mathrm{c}} \mathbf{0} \\
\exists C=C(\delta)>0, \quad w_{\delta}-C\{u\} \succeq_{\mathrm{c}} \mathbf{0} \tag{70}
\end{array}
$$

where (69) implies (66), and (70) implies (67). (At this stage, we already have all we need to prove Theorem 3, but we defer the proof to subsection V-D.) As we mentioned earlier, these new constraints are easy to test computationally. Indeed, we will develop graph theoretic methods to exactly verify the cyclic equivalence relation and partial order in due course.

However, before presenting these graph theoretic characterizations, we briefly digress and present an equivalent characterization of the cyclic equivalence relation. Our condition is inspired by the notion of "purification" introduced in Lemma 5. Indeed, notice that Lemma 5 also holds for $\simeq_{c}$ : For any counting form $w_{1}$ with rank $s \leq r$, the counting forms $w_{2}$ and $w_{3}$, defined in (63) and (64), respectively, satisfy $w_{1} \simeq_{c} w_{2} \simeq_{c} w_{3}$, and have pure rank $t \in[s, r]$. (We omit an explicit proof of this fact since it follows the argument for Lemma 5 mutatis mutandis.) Hence, for any basis clause $\{v\}$ with rank less than $r$, we clearly have $\mathbf{0} \simeq_{c}\{v\}-\{v\} \simeq{ }_{c}\{v, 0\}+\{v, 1\}+\{v, u\}-\{0, v\}-\{1, v\}-\{u, v\}$ via purification, where $\{v, z\},\{z, v\}$ denote basis clauses that are obtained by concatenating $v$ and $z \in \mathcal{Y}$. The next theorem shows that linear combinations of counting forms of this kind form a nontrivial linear subspace of counting forms that are equivalent to 0 in the cyclic sense.

Theorem 4 (Equivalent Characterization of $\simeq_{c}$ ). Consider any counting form $w$ with pure rank $s \in\{2, \ldots, r\}$. Then, $w \simeq_{c} \mathbf{0}$ if and only if $w \in \operatorname{span}\left(\left\{\rho_{v}: v \in \mathcal{Y}^{s-1}\right\}\right)$, where $\operatorname{span}(\cdot)$ denotes the linear span of its input
counting forms, and for any basis clause $\{v\}$ that has rank $s-1, \rho_{v}$ denotes the following counting form with pure rank s:

$$
\begin{equation*}
\rho_{v} \triangleq \sum_{z \in \mathcal{Y}}\{v, z\}-\{z, v\} \tag{71}
\end{equation*}
$$

Furthermore, the linear subspace $\operatorname{span}\left(\left\{\rho_{v}: v \in \mathcal{Y}^{s-1}\right\}\right)$ has dimension $3^{s-1}-1$.
Theorem 4 is established in appendix B using algebraic topological ideas that are more involved than the graph theoretic notions utilized in this section. (Note that we state this result as a theorem because it demonstrates the significance of the counting forms $\left\{\rho_{v}: v \in \mathcal{Y}^{s-1}\right\}$ as elucidated by the development in appendix B.) Using Theorem 4, we see that to verify whether a counting form $w$ with rank $s \in\{2, \ldots, r\}$ satisfies $w \simeq_{c} \mathbf{0}$, we can first purify it to obtain an equivalent counting form $\tilde{w}$ with pure rank $s$, and then test whether $\tilde{w} \simeq_{c} \mathbf{0}$ using the condition in Theorem 4. This latter test simply involves solving a linear programming feasibility problem. Moreover, to verify whether $w_{1} \simeq_{c} w_{2}$ for two counting forms with rank at most $r$, we can simply test for $w_{1}-w_{2} \simeq_{c} \mathbf{0}$ using the aforementioned procedure. Note that a specialization of the converse direction of Theorem 4 states that: If a u-only counting form $w$ with pure rank $s \in\{2, \ldots, r\}$ satisfies $w \in \operatorname{span}\left(\left\{\rho_{v}: v \in \mathcal{Y}^{s-1} \backslash\{0,1\}^{s-1}\right\}\right)$, then $w \simeq_{c} \mathbf{0}$. We present some numerical simulations based on this special case of Theorem 4 in appendix C, which provide evidence for the existence of $u$-only counting forms satisfying (69) for sufficiently small values of $\delta>0$. On a related but separate note, we believe that any counting form $w$ with pure rank $s \in\{2, \ldots, r\}$ satisfies $w \succeq_{c} \mathbf{0}$ if and only if $w=w^{\prime}+w_{0}$ for some counting form $w^{\prime}$ with pure rank $s$ and all non-negative coefficients, and some counting form $w_{0} \in \operatorname{span}\left(\left\{\rho_{v}: v \in \mathcal{Y}^{s-1}\right\}\right)$. The converse direction of this statement holds due to Theorem 4, but the forward direction is an open problem.

Finally, we turn to presenting our graph theoretic characterizations of the cyclic equivalence relation and partial order. To this end, consider any counting form $w$ with pure rank $r$ defined by the formal sum:

$$
\begin{equation*}
w=\sum_{v \in \mathcal{Y}^{r}} \alpha_{v}\{v\} \tag{72}
\end{equation*}
$$

with coefficients $\left\{\alpha_{v} \in \mathbb{R}: v \in \mathcal{Y}^{r}\right\}$. Corresponding to $w$, construct the weighted directed graph $\mathcal{G}_{r}(w)$ with vertex set $\mathcal{Y}^{r}$ (of strings with length $r$, or equivalently, basis clauses with rank $r$ ), directed edge set: ${ }^{14}$

$$
\begin{equation*}
E \triangleq\left\{(v, z) \in \mathcal{Y}^{r} \times \mathcal{Y}^{r}:\left(v_{2} \cdots v_{r}\right)=\left(z_{1} \cdots z_{r-1}\right)\right\} \tag{73}
\end{equation*}
$$

where $v=\left(v_{1} \cdots v_{r}\right) \in \mathcal{Y}^{r}$ and $z=\left(z_{1} \cdots z_{r}\right) \in \mathcal{Y}^{r}$, and weight function $\mathcal{W}: E \rightarrow \mathbb{R}$ given by:

$$
\begin{equation*}
\forall(v, z) \in E, \quad \mathcal{W}((v, z)) \triangleq \alpha_{v} \tag{74}
\end{equation*}
$$

This graph encodes all the information required to compute the cyclic potential of any given string. For instance, each vertex of $\mathcal{G}_{r}(w)$ has outdegree 3 , and transitioning between these vertices via the directed edges can be equivalently construed as sliding an $r$-length window along a string. Indeed, any path $v_{1} \rightarrow v_{2} \rightarrow \cdots \rightarrow v_{k-1} \rightarrow$ $v_{k}$ on this graph corresponds to successively visiting all possible $r$-length sub-strings of the associated string $\left(\left(v_{1}\right)_{1}\left(v_{2}\right)_{1} \cdots\left(v_{k-1}\right)_{1}\left(v_{k}\right)_{1} \cdots\left(v_{k}\right)_{r}\right) \in \mathcal{Y}^{r+k-1}$, where $\left(v_{i}\right)_{j} \in \mathcal{Y}$ denotes the $j$ th letter in the $i$ th sub-string $v_{i} \in \mathcal{Y}^{r}$ for $i \in\{1, \ldots, k\}$ and $j \in\{1, \ldots, r\}$. In fact, it is straightforward to verify that $\mathcal{G}_{r}(w)$ is strongly connected, because we can reach any vertex (or basis clause) from any other vertex via some sequence of intermediate letters in $\mathcal{Y}$. Furthermore, the coefficients of $w$ inherently assign weights to each vertex of $\mathcal{G}_{r}(w)$, and we let each directed edge of $\mathcal{G}_{r}(w)$ inherit the weight associated with its source vertex. So, for example, for the aforementioned path $v_{1} \rightarrow v_{2} \rightarrow \cdots \rightarrow v_{k-1} \rightarrow v_{k}$, the sum of the weights along the edges of this path is equal to $\alpha_{v_{1}}+\cdots+\alpha_{v_{k-1}}$. This equivalence can be used to capture the cyclic potentials of general strings. The proposition below rigorizes these intuitions, and conveys equivalent characterizations of $w \simeq_{c} \mathbf{0}$ and $w \succeq_{c} \mathbf{0}$.

Proposition 8 (Graph Theoretic Characterizations of $\simeq_{c}$ and $\succeq_{c}$ ). For any counting form $w$ with pure rank $r$, the following are true:

1) $w \simeq_{c} \mathbf{0}$ if and only if all cycles in the corresponding graph $\mathcal{G}_{r}(w)$ have zero total weight, where the total weight of any path in $\mathcal{G}_{r}(w)$ is given by the sum of the weights of the edges on the path.

[^10]2) $w \succeq_{c} \mathbf{0}$ (i.e. $w$ is cyclically non-negative) if and only if the corresponding graph $\mathcal{G}_{r}(w)$ does not contain any negative cycles (i.e. cycles with negative total weight). ${ }^{15}$

## Proof.

We only prove part 2 , because the proof of part 1 follows mutatis mutandis. To prove the forward direction of part 2 , assume that $w \succeq_{c} \mathbf{0}$. Suppose further for the sake of contradiction that there exists a negative cycle $v_{0} \rightarrow v_{1} \rightarrow$ $\cdots \rightarrow v_{m-1} \rightarrow v_{0}$ in $\mathcal{G}_{r}(w)$, which has $m \in \mathbb{N} \backslash\{0\}$ edges. (Note that the vertices $v_{1}, \ldots, v_{m-1}$ are distinct and not equal to $v_{0}$, and the case $m=1$ corresponds to a self-loop $\left.v_{0} \rightarrow v_{0}\right)$. Let $\tilde{y}=\left(\left(v_{0}\right)_{0} \cdots\left(v_{m-1}\right)_{0}\right) \in \mathcal{Y}^{m}$, where $\left(v_{i}\right)_{0}$ is the first letter of $v_{i}$ for all $i \in[k]$, and construct the associated string:

$$
y=\underbrace{(\tilde{y}, \ldots, \tilde{y})}_{r \text { times }} \in \mathcal{Y}^{r m}
$$

which concatenates $\tilde{y}$ with itself $r$ times. (This concatenation process ensures that the length of $y$ is at least $r$, so that its cyclic potential can be non-zero.) The non-negative cyclic potential of $y, w[y]$, is given by:

$$
\begin{aligned}
0 & \leq w[y]=\sum_{v \in \mathcal{Y}^{r}} \alpha_{v} \sum_{i=0}^{m-1} r \mathbb{1}\left\{v_{i}=v\right\} \\
& =r \sum_{i=0}^{m-1} \alpha_{v_{i}} \\
& =r \underbrace{\sum_{i=0}^{m-1} \mathcal{W}\left(\left(v_{i}, v_{(i+1)_{m}}\right)\right)}_{\substack{\text { total weight of cycle } \\
v_{0} \rightarrow v_{1} \rightarrow \ldots \rightarrow v_{m}}}
\end{aligned}
$$

where the first equality uses (72), Definition 5, and the construction of $y$, the second equality follows from swapping the order of summations, and the third equality uses (74). This contradicts our assumption that $v_{0} \rightarrow v_{1} \rightarrow \cdots \rightarrow$ $v_{m-1} \rightarrow v_{0}$ is a negative cycle. Hence, $\mathcal{G}_{r}(w)$ does not contain any negative cycles.

To prove the converse direction of part 2 , assume that $\mathcal{G}_{r}(w)$ does not contain any negative cycles. This implies that $\mathcal{G}_{r}(w)$ does not contain any negative circuits. To prove this, suppose for the sake of contradiction that $\mathcal{G}_{r}(w)$ contains a circuit starting and ending at the vertex $v_{1}$ with negative total weight. Then, there exists a vertex $v_{2} \neq v_{1}$ in this circuit that is visited at least twice. (If no such $v_{2}$ exists, then this circuit must be a cycle and its total weight cannot be negative.) So, we can decompose this circuit into a path from $v_{1}$ to $v_{2}$, followed by a circuit from $v_{2}$ to itself, followed by another path from $v_{2}$ back to $v_{1}$. This decomposition yields two shorter circuits, namely, one circuit from $v_{1}$ to itself via $v_{2}$, and another circuit from $v_{2}$ to itself. Moreover, at least one of these shorter circuits must have negative total weight (since the larger circuit has negative total weight). By recursively decomposing negative circuits and obtaining smaller negative circuits, we will eventually obtain a negative cycle. This yields a contradiction, and $\mathcal{G}_{r}(w)$ must not have had any negative circuits.

Now consider any string $y=\left(y_{0} \cdots y_{k-1}\right) \in \mathcal{Y}^{k}$ for any fixed $k \geq r$, and construct the associated circuit $v_{0} \rightarrow v_{1} \rightarrow \cdots \rightarrow v_{k-1} \rightarrow v_{0}$, where $v_{i}=\left(y_{(i)_{k}} \cdots y_{(i+r-1)_{k}}\right)$ for all $i \in[k]$. It is straightforward to verify that the edges of this circuit exist in $\mathcal{G}_{r}(w)$. Then, using essentially the same argument as before, the cyclic potential of $y, w[y]$, is non-negative, because:

$$
\begin{aligned}
w[y] & =\sum_{v \in \mathcal{Y}^{r}} \alpha_{v} \sum_{i=0}^{k-1} \mathbb{1}\left\{\left(y_{(i)_{k}} \cdots y_{(i+r-1)_{k}}\right)=v\right\} \\
& =\sum_{i=0}^{\sum_{i=0}^{k-1} \alpha_{v_{i}}} \\
& =\underbrace{\sum_{i=0}^{k-1} \mathcal{W}\left(\left(v_{i}, v_{(i+1)_{k}}\right)\right)}_{\substack{\text { total weight of circuit } \\
v_{0} \rightarrow v_{1} \rightarrow \cdots \rightarrow v_{k-1} \rightarrow v_{0}}}
\end{aligned}
$$

[^11]$$
\geq 0
$$
where, as before, the first equality uses (72) and Definition 5 , the second equality follows from swapping the order of summations, the third equality uses (74), and the final inequality holds because $\mathcal{G}_{r}(w)$ does not contain any negative circuits. Therefore, we obtain that $w \succeq_{c} \mathbf{0}$ using Definition 6. This completes the proof.

Several remarks are in order at this point. Firstly, as explained after Theorem 4, to check whether a general counting form $w$ with rank $s \leq r$ satisfies $w \succeq_{c} \mathbf{0}$ or $w \simeq_{c} \mathbf{0}$, we can first construct an equivalent purified counting form $\tilde{w}$ with pure rank $r$, and then apply Proposition 8 to $\tilde{w}$. Furthermore, to test whether $w_{1} \succeq_{c} w_{2}$ (respectively $w_{1} \simeq_{\mathrm{c}} w_{2}$ ) for two counting forms with rank at most $r$, we can simply test for $w_{1}-w_{2} \succeq_{\mathrm{c}} \mathbf{0}$ (respectively $w_{1}-$ $w_{2} \simeq_{c} \mathbf{0}$ ).

Secondly, it is worth mentioning that the graph $\mathcal{G}_{r}(w)$ has three vertices with self-loops: all zeros $(0 \cdots 0)$, all ones $(1 \cdots 1)$, and all $u$ 's $(u \cdots u)$. Since we will only test cyclic non-negativity of $u$-only counting forms $w$, the self-loop edges on $(0 \cdots 0)$ and $(1 \cdots 1)$ always get weights of 0 , and can never contribute to negative cycles. On the other hand, part 2 of Proposition 8 illustrates that a necessary condition for $w \succeq_{c} \mathbf{0}$ is that the weight of the self-loop edge on $(u \cdots u)$ is non-negative, i.e. if the weight of the self-loop edge on $(u \cdots u)$ is negative, then $w$ is not cyclically non-negative. Hence, when verifying whether a candidate $u$-only counting form $w_{\delta}$ with rank $r-1$ satisfies (69) and (70) for fixed values of $\delta \in\left(0, \frac{1}{2}\right)$, the coefficients associated with the basis clause $\{u \cdots u\}$ in purified versions of both $w_{\delta}-\mathcal{E}\left(w_{\delta}\right)$ and $w_{\delta}-C\{u\}$ must be non-negative (where $w_{\delta}-\mathcal{E}\left(w_{\delta}\right)$ is also $u$-only-see the discussion after Definition 3).

Lastly, we remark that the condition in part 2 of Proposition 8, which is the more important part of the proposition since we seek to verify (69) and (70), can be tested using several well-known graph shortest path algorithms. For example, we can utilize the Bellman-Ford algorithm to detect negative cycles in $\mathcal{G}_{r}(w)$ that are reachable from a certain fixed source vertex, cf. [54, Section 24.1]. In fact, since $\mathcal{G}_{r}(w)$ is a strongly connected graph, we can choose to compute shortest paths from any initial vertex of $\mathcal{G}_{r}(w)$ in order to detect negative cycles in the entire graph. ${ }^{16}$ For completeness, we refer readers to Algorithm 1 in appendix D, which provides pseudocode for a Bellman-Ford-based algorithm to determine the cyclic non-negativity of $u$-only counting forms.

## D. Proof of Theorem 3

At noted earlier, we are ready to establish Theorem 3 by tracing back through some of the results and arguments in this section. Since most of the arguments have already been carried out rigorously, we will omit many details.
Proof of Theorem 3. Fix any noise level $\delta \in\left(0, \frac{1}{2}\right)$, and assume that there exists $r \geq 2$ (with respect to which both $\succeq$ and $\succeq_{c}$ are defined) and a $u$-only counting form $w_{\delta}$ with rank $r-1$ that satisfies (69) and (70). Then, using Proposition 7, this $u$-only counting form $w_{\delta}$ also satisfies (66) and (67) (where we also utilize the fact that $\mathcal{E}\left(w_{\delta}\right)$ is a $u$-only counting form with rank $r$ ).

Now construct the stopping time:

$$
T \triangleq \inf \left\{k \in \mathbb{N} \backslash[r-1]: Y_{k, 0}, \ldots, Y_{k, r-1}, Y_{k-r+1}, \ldots, Y_{k, k} \in\{0,1\}\right\}
$$

with respect to the filtration $\left\{\mathcal{F}_{k}: k \in \mathbb{N}\right\}$. It is straightforward to verify that $\mathbb{P}(T<+\infty)=1$ because, as discussed after Definition 2 in subsection V-B, the event $A$ in (56) occurs with probability 1 due to the second Borel-Cantelli lemma, and $A \subseteq\{T<+\infty\}$. Hence, the events $\{T=m\}$ for $m \geq r-1$ partition the underlying sample space (without loss of generality). Let us condition on any one such event $\{T=m\} \in \mathcal{F}_{m}$, i.e., if the underlying probability space is $(\Omega, \mathcal{F}, \mathbb{P})$, let us operate in the smaller conditional probability space $\left(\{T=m\}, \mathcal{F}^{\prime}, \mathbb{P}_{m}\right)$ with sample space $\{T=m\}, \sigma$-algebra $\mathcal{F}^{\prime}=\{A \cap\{T=m\}: A \in \mathcal{F}\}$, and conditional probability measure $\mathbb{P}_{m}(\cdot)=\mathbb{P}(\cdot \cap\{T=m\}) / \mathbb{P}(T=m)$. This means that for all $k \geq m, Y_{k, 0}, \ldots, Y_{k, r-1}, Y_{k, k-r+1}, \ldots, Y_{k, k} \in\{0,1\}$ $\mathbb{P}_{m}$-almost surely (due to (44) and (45)). Furthermore, define the filtration $\left\{\mathcal{F}_{k}^{\prime}: k \geq m\right\}$ with $\sigma$-algebras $\mathcal{F}_{k}^{\prime}=$ $\left\{A \cap\{T=m\}: A \in \mathcal{F}_{k}\right\} \subseteq \mathcal{F}^{\prime}$.

Then, as explained after Definition 2 and Lemma 5 in subsection V-B, we obtain that $w_{\delta}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ is a superharmonic potential function that satisfies the conditions in (57), (58), and (59) mutatis mutandis:

$$
\forall k \geq m, \quad \mathbb{E}_{m}\left[\left|w_{\delta}\left(Y_{k}\right)\right|\right]<+\infty
$$

[^12]\[

$$
\begin{aligned}
\forall k \geq m, \forall y \in \mathcal{Y}^{k+1} \cap \mathcal{Y}_{r}^{*}, \quad \mathbb{E}_{m}\left[w_{\delta}\left(Y_{k+1}\right) \mid Y_{k}=y\right] & \leq w_{\delta}(y) \\
\forall y \in \mathcal{Y}_{r}^{*}, w_{\delta}(y) & \geq C\{u\}(y)
\end{aligned}
$$
\]

where $\mathbb{E}_{m}[\cdot]$ denotes the expectation with respect to $\mathbb{P}_{m}$, and the second inequality uses Proposition 5 , whose proof holds for $\mathbb{E}_{m}$ in place of $\mathbb{E}$, because $Y_{k+1}$ is conditionally independent of the event $\{T=m\}$ given $Y_{k}$ for $k \geq m$ (in the original probability space). Moreover, since $\left\{Y_{k}: k \geq m\right\}$ remains a Markov chain adapted to $\left\{\mathcal{F}_{k}^{\prime}: k \geq m\right\}$ in the conditional probability space, we have:

$$
\forall k \geq m, \mathbb{E}_{m}\left[w_{\delta}\left(Y_{k+1}\right) \mid \mathcal{F}_{k}^{\prime}\right]=\mathbb{E}_{m}\left[w_{\delta}\left(Y_{k+1}\right) \mid Y_{k}\right] \leq w_{\delta}\left(Y_{k}\right) \quad \mathbb{P}_{m} \text {-almost surely }
$$

i.e. $\left\{w_{\delta}\left(Y_{k}\right): k \geq m\right\}$ is a supermartingale adapted to $\left\{\mathcal{F}_{k}^{\prime}: k \geq m\right\}$. Equivalently, within our conditional probability space, we have shown that $w_{\delta}: \mathcal{Y}^{*} \rightarrow \mathbb{R}$ is a superharmonic function that fulfills the two requirements delineated in Conjecture 2 (mutatis mutandis) for the fixed value of $\delta$ under consideration.

Therefore, using the argument in the proof of Proposition 4 mutatis mutandis, we get:

$$
\mathbb{P}_{m}\left(\lim _{k \rightarrow \infty} N_{k}=0\right)=\mathbb{P}\left(\lim _{k \rightarrow \infty} N_{k}=0 \mid T=m\right)=1
$$

where the random variables $N_{k}$ are defined in (46). Since $m \geq r-1$ was arbitrary, taking expectations with respect to the law of $T$ yields that $\lim _{k \rightarrow \infty} N_{k}=0 \mathbb{P}$-almost surely. Finally, employing Lemma 4 shows that Conjecture 1 is true for the noise level $\delta$. This completes the proof.

## E. Linear Programming Criteria

Returning back to our discussion of counting forms, it is well-known that the graph theoretic task of computing shortest paths from a single source vertex, or detecting negative cycles reachable from this vertex, can also be solved using linear programming feasibility problems, cf. [54, Sections 24.4 and 29]. Therefore, in this subsection, we will present an efficient linear programming formulation that ensures cyclic non-negativity. This formulation can be employed to numerically compute $u$-only counting forms $w_{\delta}$ (e.g. with rank $r-1=3$ ) that satisfy (69) and (70) for different values of $\delta \in\left(0, \frac{1}{2}\right)$ as in Table I.

Consider any counting form $w$ with pure rank $r$ defined by the formal sum (72), and let $\mathcal{G}_{r}(w)$ be its corresponding weighted directed graph with vertex set $\mathcal{Y}^{r}$, edge set $E$ given by (73), and weight function $\mathcal{W}: E \rightarrow \mathbb{R}$ given by (74). Define the input and output incidence matrices, $\mathbf{B}_{\text {in }} \in\{0,1\}^{3^{r} \times 3^{r+1}}$ and $\mathbf{B}_{\text {out }} \in\{0,1\}^{3^{r} \times 3^{r+1}}$, respectively, of $\mathcal{G}_{r}(w)$ entry-wise as follows:

$$
\begin{align*}
& \forall v \in \mathcal{Y}^{r}, \forall e \in E, \quad\left[\mathbf{B}_{\text {in }}\right]_{v, e} \triangleq \mathbb{1}\{v \text { has incoming edge } e\}  \tag{75}\\
& \forall v \in \mathcal{Y}^{r}, \forall e \in E, \quad\left[\mathbf{B}_{\mathrm{out}}\right]_{v, e} \triangleq \mathbb{1}\{v \text { has outgoing edge } e\} \tag{76}
\end{align*}
$$

where the rows are indexed by vertices in $\mathcal{Y}^{r}$, and the columns are indexed by edges in $E$. (Note that a self-loop is considered both an incoming and an outgoing edge.) Using $\mathbf{B}_{\text {in }}$ and $\mathbf{B}_{\text {out }}$, we can define the oriented incidence matrix $\mathbf{B} \in\{0,1\}^{3^{r} \times 3^{r+1}}$ of $\mathcal{G}_{r}(w)$ as:

$$
\begin{equation*}
\mathbf{B} \triangleq \mathbf{B}_{\mathrm{out}}-\mathbf{B}_{\text {in }} \tag{77}
\end{equation*}
$$

which is a fundamental matrix utilized in algebraic graph theory to study the homology of graphs. ${ }^{17}$ The kernel (or nullspace) of $\mathbf{B}$, denoted $\operatorname{ker}(\mathbf{B}) \subseteq \mathbb{R}^{3^{r+1}}$, is the so called flow space of the graph $\mathcal{G}_{r}(w)$, which is (informally) the linear span of all cycles of $\mathcal{G}_{r}(w)$, cf. [55, Section 14.2]. Indeed, let us encode any path in $\mathcal{G}_{r}(w)$ as a column vector $x \in \mathbb{N}^{3^{r+1}}$ (indexed by edges in $E$ ) whose $e$ th element $x_{e} \geq 0$ denotes the number of times the edge $e \in E$ was traversed by the path. Then, we have the following lemma from [55, Section 14.2].

Lemma 6 (Circuits and Flow Space [55, Theorem 14.2.2]). If a vector $x \in \mathbb{N}^{3^{r+1}}$ encodes a circuit in $\mathcal{G}_{r}(w)$, then $x \in \operatorname{ker}(\mathbf{B})$, i.e. $x$ is a flow vector.

Using Lemma 6 and Proposition 8, we will present sufficient conditions for cyclic equivalence and non-negativity in terms of matrix relations and linear programming. To this end, as shown in appendix C, let us represent the counting form $w$ using its column vector of coefficients $\alpha=\left(\alpha_{v}: v \in \mathcal{Y}^{r}\right)=\zeta_{r}(w) \in \mathbb{R}^{3^{r}}$ akin to (94). Note

[^13]that $\alpha$ is indexed by the vertices of $\mathcal{G}_{r}(w)$ consistently with $\mathbf{B}$. The next proposition states our matrix and linear programming based criteria.

Proposition 9 (Matrix and Linear Programming Criteria for $\simeq_{c}$ and $\succeq_{c}$ ). Given any counting form $w$ with pure rank $r$ and coefficients $\alpha=\left(\alpha_{v}: v \in \mathcal{Y}^{r}\right) \in \mathbb{R}^{3^{r}}$, the following are true:

1) If there exists a vector $z \in \mathbb{R}^{3^{r}}$ such that $\mathbf{B}^{\mathrm{T}} z=\mathbf{B}_{\text {out }}^{\mathrm{T}} \alpha$, then $w \simeq_{\mathrm{c}} \mathbf{0}$.
2) If there exists a vector $z \in \mathbb{R}^{3^{r}}$ such that $\mathbf{B}^{\mathrm{T}} z \leq \mathbf{B}_{\text {out }}^{\mathrm{T}} \alpha$ entry-wise, then $w \succeq_{c} \mathbf{0}$.

## Proof.

Part 1: First, observe that for any flow vector $x \in \mathbb{N}^{3^{r+1}}$ that encodes a circuit in $\mathcal{G}_{r}(w)$, the bilinear form $\alpha^{\mathrm{T}} \mathbf{B}_{\text {out }} x$ is equal to the total weight of the circuit corresponding to $x$ due to (74). (Note that there may be multiple circuits with the same encoding $x$, but these circuits have the same total weight.) Now, suppose the edge weights $\alpha^{\mathrm{T}} \mathbf{B}_{\text {out }}$ of $\mathcal{G}_{r}(w)$ live in the row space of $\mathbf{B}$, or equivalently, suppose there exists $z \in \mathbb{R}^{3^{r}}$ such that $\mathbf{B}^{\mathrm{T}} z=\mathbf{B}_{\text {out }}^{\mathrm{T}} \alpha$. Then, for any circuit in $\mathcal{G}_{r}(w)$ encoded by the flow vector $x \in \mathbb{N}^{3^{r+1}}$, since $x \in \operatorname{ker}(\mathbf{B})$ by Lemma 6, the total weight of the circuit is $\alpha^{\mathrm{T}} \mathbf{B}_{\text {out }} x=z^{\mathrm{T}} \mathbf{B} x=0$. So, using part 1 of Proposition 8, we have $w \simeq_{c} \mathbf{0}$. This proves part 1.

Part 2: To prove part 2, consider the feasible integer LP:

$$
L_{1}=\min _{\substack{x \in \mathbb{Z}^{3^{r+1}} \\ \mathbf{B} x=\mathbf{0}, x \geq \mathbf{0}}} \alpha^{\mathrm{T}} \mathbf{B}_{\text {out }} x
$$

where inequalities among vectors hold entry-wise. Note that the constraints of this integer LP ensure that the minimization is over all non-negative integer-valued flow vectors $x$, which includes all flow vectors that encode circuits in $\mathcal{G}_{r}(w)$ by Lemma 6 . Moreover, when the flow vector $x$ encodes a circuit, the objective function $\alpha^{\mathrm{T}} \mathbf{B}_{\text {out }} x$ is equal to the total weight of this encoded circuit. Thus, using part 2 of Proposition 8 , if $L_{1}=0$ then $w \succeq_{c} \mathbf{0}$ (where we use the fact that $\mathcal{G}_{r}(w)$ has no negative cycles if and only if it has no negative circuits-see the proof of Proposition 8). ${ }^{18}$

We next relax the integer constraints of this integer LP and obtain the following feasible LP:

$$
L_{2}=\min _{\substack{x \in \mathbb{R}^{3^{r+1}} \\ \mathbf{B} x=\mathbf{0}, x \geq \mathbf{0}}} \alpha^{\mathrm{T}} \mathbf{B}_{\text {out }} x
$$

which obviously satisfies $L_{2} \leq L_{1}$. It is straightforward to verify that the dual LP of this (primal) LP is the feasibility problem (cf. [54, Section 29.4], [56, Chapter 1, Section 2]):

$$
L_{3}=\max _{\substack{z \in \mathbb{R}^{3^{r}} \\ \mathbf{B}^{\mathrm{T}} z \leq \mathbf{B}_{\text {out }}^{\mathrm{T}}}} 0
$$

where the dual variables $z$ are indexed by the vertices in $\mathcal{Y}^{r}$. Using the duality theorem (see e.g. [56, Chapter 1, Section 8], there are only two possibilities:

1) If there exists $z \in \mathbb{R}^{3^{r}}$ such that $\mathbf{B}^{\mathrm{T}} z \leq \mathbf{B}_{\text {out }}^{\mathrm{T}} \alpha$, then we have $L_{1}=L_{2}=L_{3}=0$ (via strong duality).
2) Otherwise, we have $L_{2}=-\infty$, i.e. the primal LP is unbounded.

This proves that if there exists $z \in \mathbb{R}^{3^{r}}$ such that $\mathbf{B}^{\mathrm{T}} z \leq \mathbf{B}_{\text {out }}^{\mathrm{T}} \alpha$, then $w \succeq_{\mathrm{c}} \mathbf{0}$ as desired.
It is worth briefly interpreting the sufficient condition in part 2 of Proposition 9 . The result says that the counting form $w$ is cyclically non-negative, or equivalently, all circuits in $\mathcal{G}_{r}(w)$ have non-negative total weight, if the set of edge weights $\mathbf{B}_{\text {out }}^{\mathrm{T}} \alpha$ of $\mathcal{G}_{r}(w)$ can be decomposed as follows:

$$
\begin{equation*}
\exists z \in \mathbb{R}^{3^{r}}, \exists \beta \in \mathbb{R}^{3^{r+1}}, \quad \mathbf{B}_{\text {out }}^{\mathrm{T}} \alpha=\mathbf{B}^{\mathrm{T}} z+\beta \text { and } \beta \geq \mathbf{0} \tag{78}
\end{equation*}
$$

where the set of edge weights $\mathbf{B}^{\mathrm{T}} z$ lead to all circuits in $\mathcal{G}_{r}(w)$ having zero total weight (see part 1 of Proposition 9), and the set of edge weights $\beta$ are entry-wise non-negative. On a separate note, it is also worth comparing part 1 of Proposition 9 with the converse direction of Theorem 4, because the two results are closely related.

[^14]Using part 2 of Proposition 9, we now illustrate sufficient conditions for (69) and (70). Let us represent the conditional expectation operator $\mathcal{E}$ from Definition 3 as the $3^{r} \times 3^{r-1}$ matrix $\mathbf{C}_{r-1}(\delta)$ (for $\delta \in\left[0, \frac{1}{2}\right]$ ) defined in (98) in appendix C, and let $P_{r-1}$ be the $3^{r} \times 3^{r-1}$ "purification matrix" defined in (101) in appendix C. Note that each $u$-only counting form with pure rank $s \leq r$ can be represented by a column vector of coefficients in $\mathbb{R}^{3^{s}}$ using the map $\zeta_{s}(\cdot)$ as shown in (94). So, $\mathbf{C}_{r-1}(\delta)$ and $P_{r-1}$ map coefficient vectors of $u$-only counting forms with pure rank $r-1$ into coefficient vectors of $u$-only counting forms with pure rank $r$ (see e.g. (96) in appendix C). In fact, the output vectors of $P_{r-1}$ are cyclically equivalent to the corresponding input vectors (as discussed in appendix C). We obviously assume that the entries of different coefficient vectors, and the rows and columns of $\mathbf{C}_{r-1}(\delta)$ and $P_{r-1}$, are all indexed consistently by the sets $\mathcal{Y}^{s}$ for appropriate choices of $s$. (We remark that the indexing in (95) in appendix C endows the strings in $\mathcal{Y}^{s}$ for any $s \leq r$ with a lexicographic ordering, ${ }^{19}$ which is useful for computing purposes.) Furthermore, define the vector of coefficients $\psi=\left(\psi_{v}: v \in \mathcal{Y}^{r}\right) \in \mathbb{R}^{3^{r}}$ of a purified version of the basis clause $\{u\}$ :

$$
\begin{equation*}
\forall v=\left(v_{1} \cdots v_{r}\right) \in \mathcal{Y}^{r}, \quad \psi_{v} \triangleq \mathbb{1}\left\{v_{1}=u\right\} \tag{79}
\end{equation*}
$$

It is straightforward to verify that $\{u\} \simeq_{c} \sum_{v \in \mathcal{Y}^{r}} \psi_{v}\{v\}$ (see Lemma 5 and the discussion preceding Theorem 4). The ensuing proposition portrays sufficient conditions for (69) and (70) that can be tested using an LP.

Proposition 10 (Linear Programming Criterion for Satisfying (69) and (70)). Fix any $\delta \in\left(0, \frac{1}{2}\right)$, and construct the partitioned matrices:

$$
\begin{align*}
& \xi \triangleq\left[\mathbf{0}^{\mathrm{T}} \mid \psi^{\mathrm{T}} \mathbf{B}_{\text {out }}\right]^{\mathrm{T}} \in \mathbb{R}^{2\left(3^{r+1}\right)}  \tag{80}\\
& \mathbf{A}(\delta) \triangleq\left[\begin{array}{c|c|c}
\mathbf{B}_{\text {out }}^{\mathrm{T}}\left(P_{r-1}-\mathbf{C}_{r-1}(\delta)\right) & \mathbf{B}^{\mathrm{T}} & \mathbf{0} \\
\hline \mathbf{B}_{\text {out }}^{\mathrm{T}} P_{r-1} & \mathbf{0} & \mathbf{B}^{\mathrm{T}}
\end{array}\right] \in \mathbb{R}^{2\left(3^{r+1}\right) \times\left(3^{r-1}+2\left(3^{r}\right)\right)} \tag{81}
\end{align*}
$$

where $\mathbf{0}$ denotes the zero vector or matrix of appropriate dimension (with abuse of notation). Consider the following linear programming feasibility problem:

$$
\begin{gather*}
\exists \phi(\delta)=\left[\alpha(\delta)^{\mathrm{T}} \mid z(\delta)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{3^{r-1}+2\left(3^{r}\right)} \text { with } \alpha(\delta)=\left(\alpha_{v}(\delta): v \in \mathcal{Y}^{r-1}\right) \in \mathbb{R}^{3^{r-1}} \\
\text { such that } \mathbf{A}(\delta) \phi(\delta) \geq \xi  \tag{82}\\
\text { and } \alpha_{v}(\delta)=0 \text { for all } v \in\{0,1\}^{r-1}
\end{gather*}
$$

If this LP in (82) has a feasible solution $\phi(\delta)=\phi^{*}(\delta)=\left[\alpha^{*}(\delta)^{\mathrm{T}} \mid z^{*}(\delta)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{3^{r-1}+2\left(3^{r}\right)}$, then the corresponding $u$-only counting form $w_{\delta}^{*}$ with pure rank $r-1$ :

$$
\begin{equation*}
w_{\delta}^{*}=\sum_{v \in \mathcal{Y}^{r-1}} \alpha_{v}^{*}(\delta)\{v\} \tag{83}
\end{equation*}
$$

satisfies both (69) and (70) with $C=C(\delta)=1$. Furthermore, the u-only counting form $w_{\delta}^{*}$ satisfies both (66) and (67) with $C=1$.

Proof. We seek to construct a $u$-only counting form $w_{\delta}$ with pure rank $r-1$ such that (69) and (70) hold, i.e. $w_{\delta}-\mathcal{E}\left(w_{\delta}\right) \succeq_{c} \mathbf{0}$ and $w_{\delta}-C\{u\} \succeq_{c} \mathbf{0}$ for some constant $C=C(\delta)>0$. Let $\alpha(\delta)=\left(\alpha_{v}(\delta): v \in \mathcal{Y}^{r-1}\right) \in \mathbb{R}^{3^{r-1}}$ denote the vector of coefficients of $w_{\delta}$. Using part 2 of Proposition 9, a sufficient condition for (69) is:

$$
\begin{equation*}
\exists z_{1}(\delta) \in \mathbb{R}^{3^{r}}, \quad \mathbf{B}^{\mathrm{T}} z_{1}(\delta) \leq \mathbf{B}_{\mathrm{out}}^{\mathrm{T}}\left(P_{r-1}-\mathbf{C}_{r-1}(\delta)\right) \alpha(\delta) \tag{84}
\end{equation*}
$$

because $\left(P_{r-1}-\mathbf{C}_{r-1}(\delta)\right) \alpha(\delta) \in \mathbb{R}^{3^{r}}$ is the vector of coefficients corresponding to a purified version of $w_{\delta}-\mathcal{E}\left(w_{\delta}\right)$. Similarly, using part 2 of Proposition 9, a sufficient condition for (70) is:

$$
\begin{equation*}
\exists C=C(\delta)>0, \exists z_{2}(\delta) \in \mathbb{R}^{3^{r}}, \quad \mathbf{B}^{\mathrm{T}} z_{2}(\delta) \leq \mathbf{B}_{\text {out }}^{\mathrm{T}}\left(P_{r-1} \alpha(\delta)-C \psi\right) \tag{85}
\end{equation*}
$$

because $P_{r-1} \alpha(\delta)-C \psi \in \mathbb{R}^{3^{r}}$ is the vector of coefficients corresponding to a purified version of $w_{\delta}-C\{u\}$. Since our goal is to find a $u$-only counting form $w_{\delta}$, or equivalently, a vector of coefficients $\alpha(\delta)$, that satisfies (84) and (85), the vector $\alpha(\delta)$ is also unknown in (84) and (85). Hence, we can set $C=1$ without loss of generality in

[^15](85), because we can multiply both sides of (84) and (85) by $C^{-1}$ and absorb this multiplicative constant into the unknown vectors $z_{1}(\delta), z_{2}(\delta)$, and $\alpha(\delta)$.

Next, construct the block vector:

$$
\phi(\delta) \triangleq\left[\alpha(\delta)^{\mathrm{T}}\left|-z_{1}(\delta)^{\mathrm{T}}\right|-z_{2}(\delta)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{3^{r-1}+2\left(3^{r}\right)}
$$

the block vector $\xi \in \mathbb{R}^{2\left(3^{r+1}\right)}$ in (80), and the block matrix $\mathbf{A}(\delta) \in \mathbb{R}^{2\left(3^{r+1}\right) \times\left(3^{r-1}+2\left(3^{r}\right)\right)}$ in (81). Then, it is straightforward to verify that for any fixed vector $\alpha(\delta)$, the sufficient conditions (84) and (85) with $C=1$ can be simultaneously expressed as:

$$
\exists z_{1}(\delta), z_{2}(\delta) \in \mathbb{R}^{3^{r}}, \quad \mathbf{A}(\delta) \phi(\delta) \geq \xi
$$

Moreover, since we seek to construct an $\alpha(\delta)$ satisfying this condition, our true objective is to verify the following sufficient condition:

$$
\exists \alpha(\delta) \in \mathbb{R}^{3^{r-1}}, \exists z_{1}(\delta), z_{2}(\delta) \in \mathbb{R}^{3^{r}}, \quad \mathbf{A}(\delta) \phi(\delta) \geq \xi
$$

where $\alpha(\delta)$ additionally satisfies the constraints:

$$
\forall v \in\{0,1\}^{r-1}, \quad \alpha_{v}(\delta)=0
$$

because it must correspond to a $u$-only counting form $w_{\delta}$. If these two sets of constraints hold for some feasible $\alpha(\delta)=\alpha^{*}(\delta) \in \mathbb{R}^{3^{r-1}}$, then the $u$-only counting form $w_{\delta}^{*}$ defined by the coefficients $\alpha^{*}(\delta)$ satisfies both (69) and (70) with $C=1$, which implies, via Proposition 7, that $w_{\delta}^{*}$ also satisfies both (66) and (67) with $C=1$ (see the discussion following Proposition 7). This completes the proof.

Proposition 10 can be exploited to computationally find counting forms that satisfy the conditions of Theorem 3. As noted in subsection II-B, the MATLAB and CVX based simulation results in Table I solve the LP in (82) of Proposition 10 to numerically construct $u$-only counting forms $w_{\delta}^{*}$ with pure rank 3 that satisfy (69) and (70) with $C=1$. In particular, for any noise level $\delta \in\left(0, \frac{1}{2}\right)$, the constraint matrices (80) and (81) can be explicitly written out (as expounded earlier), and the corresponding feasible vectors of coefficients $\alpha^{*}(\delta) \in \mathbb{R}^{27}$ that solve the LP in (82) (with reasonably small $\ell^{1}$-norm), depicted in Table I, define $u$-only counting forms $w_{\delta}^{*}$ with pure rank 3 via (83). Furthermore, inspired by the reasons outlined at the end of subsection V-B, we set $r=4$ for these simulations; indeed, the proof of ergodicity of the 1D PCA with NAND gates also uses counting forms with rank 3 [39, Section 2.2].

To conclude this section, we make a few other noteworthy remarks. As we indicated earlier in subsection II-B, Theorem 3 and simulations based on the other arguments in this section yield non-rigorous computer-assisted proofs of the impossibility of broadcasting on NAND 2D regular grids. Indeed, for every fixed noise level $\delta \in\left(0, \frac{1}{2}\right)$ where we can compute a feasible solution $\alpha^{*}(\delta)$ of the LP in (82), such as the values of $\delta$ that feature in Table I, Proposition 10 portrays that we can obtain a $u$-only counting form $w_{\delta}^{*}$ with pure rank 3 that satisfies both (69) and (70) with $C=1$, and Theorem 3 then implies that Conjecture 1 is true for this $\delta$. We emphasize that the computer-assisted part of this argument is not rigorous without further careful analysis, because we use floating-point arithmetic to solve the LP in (82). If we could instead solve an integer LP akin to (82) (when $\delta$ is rational) or utilize interval arithmetic, then our feasible solution $\alpha^{*}(\delta)$ would be deemed rigorous, and the discussion in section V would truly yield a computer-assisted proof.

Nevertheless, even with a rigorous computational technique, the above approach only demonstrates the impossibility of broadcasting for specific values of $\delta$. So, while this approach lends credence to our development of counting forms as a framework for constructing superharmonic potential functions, it does not enable us to establish Conjecture 1 for all $\delta \in\left(0, \frac{1}{2}\right)$. A promising future direction to remedy this issue is to develop a more sophisticated semidefinite programming formulation for the problem of constructing $u$-only counting forms satisfying (69) and (70) for all $\delta \in\left(0, \frac{1}{2}\right)$. For instance, consider the key constraint $\mathbf{A}(\delta) \phi(\delta) \geq \xi$ in the LP in (82) without fixing $\delta$. Let us restrict the vector $\phi(\delta)$ in (82) to be a polynomial vector in $\delta$ with some (sufficiently large) fixed degree and unknown coefficients. Then, since $\mathbf{C}_{r-1}(\delta)$ is a known polynomial matrix in $\delta$ with degree $2(r-1)$ (see (99) in appendix C ), the constraint $\mathbf{A}(\delta) \phi(\delta) \geq \xi$ is equivalent to simultaneously ensuring the non-negativity of a collection of univariate polynomials in $\delta$, whose coefficients are affine functions of the unknown coefficients of $\phi(\delta)$, over the interval $\left[0, \frac{1}{2}\right]$. Since the Markov-Lukács theorem provides well-known necessary and sufficient conditions for the non-negativity of univariate polynomials in terms of sum of squares (SOS) characterizations [57], [58] (also see [59, Theorems 1.21 .1 and 1.21.2]), we can represent the aforementioned simultaneous non-negativity of polynomials in $\delta$ over $\left[0, \frac{1}{2}\right]$ using an SOS program. Such SOS programs can be reformulated as semidefinite programs (SDPs) and computationally solved using standard optimization tools, cf. [60], [61]. Hence, one could computationally
establish the existence of $u$-only counting forms satisfying (69) and (70) for all $\delta \in\left(0, \frac{1}{2}\right)$ using a single SDP. This, in conjunction with rigorous interval arithmetic analysis, could be an avenue for proving Conjecture 1 in its entirety. (Additionally, we remark that this connection between SDPs and the existence of pertinent counting forms is somewhat natural, because SDPs have been very useful in constructing Lyapunov functions in many control theoretic problems, and there are limpid parallels between superharmonic functions and Lyapunov functions as discussed earlier.)

Lastly, we reiterate that while subsection II-B and section V present a detailed martingale-based approach to prove the impossibility of broadcasting on the NAND 2D regular grid, they also provide a general program for establishing the impossibility of broadcasting on 2D regular grids with other Boolean processing functions. Indeed, the arguments in section V can be easily carried out for other processing functions, such as AND or XOR, with minor cosmetic changes, e.g. the elements of the matrix $\mathbf{C}_{r-1}(\delta)$ representing $\mathcal{E}$ need to be modified accordingly.

## VI. Conclusion

In closing, we briefly summarize the main contributions of this paper. Propelled by the positive rates conjecture for 1D PCA, we conjectured in subsection I-A that broadcasting is impossible on 2D regular grids when all vertices with two inputs use a common Boolean processing function. We made considerable progress towards establishing this conjecture, and proved impossibility results for 2D regular grids with all AND and all XOR processing functions in Theorems 1 and 2, respectively. Specifically, our proof for the AND 2D regular grid utilized phase transition results concerning oriented bond percolation in 2D lattices, and our proof for the XOR 2D regular grid relied on analyzing the equivalent problem of decoding the first bit of a codeword drawn uniformly from of a linear code. Then, we showed in Theorem 3 that an impossibility result for 2D regular grids with all NAND processing functions follows from the existence of certain structured supermartingales. Moreover, in much of section V, we developed the notion of counting forms and elucidated graph theoretic and linear programming based criteria to help construct the desired superharmonic potential functions of Theorem 3. These ideas were utilized to depict several numerical examples of such potential functions in Table I. Finally, we list some fruitful directions of future research below:

1) The program in subsection II-B and section $V$ can be rigorously executed to prove the impossibility of broadcasting on the NAND 2D regular grid for all $\delta \in\left(0, \frac{1}{2}\right)$. In particular, our suggestion in subsection V-E of using SDPs to obtain a computer-assisted proof could be pursued.
2) In order to completely resolve the 2 D version of our conjecture in subsection I-A, the impossibility of broadcasting on 2D regular grids with IMP processing functions must also be established (see (6) in subsection II-B). This is another salient open problem.
3) In order to prove the 3 D part of our broader conjecture in subsection I-A that broadcasting is feasible in 3 or more dimensions for sufficiently low noise levels $\delta$, broadcasting on the majority 3D regular grid can be examined. As indicated by Proposition 3 and discussed in subsection II-D, it may be possible to draw on ideas from the analysis of Toom's NEC rule in [36] and [44] to establish the possibility of broadcasting on the majority 3D regular grid.
4) Much like the correspondence between our 2D regular grid model with NAND processing functions and the 1D PCA with NAND gates in [39], we can define a $2 \mathrm{D} 45^{\circ}$ grid model with 3 -input majority processing functions corresponding to Gray's 1D PCA with 3 -input majority gates [25, Example 5]. In particular, a 2D $45^{\circ}$ grid model has $L_{k}=2 k+1$ vertices at each level $k \in \mathbb{N}$, and every vertex has three outgoing edges that have $45^{\circ}$ separation between them. ${ }^{20}$ Furthermore, all vertices in the 2D $45^{\circ}$ grid model which are at least two positions away from the boundary have three incoming edges. Although Gray's proof sketch of the ergodicity of the 1D PCA with 3-input majority gates in [25, Section 3] shows exponentially fast convergence for sufficiently small noise levels, it includes noise on the vertices (rather than the edges) and obviously does not account for the boundary effects of 2 D grid models. It is therefore an interesting future endeavor to study broadcasting in the $2 \mathrm{D} 45^{\circ}$ grid model with 3 -input majority processing functions by building upon the reasoning in [25, Section 3].

## Appendix A

## Proof of Proposition 3

Proof. To prove this proposition, we will perform a simple projection argument. Specifically, for any $k \in \mathbb{N}$, we will project any vertex $v=\left(v_{1}, v_{2}, v_{3}\right) \in \mathcal{S}_{k}$ of the majority 3D regular grid onto its first two coordinates to obtain

[^16]a site $v \mapsto\left(v_{1}, v_{2}\right) \in \hat{\mathcal{S}}_{k}$ at time $k$ of Toom's 2D PCA with boundary conditions. (Note that this projection map is bijective, because for any site $x=\left(x_{1}, x_{2}\right) \in \hat{\mathcal{S}}_{k}$ at time $k \in \mathbb{N}$ of Toom's 2D PCA with boundary conditions, we can recover the corresponding vertex of the majority 3D regular grid via the affine map $x \mapsto\left(x_{1}, x_{2}, k-x_{1}-x_{2}\right) \in \mathcal{S}_{k}$.)

To this end, let us first couple the i.i.d. Bernoulli $(\delta)$ random variables $\left\{Z_{v, i}: v \in \mathbb{N}^{3} \backslash\{\mathbf{0}\}, i \in\{1,2,3\}\right\}$ with the i.i.d. Bernoulli $(\delta)$ random variables $\left\{Z_{k, x, v}: k \in \mathbb{N}, x \in \hat{\mathcal{S}}_{k+1}, v \in \mathcal{N}\right\}$ such that:

$$
\begin{gather*}
\forall v=\left(v_{1}, v_{2}, v_{3}\right) \in \mathbb{N}^{3} \backslash\{\mathbf{0}\}, Z_{v, i}=Z_{v_{1}+v_{2}+v_{3}-1,\left(v_{1}, v_{2}\right),-e_{i}} \text { for } i \in\{1,2\},  \tag{86}\\
\text { and } Z_{v, 3}=Z_{v_{1}+v_{2}+v_{3}-1,\left(v_{1}, v_{2}\right), \mathbf{0}},
\end{gather*}
$$

or equivalently:

$$
\begin{aligned}
\forall k \in \mathbb{N}, \forall x=\left(x_{1}, x_{2}\right) \in \hat{\mathcal{S}}_{k+1}, \quad Z_{k, x,-e_{i}} & =Z_{\left(x_{1}, x_{2}, k+1-x_{2}-x_{2}\right), i} \text { for } i \in\{1,2\}, \\
\text { and } Z_{k, x, \mathbf{0}} & =Z_{\left(x_{1}, x_{2}, k+1-x_{2}-x_{2}\right), 3}
\end{aligned}
$$

almost surely. The rest of the proof proceeds by strong induction.
Notice that the base case, $X_{\mathbf{0}}=\xi_{0}(\mathbf{0})$ almost surely, holds by assumption. Suppose further that the inductive hypothesis:

$$
\begin{equation*}
\forall v=\left(v_{1}, v_{2}, v_{3}\right) \in \bigcup_{m=0}^{k} \mathcal{S}_{m}, \quad X_{v}=\xi_{v_{1}+v_{2}+v_{3}}\left(\left(v_{1}, v_{2}\right)\right) \quad \text { almost surely } \tag{87}
\end{equation*}
$$

holds for some $k \in \mathbb{N}$. Now consider any vertex $v=\left(v_{1}, v_{2}, v_{3}\right) \in \mathcal{S}_{k+1}$. There are several cases for us to verify:

1) If $v_{1}=v_{2}=0$ and $v_{3}=k+1$, then we have almost surely:

$$
X_{v}=X_{v-e_{3}} \oplus Z_{v, 3}=\xi_{k}(\mathbf{0}) \oplus Z_{k, \mathbf{0}, \mathbf{0}}=\xi_{k+1}(\mathbf{0})
$$

using (10), (87), (86), and (13).
2) If there exists $i \in\{1,2\}$ such that $v_{i}=k+1$, then we have almost surely:

$$
X_{v}=X_{v-e_{i}} \oplus Z_{v, i}=\xi_{k}\left(k e_{i}\right) \oplus Z_{k,(k+1) e_{i},-e_{i}}=\xi_{k+1}\left(\left(v_{1}, v_{2}\right)\right)
$$

using (10), (87), (86), and (14).
3) If there exists $i \in\{1,2\}$ such that $v_{i}+v_{3}=k+1$, then we have almost surely:

$$
X_{v}= \begin{cases}X_{v-e_{3}} \oplus Z_{v, 3}=\xi_{k}\left(\left(v_{1}, v_{2}\right)\right) \oplus Z_{k,\left(v_{1}, v_{2}\right), \mathbf{0},} \quad \text { with probability } \frac{1}{2} \\ X_{v-e_{i}} \oplus Z_{v, i}=\xi_{k}\left(\left(v_{1}, v_{2}\right)-e_{i}\right) \oplus Z_{k,\left(v_{1}, v_{2}\right),-e_{i}}, & \text { with probability } \frac{1}{2}=\xi_{k+1}\left(\left(v_{1}, v_{2}\right)\right)\end{cases}
$$

using (11), (87), (86), and (15).
4) If $v_{1}+v_{2}=k+1$ and $v_{3}=0$, then we have almost surely:

$$
X_{v}=\left\{\begin{array}{l}
X_{v-e_{1}} \oplus Z_{v, 1}=\xi_{k}\left(\left(v_{1}-1, v_{2}\right)\right) \oplus Z_{k,\left(v_{1}, v_{2}\right),-e_{1}}, \text { with probability } \frac{1}{2}=\xi_{k+1}\left(\left(v_{1}, v_{2}\right)\right) \\
X_{v-e_{2}} \oplus Z_{v, 2}=\xi_{k}\left(\left(v_{1}, v_{2}-1\right)\right) \oplus Z_{k,\left(v_{1}, v_{2}\right),-e_{2}}, \text { with probability } \frac{1}{2}=\xi_{k}
\end{array}\right.
$$

using (11), (87), (86), and (16).
5) If $v_{1}, v_{2}, v_{3}>0$, then we have almost surely:

$$
\begin{aligned}
X_{v} & =\operatorname{majority}\left(X_{v-e_{1}} \oplus Z_{v, 1}, X_{v-e_{2}} \oplus Z_{v, 2}, X_{v-e_{3}} \oplus Z_{v, 3}\right) \\
& =\operatorname{majority}\left(\xi_{k}\left(\left(v_{1}-1, v_{2}\right)\right) \oplus Z_{k,\left(v_{1}, v_{2}\right),-e_{1}}, \xi_{k}\left(\left(v_{1}, v_{2}-1\right)\right) \oplus Z_{k,\left(v_{1}, v_{2}\right),-e_{2}}\right. \\
& \left.\quad \xi_{k}\left(\left(v_{1}, v_{2}\right)\right) \oplus Z_{k,\left(v_{1}, v_{2}\right), \mathbf{0}}\right) \\
& =\xi_{k+1}\left(\left(v_{1}, v_{2}\right)\right)
\end{aligned}
$$

using (12), (87), (86), and (17).
This shows that for every $v=\left(v_{1}, v_{2}, v_{3}\right) \in \mathcal{S}_{k+1}$, we have $X_{v}=\xi_{k+1}\left(\left(v_{1}, v_{2}\right)\right)$ almost surely. Therefore, our coupling of the i.i.d. Bernoulli $(\delta)$ random variables in (86) establishes the proposition statement by induction.

## Appendix B <br> Proof of Theorem 4

We will prove Theorem 4 in this appendix. Throughout section $V$, we intentionally circumvented an exposition of the algebraic topological perspective of counting forms since it does not help us establish Theorem 3 or perform the simulations illustrated in Table I. However, in order to derive Theorem 4 and elucidate the elegant intuition behind the counting forms $\left\{\rho_{v}: v \in \mathcal{Y}^{s-1}\right\}$ in (71), we have to introduce some basic ideas from algebraic graph theory. Although some of these ideas can be found in or derived from standard expositions, cf. [55], [62], we develop the details here in a manner that is directly relevant to us.

Consider a strongly connected finite directed graph $\mathcal{G}=(\mathcal{V}, E)$ with vertex set $\mathcal{V}$ and edge set $E$. (Note that $\mathcal{G}$ has no isolated vertices, but $\mathcal{G}$ may contain self-loop edges.) Given $\mathcal{G}$, we begin with a list of useful definitions:

1) For any codomain $S \subseteq \mathbb{R}$, we refer to a map $f: E \rightarrow S$ as an $S$-edge function. Moreover, we refer to $\mathbb{R}$-edge functions as simply edge functions.
2) An edge function $f$ is said to be closed if for every directed circuit $\left(e_{1}, \ldots, e_{n}\right) \in E^{n}$ in $\mathcal{G},{ }^{21}$ we have:

$$
\begin{equation*}
\sum_{k=1}^{n} f\left(e_{k}\right)=0 \tag{88}
\end{equation*}
$$

where the length $n \in \mathbb{N}$ of the circuit is arbitrary.
3) For the fields $S=\mathbb{R}$ and $S=\mathbb{Q}$, the set of all $S$-edge functions forms an inner product space over $S$ with inner product given by:

$$
\begin{equation*}
\langle f, g\rangle \triangleq \sum_{e \in E} f(e) g(e) \tag{89}
\end{equation*}
$$

for all pairs of $S$-edge functions $f, g$.
4) The divergence of an edge function $f$ at a vertex $v \in \mathcal{V}$ is defined as:

$$
\begin{equation*}
(\operatorname{div} f)(v) \triangleq \sum_{\substack{e \in E: \\ v \text { has incoming edge } e}} f(e)-\sum_{\substack{e^{\prime} \in E: \\ v \text { has outgoing edge } e^{\prime}}} f\left(e^{\prime}\right) \tag{90}
\end{equation*}
$$

Furthermore, we write $\operatorname{div} f=0$ to denote that $(\operatorname{div} f)(v)=0$ for all vertices $v \in \mathcal{V}$.
5) For every vertex $v \in \mathcal{V}$, we define an associated $\mathbb{Z}$-edge function $\varrho_{v}$ as follows:

$$
\begin{equation*}
\forall e \in E, \varrho_{v}(e) \triangleq \mathbb{1}\{v \text { has incoming edge } e\}-\mathbb{1}\{v \text { has outgoing edge } e\} \tag{91}
\end{equation*}
$$

These edge functions are actually gradients of characteristic (or indicator) functions of vertices in $\mathcal{V}$. They can also be perceived as rows of an oriented incidence matrix of $\mathcal{G}$.
6) Lastly, for every directed circuit $H=\left(e_{1}, \ldots, e_{n}\right) \in E^{n}$ in $\mathcal{G}$ (where $n \in \mathbb{N}$ is arbitrary), we define an associated $\mathbb{N}$-edge function $f_{H}$ as follows:

$$
\begin{equation*}
\forall e \in E, \quad f_{H}(e) \triangleq \sum_{k=1}^{n} \mathbb{1}\left\{e_{k}=e\right\} \tag{92}
\end{equation*}
$$

$f_{H}$ encodes the circuit $H$ by counting the number of times each edge is traversed. (This parallels the encoding used in Lemma 6.) Moreover, since the number of incoming edges equals the number of outgoing edges for every vertex in a circuit, it is straightforward to verify that div $f_{H}=0$.
We will now state the result that includes Theorem 4 as a special case (the reduction is given at the end of this appendix).

Theorem 4B (Closed $\mathbb{R}$-Edge Functions). Given any strongly connected finite directed graph $\mathcal{G}$, an edge function $f$ is closed if and only if there exist coefficients $\alpha_{v} \in \mathbb{R}$ for $v \in \mathcal{V}$ such that:

$$
f=\sum_{v \in \mathcal{V}} \alpha_{v} \varrho_{v}
$$

The dimension of the set of closed edge functions equals $|\mathcal{V}|-1$.

[^17]Proof. We are trying to prove that the image of the linear map $G: \mathbb{R}^{\mathcal{V}} \rightarrow \mathbb{R}^{E}$ given by:

$$
\mathrm{G}(\alpha) \triangleq \sum_{v \in \mathcal{V}} \alpha_{v} \varrho_{v}
$$

coincides with the kernel of the map $\mathrm{H}: \mathbb{R}^{E} \rightarrow \mathbb{R}^{m}$ mapping an edge function to its evaluation on each of (finitely many, $m$ ) circuits in $\mathcal{G}$. Since $\mathbb{R}$ can be seen as an (infinite dimensional) $\mathbb{Q}$-vector space, it is a flat $\mathbb{Q}$-module over $\mathbb{Q}$. Thus, by tensoring the exact sequence formed by $G, H$ with $\mathbb{R}$, we see that equality of image of $G$ with the kernel of H can be checked over $\mathbb{Q}$. From now on we restrict to $\mathbb{Q}$-edge functions.

Denote:

$$
V_{1}=\operatorname{span}\left(\left\{\varrho_{v}: v \in V\right\}\right), \quad V_{2}=\{f: \operatorname{div} f=0\}
$$

We claim that:

$$
\begin{equation*}
V_{1}^{\perp}=V_{2} \tag{93}
\end{equation*}
$$

or $f \in V_{1}$ if and only if $\langle f, g\rangle=0$ for all $g \in V_{2}$. Indeed, from the definition:

$$
(\operatorname{div} g)(v)=0 \quad \Longleftrightarrow \quad\left\langle\varrho_{v}, g\right\rangle=0
$$

In view of (93) we see that the statement of the theorem (over $\mathbb{Q}$ ) is equivalent to the following claim: $A \mathbb{Q}$-edge function $g$ satisfies $\operatorname{div} g=0$ if and only if $g$ is a finite linear combination of $\left\{f_{H}: H\right.$ is a circuit in $\left.\mathcal{G}\right\}$ with rational coefficients. One direction is clear: $\operatorname{div} f_{H}=0$ for any circuit $H$ in $\mathcal{G}$ since every vertex has the same number of incoming and outgoing edges in a circuit.

For the converse direction, first we consider a non-zero $\mathbb{N}$-edge function $g$ satisfying div $g=0$. Take any edge $e_{1} \in E$ such that $g\left(e_{1}\right)>0$, and let $e_{1}$ have destination vertex $v \in \mathcal{V}$. Then, since $(\operatorname{div} g)(v)=0$, there exists an edge $e_{2} \in E$ with source vertex $v$ such that $g\left(e_{2}\right)>0$. We can repeatedly generate new edges $e \in E$ with $g(e)>0$ using this argument until we revisit an already visited vertex. (This must happen because $\mathcal{G}$ is finite.) This yields a directed cycle $H=\left(e_{1}^{\prime}, \ldots, e_{n}^{\prime}\right) \in E^{n}$ with some length $n \in \mathbb{N}$ such that $g\left(e_{i}^{\prime}\right)>0$ for all $i \in\{1, \ldots, n\}$. Notice that $g-f_{H}$ is also an $\mathbb{N}$-edge function. So, we can iterate this procedure. This shows that any $\mathbb{N}$-edge function $g$ satisfies $\operatorname{div} g=0$ if and only if $g$ can be written as:

$$
g=\sum_{\text {directed cycles } H} \beta_{H} f_{H}
$$

for some choice of coefficients $\beta_{H} \in \mathbb{N}$, where the sum is over all directed cycles of $\mathcal{G} .{ }^{22}$
Now suppose that $g$ is a $\mathbb{Q}$-edge function satisfying $\operatorname{div} g=0$ (as in the above claim). Let $J$ be a directed circuit passing through every edge of $\mathcal{G}$, which exists because $\mathcal{G}$ is strongly connected, and let $f_{J}$ be its corresponding $\mathbb{N}$ edge function. Then, since $\operatorname{div} f_{J}=0$ and $f_{J}(e)>0$ for all $e \in E$, for some sufficiently large integers $k_{1}, k_{2} \in \mathbb{N}$, $g^{\prime}=k_{1} g+k_{2} f_{J}$ is an $\mathbb{N}$-edge function satisfying $\operatorname{div} g^{\prime}=0$. Thus, $g^{\prime}$ can be represented as:

$$
k_{1} g+k_{2} f_{J}=g^{\prime}=\sum_{\text {directed cycles } H} \beta_{H} f_{H}
$$

for some choice of coefficients $\beta_{H} \in \mathbb{N}$, as argued above. This implies that $g$ is a finite linear combination of $\left\{f_{H}: H\right.$ is a circuit in $\left.\mathcal{G}\right\}$ with rational coefficients. This establishes the claim.

The statement about dimension follows from [55, Theorem 8.3.1], which shows $\operatorname{dim} V_{1}=|\mathcal{V}|-1$, after we use the facts that $\mathcal{G}$ is strongly connected and that $\left\{\varrho_{v}: v \in \mathcal{V}\right\}$ define the rows of the oriented incidence matrix of $\mathcal{G}$. This completes the proof.

Proof of Theorem 4. Akin to (72), consider any counting form $w$ with pure rank $s$ defined by the formal sum:

$$
w=\sum_{e \in \mathcal{Y}^{s}} \alpha_{e}\{e\}
$$

with coefficients $\left\{\alpha_{e} \in \mathbb{R}: e \in \mathcal{Y}^{s}\right\}$. Let $\mathcal{G}$ be the strongly connected directed graph with vertex set $\mathcal{Y}^{s-1}$ and directed edge set $\mathcal{Y}^{s}$, where any edge $\left(v_{1} \cdots v_{s}\right) \in \mathcal{Y}^{s}$ has source vertex $\left(v_{1} \cdots v_{s-1}\right)$ and destination vertex $\left(v_{2} \cdots v_{s}\right)$. Note that $\mathcal{G}$ has the same underlying structure as the graph constructed in subsection V-C, but it identifies

[^18]edges (rather than vertices) with basis clauses of rank $s .{ }^{23}$ Define the edge function $f: \mathcal{Y}^{s} \rightarrow \mathbb{R}$ corresponding to $w$ such that:
$$
\forall e \in \mathcal{Y}^{s}, \quad f(e)=\alpha_{e}
$$

By treating $f$ as a weight function on the edges of $\mathcal{G}$, the proof of Proposition 8 in subsection V-C, mutatis mutandis, shows that $w \simeq_{c} \mathbf{0}$ if and only if $f$ is closed. (We omit the details of this argument for brevity.) Hence, applying Theorem 4B yields that $w \simeq_{c} \mathbf{0}$ if and only if $f \in \operatorname{span}\left(\left\{\varrho_{v}: v \in \mathcal{Y}^{s-1}\right\}\right)$. Now, using the structure of $\mathcal{G}$ and the definitions in (71) and (91), it is easy to see that for all $v \in \mathcal{Y}^{s-1}$, the $\mathbb{Z}$-edge function $-\varrho_{v}$ defines the coefficients of the counting form $\rho_{v}$ :

$$
\rho_{v}=-\sum_{e \in \mathcal{Y}^{s}} \varrho_{v}(e)\{e\}
$$

Therefore, since $f$ encodes the coefficients of $w, w \simeq_{c} \mathbf{0}$ if and only if $w \in \operatorname{span}\left(\left\{\rho_{v}: v \in \mathcal{Y}^{s-1}\right\}\right)$. Lastly, by Theorem 4B, we get that $\operatorname{span}\left(\left\{\rho_{v}: v \in \mathcal{Y}^{s-1}\right\}\right.$ ) has dimension $\left|\mathcal{Y}^{s-1}\right|-1=3^{s-1}-1$ (using (43)). This completes the proof.

## Appendix C

## Necessary Conditions to Satisfy (69) for Sufficiently Small Noise

In this appendix, we consider the setting of sufficiently small noise $\delta>0$, which is intuitively the most challenging regime to establish the impossibility of broadcasting in. In particular, we will computationally construct a $u$-only counting form with rank 3 which provides evidence for the existence of $u$-only counting forms that satisfy the supermartingale condition (69) for all sufficiently small $\delta>0$. (Note that we do not consider the constraint (70) in this appendix.)

To present our computer-assisted construction, define the function $T: \mathcal{Y} \rightarrow\{0,1,2\}, T(y)=2 \mathbb{1}\{y=u\}+$ $y \mathbb{1}\{y \in\{0,1\}\}$, which changes $u$ 's into 2 's. Moreover, for the linear subspaces of counting forms with pure rank $s \in \mathbb{N} \backslash\{0\}$, define the ensuing maps that equivalently represent these counting forms as column vectors in $\mathbb{R}^{3^{s}}$ :

$$
\begin{align*}
& w=\sum_{v \in \mathcal{Y}^{s}} \alpha_{v}\{v\} \quad \mapsto \quad \zeta_{s}(w) \triangleq\left[\tilde{\alpha}(0) \cdots \tilde{\alpha}\left(3^{s}-1\right)\right]^{\mathrm{T}} \in \mathbb{R}^{3^{s}}  \tag{94}\\
& \forall v=\left(v_{s-1} \cdots v_{0}\right) \in \mathcal{Y}^{s}, \tilde{\alpha}\left(\sum_{i=0}^{s-1} T\left(v_{i}\right) 3^{i}\right)=\alpha_{v} \tag{95}
\end{align*}
$$

where $\left\{\alpha_{v} \in \mathbb{R}: v \in \mathcal{Y}^{s}\right\}$ denote the coefficients of the counting form $w$ with pure rank $s$, and we write these coefficients as column vectors whose indices are the lexicographically ordered $s$-length strings interpreted as ternary representations of non-negative integers.

Now consider the conditional expectation operator $\mathcal{E}$ in Definition 3 acting on the linear subspace of all counting forms with pure rank $s$. Then, since the codomain of $\mathcal{E}$ is the linear subspace of all counting forms with pure rank $s+1$, we can equivalently represent $\mathcal{E}$ using a $3^{s+1} \times 3^{s}$ matrix $\mathbf{C}_{s}(\delta)$ such that for every counting form $w$ with pure rank $s$, we have:

$$
\begin{equation*}
\zeta_{s+1}(\mathcal{E}(w))=\mathbf{C}_{s}(\delta) \zeta_{s}(w) \tag{96}
\end{equation*}
$$

Note that when $s=1$, the matrix $\mathbf{C}_{1}(\delta)$ is given by:

[^19]where we label the indices of the rows and columns using $\mathcal{Y}^{2}$ and $\mathcal{Y}$, respectively, for illustrative purposes. (Note, for example, that the rows are actually indexed by [9] since each $\left(v_{1} v_{0}\right) \in \mathcal{Y}^{2}$ corresponds to a non-negative integer $T\left(v_{0}\right)+3 T\left(v_{1}\right)$ using (95).) Furthermore, when $s \geq 2$, for every $v=\left(v_{s} \cdots v_{0}\right) \in \mathcal{Y}^{s+1}$ and every $y=\left(y_{s-1} \cdots y_{0}\right) \in \mathcal{Y}^{s}$, the $(a, b)$ th element of the matrix $\mathbf{C}_{s}(\delta)$, with $a=\sum_{i=0}^{s} T\left(v_{i}\right) 3^{i}$ and $b=\sum_{j=0}^{s-1} T\left(y_{j}\right) 3^{j}$, is given by:
\[

$$
\begin{equation*}
\forall \delta \in\left[0, \frac{1}{2}\right],\left[\mathbf{C}_{s}(\delta)\right]_{a, b} \triangleq \prod_{k=0}^{s-1}\left[\mathbf{C}_{1}(\delta)\right]_{T\left(v_{k}\right)+3 T\left(v_{k+1}\right), T\left(y_{k}\right)}=\mathbb{P}\left(\left(Y_{s+1,1}, \ldots, Y_{s+1, s}\right)=v \mid Y_{s}=y\right) \tag{98}
\end{equation*}
$$

\]

according to Definition 3. Since $\mathbf{C}_{1}(\delta)$ is a quadratic polynomial matrix in $\delta$ (see (97)), it follows that $\mathbf{C}_{s}(\delta)$ is a polynomial matrix in $\delta$ with degree $2 s$, and we may write it as:

$$
\begin{equation*}
\forall \delta \in\left[0, \frac{1}{2}\right], \quad \mathbf{C}_{s}(\delta)=\sum_{k=0}^{2 s} \mathbf{C}_{s}^{(k)} \delta^{k} \tag{99}
\end{equation*}
$$

where each $3^{s+1} \times 3^{s}$ coefficient matrix $\mathbf{C}_{s}^{(k)}$ does not depend on $\delta$ (and is explicitly known).
Next, suppose we seek to find a $u$-only counting form $w_{\delta}$ with pure rank $s$ satisfying (69) that can be represented as a linear polynomial vector:

$$
\begin{equation*}
\forall \delta \in\left[0, \frac{1}{2}\right], \zeta_{s}\left(w_{\delta}\right)=\hat{w}_{0}+\hat{w}_{1} \delta \tag{100}
\end{equation*}
$$

where $\hat{w}_{0}, \hat{w}_{1} \in \mathbb{R}^{3^{s}}$ do not depend on $\delta$. Define the $3^{s+1} \times 3^{s}$ "purification matrix" $P_{s}$ via (see Lemma 5 and the discussion preceding Theorem 4):

$$
P_{s} \triangleq I_{s} \otimes\left[\begin{array}{lll}
1 & 1 & 1 \tag{101}
\end{array}\right]^{\mathrm{T}}
$$

where $I_{s}$ denotes the $3^{s} \times 3^{s}$ identity matrix, and $\otimes$ denotes the Kronecker product of matrices. Note that for any $u$-only counting form $w$ with pure rank $s, P_{s} \zeta_{s}(w)$ produces the coefficients of a $u$-only counting form with pure rank $s+1$ that is cyclically equivalent to $w$. To satisfy the supermartingale condition (69), $w_{\delta}-\mathcal{E}\left(w_{\delta}\right) \succeq_{\mathrm{c}} \mathbf{0}$, observe that it suffices to ensure that:

$$
\begin{equation*}
\exists z_{\delta} \in \operatorname{span}\left(\left\{\rho_{v}: v \in \mathcal{Y}^{s} \backslash\{0,1\}^{s}\right\}\right), \underbrace{P_{s} \zeta_{s}\left(w_{\delta}\right)}_{\simeq_{c} w_{\delta}}-\underbrace{\mathbf{C}_{s}(\delta) \zeta_{s}\left(w_{\delta}\right)}_{=\zeta_{s+1}\left(\mathcal{E}\left(w_{\delta}\right)\right)}+\underbrace{\zeta_{s+1}\left(z_{\delta}\right)}_{\substack{\text { does not change } \\ w_{\delta}-\mathcal{E}\left(w_{\delta}\right)}} \geq \mathbf{0} \tag{102}
\end{equation*}
$$

for all $\delta \in\left[0, \frac{1}{2}\right]$, where we utilize the special case of the converse direction of Theorem 4 for $u$-only counting forms (and (96)), and the $\geq$ relation holds entry-wise. Indeed, if all the coefficients of a counting form that is cyclically equivalent to $w_{\delta}-\mathcal{E}\left(w_{\delta}\right)$ are non-negative, then $w_{\delta}-\mathcal{E}\left(w_{\delta}\right) \succeq_{c} \mathbf{0}$ using Definitions 5 and 6 . To simplify (102) further, observe using (96), (99), and (100) that:

$$
\begin{align*}
\zeta_{s+1}\left(\mathcal{E}\left(w_{\delta}\right)\right) & =\left(\sum_{k=0}^{2 s} \mathbf{C}_{s}^{(k)} \delta^{k}\right)\left(\hat{w}_{0}+\hat{w}_{1} \delta\right)=\mathbf{C}_{s}^{(0)} \hat{w}_{0}+\left(\mathbf{C}_{s}^{(0)} \hat{w}_{1}+\mathbf{C}_{s}^{(1)} \hat{w}_{0}\right) \delta+O\left(\delta^{2}\right)  \tag{103}\\
P_{s} \zeta_{s}\left(w_{\delta}\right) & =P_{s} \hat{w}_{0}+P_{s} \hat{w}_{1} \delta \tag{104}
\end{align*}
$$

where $O\left(\delta^{2}\right)$ is the standard big- $O$ asymptotic notation (as $\delta \rightarrow 0^{+}$). This implies that:

$$
\begin{equation*}
P_{s} \zeta_{s}\left(w_{\delta}\right)-\zeta_{s+1}\left(\mathcal{E}\left(w_{\delta}\right)\right)=\left(P_{s}-\mathbf{C}_{s}^{(0)}\right) \hat{w}_{0}+\left(\left(P_{s}-\mathbf{C}_{s}^{(0)}\right) \hat{w}_{1}-\mathbf{C}_{s}^{(1)} \hat{w}_{0}\right) \delta+O\left(\delta^{2}\right) \tag{105}
\end{equation*}
$$

Furthermore, every $u$-only counting form $z_{\delta} \in \operatorname{span}\left(\left\{\rho_{v}: v \in \mathcal{Y}^{s} \backslash\{0,1\}^{s}\right\}\right)$ can be represented as $\zeta_{s+1}\left(z_{\delta}\right)=$ $\mathbf{D}_{s} \tilde{x}_{\delta}$ for some $\delta$-dependent vector $\tilde{x}_{\delta} \in \mathbb{R}^{3^{s}-2^{s}}$, where $\mathbf{D}_{s}$ denotes the (explicitly known and $\delta$-independent) $3^{s+1} \times\left(3^{s}-2^{s}\right)$ matrix whose columns are given by:

$$
\begin{equation*}
\mathbf{D}_{s}=\left[\zeta_{s+1}\left(\rho_{v}\right): v \in \mathcal{Y}^{s} \backslash\{0,1\}^{s}\right] \tag{106}
\end{equation*}
$$

Suppose $\tilde{x}_{\delta}=x_{0}+x_{1} \delta$ for $x_{0}, x_{1} \in \mathbb{R}^{3^{s}-2^{s}}$ that do not depend on $\delta$. Then, in the limit when $\delta \rightarrow 0^{+}$, using (105) and (106), (102) can be written as:

$$
\begin{equation*}
\exists x_{0}, x_{1} \in \mathbb{R}^{3^{s}-2^{s}},\left(\left(P_{s}-\mathbf{C}_{s}^{(0)}\right) \hat{w}_{0}+\mathbf{D}_{s} x_{0}\right)+\left(\left(P_{s}-\mathbf{C}_{s}^{(0)}\right) \hat{w}_{1}-\mathbf{C}_{s}^{(1)} \hat{w}_{0}+\mathbf{D}_{s} x_{1}\right) \delta \geq \mathbf{0} \tag{107}
\end{equation*}
$$

where we neglect the $O\left(\delta^{2}\right)$ terms.

To simplify (107) further, let us set $s=3$. So, we seek to find a $u$-only counting form $w_{\delta}$ of the form (100) with pure rank 3 , that satisfies (69) up to first order in the limit as $\delta \rightarrow 0^{+}$. Recall the $u$-only harmonic counting form $w_{*}$ with rank 3 defined in (55), and notice that the proof of Proposition 6 can be easily modified (using purification with respect to $\simeq_{c}$-see the discussion preceding Theorem 4) to yield $w_{*} \simeq_{c} \mathcal{E}\left(w_{*}\right)$ when $\delta=0$. Hence, let us restrict ourselves to $u$-only counting forms $w_{\delta}$ with some fixed $w_{0}=w_{0}^{*} \simeq_{c} w_{*}$ so that $w_{0}^{*} \simeq_{c} \mathcal{E}\left(w_{0}^{*}\right)$, where $w_{0}^{*}$ is $u$-only and has pure rank 3 . Specifically, we use:

$$
\hat{w}_{0}=\hat{w}_{0}^{*} \triangleq \zeta_{3}\left(w_{0}^{*}\right)=\left[\begin{array}{lllllllllllllllllllllllll}
0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 4 & 3 & 3 & 2 \tag{108}
\end{array} 2\right]^{\mathrm{T}} \in \mathbb{R}^{27}
$$

With this choice of $\hat{w}_{0}=\hat{w}_{0}^{*}$, since $w_{0}^{*} \simeq_{c} \mathcal{E}\left(w_{0}^{*}\right)$, we know that the counting form corresponding to $\left(P_{3}-\mathbf{C}_{3}^{(0)}\right) \hat{w}_{0}^{*}+$ $\mathbf{D}_{3} x_{0}$ is $\simeq_{c} \mathbf{0}$ in (107). Therefore, in the limit when $\delta \rightarrow 0^{+}$, the problem of finding $\hat{w}_{1}$ such that (107) is satisfied can be recast as:

$$
\text { Find } \hat{w}_{1} \in \mathbb{R}^{27} \text { and } x_{1} \in \mathbb{R}^{19} \text { such that: }
$$

$$
\begin{align*}
& \left(P_{3}-\mathbf{C}_{3}^{(0)}\right) \hat{w}_{1}-\mathbf{C}_{3}^{(1)} \hat{w}_{0}^{*}+\mathbf{D}_{3} x_{1} \geq \mathbf{0}  \tag{109}\\
& \forall\left(v_{2} v_{1} v_{0}\right) \in\{0,1\}^{3}, \quad\left(\hat{w}_{1}\right)_{v_{0}+3 v_{1}+9 v_{2}}=0
\end{align*}
$$

where $\left(\hat{w}_{1}\right)_{i}$ denotes the $i$ th entry of $\hat{w}_{1}$ for $i \in[27]$, the matrices $P_{3}-\mathbf{C}_{3}^{(0)} \in \mathbb{R}^{81 \times 27}, \mathbf{D}_{3} \in \mathbb{R}^{81 \times 19}$ and the vector $\mathbf{C}_{3}^{(1)} \hat{w}_{0}^{*} \in \mathbb{R}^{81}$ are known, and the equality constraints in (109) ensure that $\hat{w}_{1}$ corresponds to a $u$-only counting form.

The problem in (109) is a linear programming feasibility problem. Using MATLAB, we can solve (109) (with additional constraints) to obtain the following integer-valued solution:

$$
\hat{w}_{1}=\hat{w}_{1}^{*} \triangleq\left[\begin{array}{llllllllllllllllllllllll}
0 & 0 & 2 & 0 & 0 & 4 & 4 & 4 & 3 & 0 & 0 & 4 & 0 & 0 & 4 & 4 & 4 & 4 & 2 & 4 & 4 & 4 & 4 & 3 \tag{110}
\end{array}\right]^{\mathrm{T}} \in \mathbb{N}^{27}
$$

which has a corresponding integer-valued solution $x_{1}=x_{1}^{*} \in \mathbb{Z}^{19}$ that we do not reproduce here. Therefore, the $u$-only counting form $w_{\delta}^{*} \triangleq \zeta_{3}^{-1}\left(\hat{w}_{0}^{*}+\hat{w}_{1}^{*} \delta\right)$ with pure rank 3 has the property that it satisfies (69) up to first order as $\delta \rightarrow 0^{+}$. This provides numerical evidence for the existence of $u$-only counting forms with rank 3 satisfying (69) for all values of $\delta$ in a sufficiently small neighborhood of 0 . Indeed, if we could construct solutions $\hat{w}_{1}^{*}$ and $x_{1}^{*}$ that satisfied the entry-wise strict inequality:

$$
\begin{equation*}
\left(P_{3}-\mathbf{C}_{3}^{(0)}\right) \hat{w}_{1}^{*}-\mathbf{C}_{3}^{(1)} \hat{w}_{0}^{*}+\mathbf{D}_{3} x_{1}^{*}>\mathbf{0} \tag{111}
\end{equation*}
$$

then we would obtain that for all sufficiently small $\delta>0$ :

$$
\begin{equation*}
\left(\left(P_{3}-\mathbf{C}_{3}^{(0)}\right) \hat{w}_{1}^{*}-\mathbf{C}_{3}^{(1)} \hat{w}_{0}^{*}+\mathbf{D}_{3} x_{1}^{*}\right) \delta+O\left(\delta^{2}\right) \geq \mathbf{0} \tag{112}
\end{equation*}
$$

which includes all the higher order terms in (105). This would in turn prove that $w_{\delta}^{*}$ satisfies (69) for all sufficiently small $\delta>0$. Since our solution (110) does not satisfy (111), it only suggests the existence of $u$-only counting forms with rank 3 that satisfy (69) for all sufficiently small $\delta>0$.

## Appendix D

## Algorithm to Detect Cyclically Non-negative Counting Forms

In this appendix, we present some pseudocode in Algorithm 1 for an algorithm that computes whether a given $u$-only counting form with pure rank $r$ is cyclically non-negative in the sense of Definition 6 . Specifically, given the coefficients of a $u$-only counting form $w$ with pure rank $r$ as input, Algorithm 1 first constructs the directed graph $\mathcal{G}_{r}(w)$ by adding three outgoing edges to each vertex in $\mathcal{Y}^{r}$, even if the coefficient corresponding to the vertex (or basis clause) is zero, i.e. even if the outgoing edges are assigned zero weight. (Indeed, edges with zero weight are not equivalent to non-existent edges, because the latter kind of edges can be construed as having infinite weight.) Then, Algorithm 1 executes the Bellman-Ford algorithm, cf. [54, Section 24.1], to find negative cycles in $\mathcal{G}_{r}(w)$ with edge weights given by the relation (74) and source vertex chosen as $(u \cdots u)$ (all $u$ 's), which is a reasonable choice since we only consider $u$-only counting forms $w$. The correctness of this algorithm follows from part 2 of Proposition 8 and the correctness of the Bellman-Ford algorithm (in detecting the existence of negative cycles). We also remark that it is straightforward to modify Algorithm 1 to return an explicit negative cycle in $\mathcal{G}_{r}(w)$ when such cycles exist.

Since the correctness of the Bellman-Ford algorithm is typically proved for directed simple graphs, it is worth explicitly mentioning that the self-loops on the vertices $(0 \cdots 0),(1 \cdots 1)$, and $(u \cdots u)$ in $\mathcal{G}_{r}(w)$ do not affect the

```
Algorithm 1 Algorithm to determine whether a \(u\)-only counting form with pure rank is cyclically non-negative.
Input 1: A rank \(r \in \mathbb{N} \backslash\{0\}\)
Input 2: An array of coefficients \(\left(\alpha[v] \in \mathbb{R}: v \in \mathcal{Y}^{r}\right)\) corresponding to a \(u\)-only counting form \(w\) with pure rank
    \(r\), where \(\alpha[v]=0\) if \(v \in\{0,1\}^{r} \quad \triangleright \alpha[v]\) is the coefficient of the basis clause \(\{v\}\) for \(w\)
Boolean Output: true (if \(w \succeq_{c} \mathbf{0}\) ) or false (otherwise)
    Stage 1
    Initialization: \(E=\{ \}\)
    for all source vertices \(v=\left(v_{1} \cdots v_{r}\right) \in \mathcal{Y}^{r}\) do
        for all letters \(y \in \mathcal{Y}\) do
                \(v^{(y)}=\left(v_{2} \cdots v_{r} y\right) \quad \triangleright\) Construct destination vertex
                \(E . \operatorname{INSERT}\left\{\left(v, v^{(y)}\right)\right\} \quad \triangleright\) Insert directed edge into \(E\)
        end for
    end for
    Stage \(2 \triangleright\) Bellman-Ford algorithm to detect negative cycles reachable from \((u \cdots u)\)
    Initialization: distances \([v] \leftarrow+\infty\) for all \(v \in \mathcal{Y}^{r} \quad \triangleright\) Shortest path lengths from \((u \cdots u)\) to \(v\)
    Initialization: distances \([(u \cdots u)] \leftarrow 0 \quad \triangleright(u \cdots u)\) is the source vertex
    for \(i=1\) to \(3^{r}-1\) do \(\quad \triangleright\) Loop to relax edges
        for all edges \(\left(v_{1}, v_{2}\right) \in E\) do
            if distances \(\left[v_{1}\right]+\alpha\left[v_{1}\right]<\operatorname{distances}\left[v_{2}\right]\) then \(\quad \triangleright\) Edge weights are given by the relation (74)
                distances \(\left[v_{2}\right] \leftarrow \operatorname{distances}\left[v_{1}\right]+\alpha\left[v_{1}\right]\)
            end if
        end for
    end for
    for all edges \(\left(v_{1}, v_{2}\right) \in E\) do \(\quad \triangleright\) Loop to detect negative cycles reachable from \((u \cdots u)\)
        if distances \(\left[v_{1}\right]+\alpha\left[v_{1}\right]<\operatorname{distances}\left[v_{2}\right]\) then
            return false \(\quad \triangleright\) Negative cycle found!
        end if
    end for
    return true \(\quad \triangleright\) No negative cycles found
```

correctness of this algorithm. As mentioned earlier in subsection V-C, the self-loop edges on $(0 \cdots 0)$ and $(1 \cdots 1)$ have weight 0 , and never satisfy the if clause conditions in steps 12 and 18 of Algorithm 1 . Likewise, when the self-loop edge on $(u \cdots u)$ has non-negative weight, it also never satisfies the if clause conditions in steps 12 and 18. Hence, the algorithm runs correctly in this scenario. On the other hand, if the self-loop edge on $(u \cdots u)$ has negative weight, then it does satisfy the if clause condition in step 18 , and the algorithm correctly discovers a negative cycle.

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[^0]:    ${ }^{1}$ The cases $\delta=0$ and $\delta=\frac{1}{2}$ are uninteresting because the former corresponds to a deterministic grid and the latter corresponds to an independent grid.
    ${ }^{2}$ We can similarly define a more general model where every vertex $X_{k, j}$ has its own Boolean processing function $f_{k, j}$, but we will only analyze instances of the simpler model presented here.
    ${ }^{3}$ Likewise, we say that reconstruction is possible, or "broadcasting is possible," when $\lim _{k \rightarrow \infty} \mathbb{P}\left(h_{\mathrm{ML}}^{k}\left(X_{k}\right) \neq X_{0}\right)<\frac{1}{2}$, or equivalently, $\lim _{k \rightarrow \infty}\left\|P_{X_{k}}^{+}-P_{X_{k}}^{-}\right\|_{\mathrm{TV}}>0$.

[^1]:    ${ }^{4}$ As Gray explains in [25, Section 3], his proof of ergodicity is not complete; he is "very detailed for certain parts of the argument and very sketchy in others" [25]. Although the references in [25] indicate that Gray was preparing a paper with the complete proof, this paper was never published to our knowledge. So, the ergodicity of 1D PCA with 3-input majority gates has not been rigorously established.

[^2]:    ${ }^{5}$ This implies that $\mathcal{G}$ is actually a multigraph since we permit multiple edges to exist between two vertices in successive levels. As explained in [2, Section I-C], we can perceive the Bayesian network defined on this multigraph as a true DAG by constructing auxiliary vertices.

[^3]:    ${ }^{6} \mathrm{We}$ also refer interested readers to [42], [43], and the references therein for more on the related theory of strong data processing inequalities.

[^4]:    ${ }^{7}$ In this case, the transition function and interaction neighborhood really yield a South-West-Center (SWC) rule rather than Toom's NEC rule, but we will still use the historically inspired nomenclature.

[^5]:    ${ }^{8}$ This idea stems from the study of Fortuin-Kasteleyn random cluster representations of Ising models [46].

[^6]:    ${ }^{9}$ It is worth mentioning that in the ensuing coding problem in (41), if such a codeword does not exist, we can also decode the first codeword bit with zero probability of error because all codewords must have the first bit equal to 0 .

[^7]:    ${ }^{10}$ Indeed, the ML decoder for $X_{0,0}$ based on $Y_{k}^{\prime}$ has a smaller (or equal) probability of error than the decoder which first translates $Y_{k}^{\prime}$ into $Y_{k}$ by replacing every e with an independent Bernoulli $\left(\frac{1}{2}\right)$ bit, and then applies the ML decoder for $X_{0,0}$ based on $Y_{k}$ as in the coding problem (41). We also remark that the fact that a $\operatorname{BEC}(2 \delta)$ is "less noisy" than a $\operatorname{BSC}(\delta)$ is well-known in information theory, cf. [43, Section 6, Equation (16)].

[^8]:    ${ }^{11}$ Equivalently, we let $0=(0,0)$ and $1=(1,1)$ with abuse of notation, and we let $u=\{(0,1),(1,0)\}$.
    ${ }^{12}$ Note that the NAND of two $u$ 's could be 1 , e.g. $(1,0)$ and $(0,1)$ pass through a NAND gate to produce $(1,1)$. However, we conservatively treat the NAND of two $u$ 's as unknown, i.e. $u$.

[^9]:    ${ }^{13}$ As shown later in (53), it is easy to use (49) to lower bound $\mathbb{P}\left(T_{k+1}<+\infty\right)$, but this does not immediately lower bound the term $\mathbb{P}\left(T_{k+1}<+\infty \mid A_{k, 0}^{c}\right)$ in (50). So, we must apply a different approach here

[^10]:    ${ }^{14}$ Here, the notation for an edge $(v, z) \in \mathcal{Y}^{r} \times \mathcal{Y}^{r}$ should not be confused with the notation for concatenation of the strings $v$ and $z$; indeed, the correct meaning of $(v, z)$ should be clear from context in the sequel.

[^11]:    ${ }^{15}$ In this paper, we refer to directed paths that begin and end at the same vertex as (directed) circuits, and we refer to circuits with no repeated vertices or edges as (directed) cycles.

[^12]:    ${ }^{16}$ For directed graphs that are not strongly connected, one could either run the Bellman-Ford algorithm several times with different choices of source vertices, or run the Floyd-Warshall algorithm, which can detect negative cycles reachable from any source vertex in the graph, cf. [54, Section 25.2].

[^13]:    ${ }^{17}$ Indeed, such matrices define boundary operators in the study of abstract simplicial complices in algebraic topology. Moreover, the dual Gramian matrix corresponding to the oriented incidence matrix of a directed simple graph is equal to the unnormalized Laplacian matrix of the associated undirected graph obtained by removing edge orientations (see e.g. [55, Section 8.3]).

[^14]:    ${ }^{18}$ Notice further that, since the constraint set is invariant to scaling with non-negative integers, $L_{1}<0$ implies $L_{1}=-\infty$, i.e. if there exists $x \in \mathbb{Z}^{3^{r+1}}$ satisfying $\mathbf{B} x=\mathbf{0}$ and $x \geq \mathbf{0}$ such that $\alpha^{\mathrm{T}} \mathbf{B}_{\text {out }} x<0$, then the integer LP is unbounded. Similarly, $L_{1} \geq 0$ implies $L_{1}=0$, because the zero vector is a feasible solution of the integer LP. Thus, there are only two possible values of $L_{1}$, viz. $L_{1} \in\{0,-\infty\}$.

[^15]:    ${ }^{19}$ Specifically, we index the entries of vectors in $\mathbb{R}^{3^{s}}$, or the rows and columns of appropriately sized matrices, using $\left[3^{s}\right]$ and assume the following bijective correspondence between $\left[3^{s}\right]$ and strings (or basis clauses) in $\mathcal{Y}^{s}$ : For each $v^{\prime} \in\left[3^{s}\right]$, we first write $v^{\prime}$ in its $s$-length ternary representation, and then replace each 2 in this representation with a $u$. The resulting string $v$ belongs to $\mathcal{Y}^{s}$, and defines the correspondence $\left[3^{s}\right] \ni v^{\prime} \leftrightarrow v \in \mathcal{Y}^{s}$.

[^16]:    ${ }^{20}$ In contrast, every vertex in the 2D regular grid model has two outgoing edges that have $90^{\circ}$ separation between them (see Figure 1 ).

[^17]:    ${ }^{21}$ Note that the source vertex of $e_{i+1}$ is equal to the destination vertex of $e_{i}$ for all $i \in\{1, \ldots, n-1\}$, and the source vertex of $e_{1}$ is equal to the destination vertex of $e_{n}$.

[^18]:    ${ }^{22}$ This representation of $g$ can be construed as the converse of Lemma 6, cf. [55, Theorem 14.2.2, Corollary 14.2.3].

[^19]:    ${ }^{23}$ We used an alternative graph construction in our development in subsections V-C and V-E since it was arguably more natural.

