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A Partial Order Approach to Decentralized Control of Spatially Invariant Systems

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***Abstract*— In this paper, we consider a class of spatially distributed systems which have a special property known as spatial invariance. It is well-known that for such problems, the problem of designing decentralized controllers is hard. In this paper, we generalize some previously known results and show that for a certain class of problems, the control problem has a convex reformulation. We employ the notion of partially-ordered sets and the associated notion of incidence algebras to introduce a class of systems called poset causal systems. We show that poset causal systems are a fairly large class of systems that properly include some other classes of systems studied in the literature (namely cone-causal and funnel causal systems). Finally we show that the set of poset-causal controllers for poset-causal plants are amenable to a convex parameterization.**

I. INTRODUCTION

Many important control problems today are large-scale, complex and decentralized. The lack of availability of global state information makes the implementation of classical centralized controllers practically infeasible. This has led several researchers to the study of decentralized control. Examples of large-scale control systems include flocks of aerial vehicles and the power distribution grid.

It is well-known that in general decentralized control is a hard problem [15]. Blondel and Tsitsiklis [3] have shown that certain instances of such problems are in fact intractable. On the other hand, Voulgaris [13], [14] presented several cases where decentralized control is in fact tractable. Rotkowitz and Lall [10] have presented a criterion known as quadratic invariance that characterizes a class of problems in decentralized control that have the property that problems become convex in the Youla parameter. Shah and Parrilo [11] have studied the decentralized control problem using partially-ordered sets (or *posets*) and shown that several interesting classes of decentralized communication structures may be modeled using posets. This approach also has the appealing property that problems are convex in the Youla-parameter, a step in the direction towards computational tractability.

Many practical control problems are also naturally *spatially distributed*, i.e. the overall subsystem is composed of many subsystems, each of which is at a different spatial location. Spatially distributed systems (and the related notion of distributed parameter systems) have also been extensively studied (see [6], [2], [4], [5]). It is natural to study decentralized control in the spatially distributed setting since many spatially distributed systems are also large scale and lumped in the sense that the controller may interface with the system at only a relatively small number of spatial coordinates and thus may face natural communication constraints. The problem of decentralized control of spatially distributed systems becomes considerably simpler when the system has a property known as *spatial invariance*. Intuitively, this means that the overall system is not only time-invariant but also invariant under spatial translations. Such systems have been studied in some detail by Bamieh et. al. [7], [1].

In this paper we propose to study the problem of decentralized control of spatially invariant distributed systems based on the partial order framework developed in [11]. We show that this framework allows one to study several interesting classes of decentralized problems. To study communication structures for spatially invariant systems, it is sufficient to study the *spatio-temporal impulse response*, which constitutes the impulse response in the joint spatial and temporal domain (denoted by $h(x,t)$, where x is the spatial domain and t is the temporal domain) when the system reacts to an impulse at the origin $(x,t) = (0,0)$. The *support* of $h(x,t)$ determines the communication structure of the system.

We study the support of the impulse response using a poset-based approach. Systems which are amenable to this approach are called *poset-causal* systems. More concretely, we impose a poset on the domain of the impulse response. Posets, which are combinatorial objects come with associated algebraic objects known as incidence algebras [8]. We show that impulse responses for poset-causal systems belong to the incidence algebra. Due to the algebraic properties of the incidence algebra, poset causal systems can be shown to be closed under composition. This closure property allows us to parameterize the set of controllers that are also poset-causal. Our main contributions in this paper are the following:

1) We introduce the framework of posets to study decentralized control of spatially distributed systems.
2) We show that this framework generalizes some previously know results about classes of problems that are closed under composition. Specifically, we show that the main result in [1] regarding the fact that funnel causal impulse responses are closed under convolution are a special case of a similar result that holds for poset causal impulse responses. Indeed, we show that

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funnel causal systems are *proper* subsets of poset-causal systems.

3) We show that this closure property allows one to parameterize the set of poset-causal controllers.
4) One can also view the result in this paper as a generalization of our previous result in [11] where we use the poset paradigm for synthesis of controllers for finite dimensional controllers.

The key property that needs to be identified to generalize funnel causality into this poset framework is subadditivity. This property has also been recognized independently by Rotkowitz et al [9].

The rest of this paper is organized as follows. In Section II we provide the preliminaries regarding posets and incidence algebras, and the basic notions of spatially invariant systems that are necessary to read the rest of this paper. In section III we study the poset formulation of our problem. We study the relationship between poset causal systems and funnel causal systems. We also study several interesting examples of poset causal systems. In section IV we examine how the set of all poset causal controllers may be parameterized. In section V we conclude the paper.

II. PRELIMINARIES

A. Order-theoretic Preliminaries

Definition 1: A partially ordered set (or *poset*) consists of a set P along with a binary relation $\preceq$ which is reflexive, antisymmetric and transitive.
Posets may be defined on finite or infinite sets. The following is an example of a *finite poset*.

Example 1: An example of a poset with three elements (i.e. $\mathcal{P} = \{a, b, c\}$) with order relations $a \preceq b$ and $a \preceq c$ is shown in Figure 1.

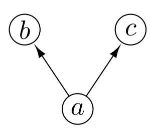

Fig. 1. A poset on the set $\{1, 2, 3\}$.

As already mentioned, the poset in Example 1 is finite. In this paper, we are interested in decentralized control of spatio-temporal systems. In Section III we will formally define a poset structure on the spatio-temporal variables, which will allow us to model a class of decentralization constraints. More concretely, if $(x, t) \in \mathbb{F}^n \times \mathbb{T}$ are the spatio-temporal variables, then the partial order relation will be a relation on the set $\mathbb{F}^n \times \mathbb{T}$. This poset will be an *infinite poset* since the underlying set is infinite.

Definition 2: Let $\mathcal{P}$ be a poset. Let $\mathbb{Q}$ be a ring. The set of all functions

$$f : P \times P \to \mathbb{Q}$$

with the property that $f(x, y) = 0$ if $x \not\preceq y$ is called the *incidence algebra* of $\mathcal{P}$ over $\mathbb{Q}$. It is denoted by $\mathcal{I}_{\mathcal{P}}(\mathbb{Q})$. If the ring and the poset are clear from the context, we will simply denote this by $\mathcal{I}$ (we will usually work over $\mathbb{R}$ or $\mathbb{Z}$).

For a finite poset $\mathcal{P}$, the set of functions in the incidence algebra may be thought of as matrices with a specific sparsity pattern given by the order relations of the poset.

Definition 3: Let $\mathcal{P}$ be a poset. The function $\zeta(P) \in I_{\mathcal{P}}(\mathbb{Q})$ defined by

$$\zeta(P)(x, y) = \begin{cases} 0, & \text{if } x \not\preceq y \\ 1, & \text{otherwise} \end{cases}$$

is called the *zeta-function* of $\mathcal{P}$.
Clearly the zeta-function of the poset is a member of the incidence algebra

Example 2: The matrix representation of the zeta function for the poset from Example 1 is as follows:

$$\zeta_{\mathcal{P}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The incidence algebra is the set of all matrices in $\mathbb{Q}^{3\times 3}$ which have the same sparsity pattern as its zeta function.
Given two functions $f, g \in I_{\mathcal{P}}(\mathbb{Q})$, their sum $f+g$ and scalar multiplication cf are defined as usual. The product $h = f \cdot g$ is defined as follows:

$$h(x, y) = \sum_{z\in\mathcal{P}} f(x, z)g(z, y).$$

As mentioned above, we will frequently think of the functions in the incidence algebra of a poset as square matrices (of appropriate dimension) inheriting a sparsity pattern dictated by the poset. The above definition of function multiplication is such that in the finite case it is consistent with matrix multiplication. In the infinite case, we will see that by replacing the sum by an integral, we will be able to interpret the above definition also as a convolution.

Theorem 1: Let $\mathcal{P}$ be a poset. Under the usual definition of addition, and multiplication as defined in (1) the incidence algebra is an associative algebra (i.e. it is closed under addition, scalar multiplication and function multiplication).

Proof: Closure under addition and scalar multiplication is obvious. Let $f, g \in \mathcal{I}$. Consider elements x, y such that $x \not\preceq y$, so that $f(x, y) = g(x, y) = 0$. Indeed if $x \not\preceq y$, there cannot exist a z such that $x \preceq z \preceq y$. Hence, in the above sum, either $f(x, z) = 0$ or $g(z, y) = 0$ for every z, and thus $h(x, y) = 0$. ∎

A standard corollary of this theorem is the following.

Corollary 1: Suppose $A \in \mathcal{I}$ is invertible. Then $A^{-1} \in \mathcal{I}$.

B. Control-theoretic preliminaries

In this section, we introduce the notion of spatially invariant systems. These are a class of distributed parameter infinite-dimensional systems that evolve along spatio-temporal coordinates $(x, t) \in \mathbb{G} \times \mathbb{T}$ where $\mathbb{G}$ is the spatial domain and $\mathbb{T}$ is the temporal domain. We assume that $\mathbb{G} = \mathbb{F}^n$, with $\mathbb{F}$ chosen to be either $\mathbb{Z}$ or $\mathbb{R}$, and a similar choice is made for $\mathbb{T}$. In this paper, we study the class of systems that are *spatially invariant*, i.e. spatio-temporal systems that are invariant under translations along the spatial coordinate. Just as LTI systems are characterized by impulse responses (such a description being possible due to the time invariance property), spatially invariant systems can be completely described by a spatio-temporal impulse response $\psi(x,t)$. We will typically be thinking of spatio-temporal responses that are described by linear PDEs, for example the wave equation:

$$\partial_t^2\psi(x,t) = c^2\partial_x^2\psi(x,t) + u(x,t). \quad (1)$$

This has a state space description:

$$\begin{aligned} \partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} &= \begin{bmatrix} 0 & I \\ c^2\partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\ \psi &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \end{aligned}$$

The impulse response of a spatially invariant system will have some support on the (x,t) plane. We assume that this support is characterized by a *support function* $f : \mathbb{G} \to \mathbb{T}$ with the property that $\psi(x,t) = 0$ whenever $t < f(x)$.

III. Communication Constraints and Posets

In this section, we show how communication constraints among the spatial coordinates can be naturally modeled in the language of partially ordered sets.

A. Partial Order Formulation

Let $x \in \mathbb{F}^n$ be the spatial variable and $t \in \mathbb{T}$ denote the temporal variable. Let $f : \mathbb{F}^n \to \mathbb{T}$ be the support function. We define a partial order on the tuple (x,t) as follows:

Definition 4:
The relation $(x_1,t_1) \preceq (x_2,t_2)$ holds if

1. $t_1 \le t_2$ (in the standard ordering on $\mathbb{T}$),
2. $f(x_2 - x_1) \le t_2 - t_1$ (in the standard ordering on $\mathbb{T}$).

Proposition 1: Suppose the support function $f : \mathbb{F}^n \to \mathbb{T}$ satisfies the following properties:

$$\begin{array}{ll} 1. & f(0) = 0, \\ 2. & f(x) > 0 \ \text{ for } \ x \neq 0, \\ 3. & f(x_1 + x_2) \le f(x_1) + f(x_2) \\ & \qquad \text{for all } x_1, x_2 \in \mathbb{K}^n \ \text{(subadditivity).} \end{array} \quad (2)$$

Then the relation $\preceq$ in Definition 4 is a partial order relation.

Proof: See [12]. ■

Once a partial order is defined on the space, one can think of the space as a poset $\mathcal{P} = (P, \preceq)$ (in our case the set $P = \mathbb{F}^n \times \mathbb{T}$). By defining a multiplication rule on functions of the form $h : P \times P \to \mathbb{R}$ one can define the incidence algebra associated with the poset.

Rather than considering all functions of the form $h : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, which are of the form $h((x_1,t_1),(x_2,t_2))$ we restrict our attention to those functions which are *spatially and temporally invariant*, i,e. the value of the function depends only on $x_1 - x_2$ and $t_1 - t_2$. More precisely, these functions are of the form $h((x_1,t_1),(x_2,t_2)) = h'(x_1 - x_2, t_1 - t_2)$.

Definition 5: The set of functions $h : \mathcal{P} \times \mathcal{P} \to \mathbb{G}$ with the property that :

$$\begin{array}{ll} 1. & h((x_1,t_1),(x_2,t_2)) = h'(x_1 - x_2, t_1 - t_2) \\ & \qquad\qquad \text{(called spatially invariance)} \\ 2. & h'(x_1 - x_2, t_1 - t_2) = 0 \ \text{ for } \ (x_1,t_1) \npreceq (x_2,t_2) \\ & \qquad\qquad \text{(called order sparsity).} \end{array}$$

is called the *spatially invariant incidence algebra* with respect to the support function f. It is denoted by $\mathcal{I}_f$.

Remark Note that a function in the spatially invariant incidence algebra $h(x_1 - x_2, t_1 - t_2)$ is nonzero only if $f(x_2 - x_1) \le t_2 - t_1$.

We now justify the reason for calling the object defined in Definition 5 an algebra. We show next that one can define a natural multiplication operation on this object, and that the object is closed under this multiplication, justifying its description as an "algebra".

We first define the multiplication operation.

Definition 6: Let $h_1(x_1 - x_2, t_1 - t_2), h_2(x_1 - x_2, t_1 - t_2) \in \mathcal{I}_f$ be two incidence functions in the spatially invariant incidence algebra. Then,

$$\begin{aligned} & h_3((x_1,t_1),(x_2,t_2)) \\ & \quad \doteq h_1(x_1 - x_2, t_1 - t_2) \star h_2(x_1 - x_2, t_1 - t_2) \\ & \quad \doteq \int_{\mathbb{T}}\int_{\mathbb{G}} h_1(x_1 - x, t_1 - t) h_2(x - x_2, t - t_2) dx dt. \end{aligned} \quad (3)$$

Remark Note that in Definition 6, if either $\mathbb{K}$ or $\mathbb{F}$ is a discrete set, then the integration is replaced by a summation over the discrete set.

We now show the closure property of the incidence algebra.

Proposition 2: Let $h_1, h_2 \in \mathcal{I}_f$ be two functions in the spatially invariant incidence algebra. Then the following statements are true:

$$\begin{array}{ll} (a) & h_1 + h_2 \in \mathcal{I}_f, \\ (b) & \text{For every scalar } \ c, c \cdot h_1 \in \mathcal{I}_f, \\ (c) & h_1 \star h_2 \in \mathcal{I}_f. \end{array}$$

Proof: See [12]. ■

Definition 7: Given a spatially invariant distributed system with impulse response $h(x,t)$, the system is said to be *poset-causal* if the impulse response satisfies order sparsity with respect to a function f satisfying the conditions $1, 2$ and 3 of Proposition 1.

Since we defined multiplication in such a way that it is consistent with convolution of impulse response functions, we get the following important theorem as a direct corollary of Proposition 2:

Theorem 2: The composition of two spatially invariant poset-causal systems is also spatially invariant and poset-causal.

We can exploit this fact in the synthesis of certain structured decentralized controllers.

B. Relation to Funnel Causality

In [1], the authors introduce a specific class of communication constraints for spatially invariant systems. They call such systems *funnel causal* systems. In their paper, the authors show that convolution of funnel causal impulse responses are also funnel causal, and that such systems are thus closed under composition. Finally, the authors show that due to this closure property, the set of all stabilizing funnel causal controllers can be described in the Youla domain in a convex fashion, thus making it amenable to optimization.

Our results are closely related to (and in fact generalize) these results by Bamieh and Voulgaris. We show in this subsection that the statement "composition of funnel-causal systems is funnel causal" is *essentially a statement about poset causal systems*. We show that funnel causal systems are a sub-class of poset causal systems, i.e. if a system is funnel causal, one can construct a poset and an associated incidence algebra that contains the impulse response of the given system. In fact funnel causal systems form a *proper subset* of poset causal systems, indeed in the next subsection we will provide examples of poset-causal systems that are not funnel causal.

In this section we will show that Theorem 2 completely generalizes the result of Bamieh [1]. The outline of the argument is as follows. Funnel causal systems are defined in terms of concave support functions (in one dimension), and poset-causal systems are defined in terms of sub-additive support functions as defined in (2). We first show that for functions in one dimension, concave functions are subadditive. Thus, if f is concave (thus funnel causal), f is sub additive and by Proposition 1 the system is poset causal. Proposition 1 shows that such systems have a naturally associated poset and incidence algebra. By Theorem 2 poset-causal (thus funnel-causal) systems are closed under composition. In [1], the authors define funnel causal systems (and the related notion of propagation functions, which are similar to the notion of support functions we introduced earlier in this paper) in the following way.

Definition 8: A scalar valued function $f(x)$ is said to be a *propagation function* if f is nonnegative, $f(0) = 0$ and such that $\{f(x), x \geq 0\}$ and $\{f(x), x \leq 0\}$ are concave respectively.

Definition 9: A systems is said to have the property of *funnel causality* if its impulse response is such that

$$h(x,t) = 0, \qquad \text{for } t < f(x),$$

where $f(x)$ is a propagation function.
Essentially, the spatio-temporal impulse response is supported in a funnel shaped region. We next show that such functions are in fact subadditive, hence they can be endowed with a partial order with respect to the propagation function f.

Proposition 3: If $f : \mathbb{R} \to \mathbb{R}$ is such that $f(0) = 0$, $f(x) > 0$ for $x \neq 0$ and $\{f(x), x \geq 0\}$, $\{f(x), x \leq 0\}$ are concave, then f is subadditive.

Proof: See [12] ■

As a corollary, we obtain the following result by Bamieh [1, Lemma 1].

Corollary 2: Composition of spatially invariant funnel causal systems is also spatially invariant and funnel causal.

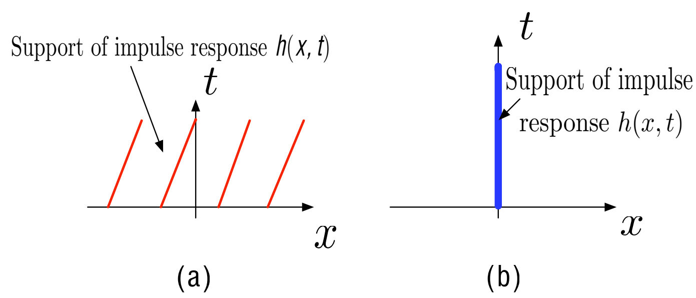

Fig. 2. Examples of poset causal systems. (a) A centralized causal system. (b) A completely decentralized causal system.

Proof: See [12]. ■

C. Examples of Poset Causal Systems

In this subsection, we consider some examples of poset-causal systems to show how some interesting communication structures can be modeled via this poset framework.

Example 3: Some extreme examples of spatially invariant systems are examples of completely decentralized problems and fully centralized problems as illustrated in Fig. 2.

Example 4: A class of systems that has been studied in the literature corresponds to the case where the support function $f(x) = c|x|$ where x is understood to be one-dimensional. Such systems have been called *cone causal* systems. Note that $f(0) = 0$, $f(x) > 0$ for $x \neq 0$. Subadditivity of f follows from the triangle inequality (alternatively, from the concavity of $|x|$). Hence, f satisfies all the conditions to prescribe a partial order on $\mathbb{G} \times \mathbb{T}$. Such systems draw motivation from the following interpretation. Suppose the system responds to an impulse at the origin. Then $h(x,t)$ is going to be supported on a *light cone* originating at the origin with speed of light equal to c. In other words the system has a constant (but finite) speed of signal propagation. Such examples arise naturally in physical systems. For example, linear wave equations have this property, as shown in [1].

Example 5: Another class of poset causal systems that have been studied in the literature are funnel causal systems. As described in Section III-B funnel causal systems are subclasses of poset causal systems. For more details and examples, the reader is referred to [1].

Example 6: For cases where the spatial domain is multi-dimensional (for example $\mathbb{R}^n$), examples of classes of systems closed under convolution have not been studied, to the best of our knowledge. (For example. the result for funnel-causal systems, closure under convolution only holds for one dimension). The advantage of our approach is that it abstracts the essential property for convolutional closure to hold. This essential property that we identify is subadditivity, which arises naturally for many classes of (multi-dimensional) functions.

A natural class of support functions $f(x)$ are all *norms* $f(\cdot) = \|\cdot\|_p$ for $p \geq 1$. Clearly, $f(0) = 0$ and $f(x) > 0$ for $x \neq 0$ by definition of a norm. Also, by the triangle

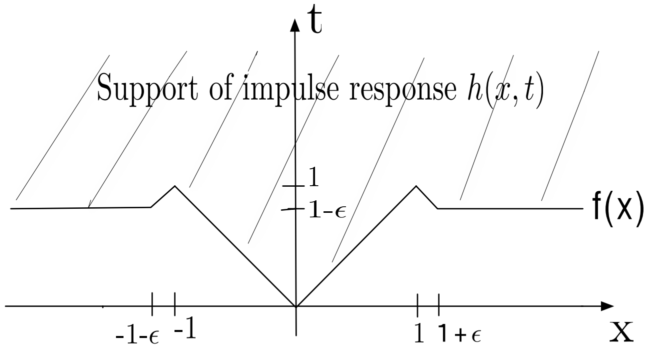

Fig. 3. A sub-additive non-concave support function.

inequality for norms,

$$f(x_1+x_2) = \|x_1+x_2\|_p \leq \|x_1\|_p+\|x_2\|_p = f(x_1)+f(x_2).$$

Again, since f satisfies all the properties necessary to impose a poset structure on the system, the incidence algebra based argument tell us that the corresponding impulses will be closed under convolution.

It is interesting to note that, in [1], the authors identified concavity of f as being the property essential to having convolutional closure. In this example, norms in general are not concave, on the contrary, they are *convex*, yet we have convolutional closure. This further strengthens the argument that sub-additivity is a more fundamental property. (In the one-dimensional case, of course, all induced norms coincide with the absolute value function, which is both concave and convex, hence this distinction becomes uninteresting).

In the next example we further investigate the relationship between funnel causality and poset causality. As already mentioned, the property at the heart of funnel causality is concavity, whereas the property at the heart of poset causality is sub-additivity. We have already shown that on one dimension, concavity implies sub-additivity. It is natural to wonder wether the converse is true, i.e. wether all subaddtive functions are concave. In higher dimensions, this is not true since (as explained in Example 6) p-norms for $p > 1$ are nonconcave but subadditive. In the one dimensional case, this counter-example clearly does not work, since the absolute value function is concave. Example 7 below is an example of a sub-additive function which is not concave (nor even convex).

Example 7: Consider the function $f : \mathbb{R} \to \mathbb{R}$ (see Fig. 3) given by

$$f(x) = \begin{cases} |x| \text{ for } |x| \leq 1 \\ 2-x \text{ for } 1 < x \leq 1+\epsilon \\ 2+x \text{ for } -1-\epsilon \leq x < -1 \\ 1-\epsilon \text{ for } |x| > 1+\epsilon. \end{cases}$$

Here we assume that ϵ is a sufficiently small positive number, say $0 < \epsilon < \frac{1}{4}$. Clearly this function is nonconcave, a straight-forward verification of several cases shows that this function is indeed subadditive.

IV. Parameterization of Poset Causal Controllers

The problem of designing optimal poset-causal controllers for a poset causal system with respect to some support function f is possible using the closure property of poset causal systems. We discuss this briefly in this section. We are interested in solving the following optimal control problem:

$$\begin{array}{ll} \inf & \|F(G,K)\| \\ \text{subject to} & K \text{ stabilizing} \\ & K \in S_f. \end{array} \quad (4)$$

As a consequence of Theorem 2, we know that the composition of two poset causal systems is also poset causal. If one has a coprime factorization of the plant G, with $G = NM^{-1}$ such that $XM - YN = I$, where X, Y, M, N are all spatially invariant and poset causal, then the set of all spatially invariant poset causal controllers is parameterized by

$$K = (Y + MQ)(X + NQ)^{-1}$$

with Q stable, spatially invariant and poset causal. Using this parametrization it is possible to show that problem (8) reduces to the following convex optimization problem.

$$\begin{array}{ll} \inf & \|H - UQV\| \\ \text{subject to} & Q \text{ stable} \\ & Q \in S_f, \end{array} \quad (5)$$

where H, U, V depend only on the description of the specified plant G. Note that this problem is still an infinite dimensional problem (albeit convex) and exact computation of the optimal controller may still be hard.

V. Conclusion

In this paper we studied the problem of decentralized control of spatially invariant distributed systems based on a poset framework. We generalized some previously known results regarding funnel causal systems using this framework. We also studied some interesting examples of systems that can be modeled using the poset framework. Finally, we showed how poset causal controllers for poset causal systems are amenable to a convex parameterization.

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